JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **25**, No. 1, February 2012

NOTES ON MODULAR ORDERED SETS

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Abstract. Generalizing modular lattices, a concept of modular ordered sets was introduced by Chajda and Rachunek. In this paper, we characterize modular ordered sets as those partially ordered set P satisfying that for $a, b, c \in P$ with $b \leq c$, $l(a, b) = l(a, c)$ and $u(a, b) = u(a, c)$ imply $b = c$. Using this, we obtain a sufficient condition for them. We also discuss the modularity of the Dedekind-MacNeille completions of ordered sets.

1. Introduction

The concept of modular lattices was introduced by Dedekind. It is well known that the class of modular lattices contains that of distributive lattices and that the lattice of normal subgroups (ideals, resp.) of a group (ring, resp.) is modular. In order to generalize the concept of modular lattices in the setting of partially ordered sets, there have been several attempts for modular ordered sets, e.g. in [2], [5] among others.

In this note we study modular ordered sets introduced by Chajda and Rachunek ([2]). We recall that a lattice is modular iff for any a, b, c in the lattice with $b \leq c$, $a \wedge b = a \wedge c$ and $a \vee b = a \vee c$ implies $b = c$. We extend this characterization of modular lattices to modular ordered sets. Using three point ordered set N_1 with a chain of two points and an element which is not comparable with both elements of the chain, we also have a forbidden configuration for modular ordered sets. Indeed an ordered set $(P, \leq)$ is modular if $N_1 \not\hookrightarrow P$ (see Figure 1.1 for N_1). But the converse need not be true and hence the class of modular ordered sets need not be hereditary. It is contrary to the fact that the class of modular lattices is equational, which implies that the class is hereditary and productive.

Received November 18, 2011; Revised January 26, 2012; Accepted February 02, 2012.

2010 Mathematics Subject Classification: Primary 06A06, 06C05, 06B23.

Key words and phrases: modular ordered sets, Dedekind-MacNeille completions.

This work was supported by the Soongsil University Research Fund.

The Dedekind-MacNeille completion $(DM(P), \subseteq)$ of a partially ordered set $(P, \leq)$ is the smallest complete lattice containing P. We show that the Dedekind-MacNeille completions of six forbidden ordered sets for moular ordered sets given in [2] are not modular.

We now introduce some notations for the remaining part of our note. In the following, $(P, \leq)$ always denotes a partially ordered set. For $a \in P$, we write $\uparrow a = \{x \in P \mid a \leq x\}$ and dually $\downarrow a = \{x \in P \mid x \leq a\}$. For $S \subseteq P$, we define $l(S) = \{x \in P \mid x \leq s \text{ for all } s \in S\} = \bigcap\{\downarrow s \mid s \in S\}$, which is called the *lower set* of S and $u(S) = \{x \in P \mid s \leq x \text{ for all } s \in S\} = \bigcap\{\uparrow s \mid s \in S\}$ the *upper set* of S dually. If $S = \{s_1, s_2, \cdots, s_n\}$ is a finite subset of P, then we write $l(S) = l(s_1, s_2, \cdots, s_n)$ and $u(S) = u(s_1, s_2, \cdots, s_n)$ briefly. Thus $l(s_1, s_2, \cdots, s_n) = \bigcap_{i=1}^{n} \downarrow s_i$ and $u(s_1, s_2, \cdots, s_n) = \bigcap_{i=1}^{n} \uparrow s_i$. Further, For $a \in P$ and $S \subseteq P$, $l(\{a\} \bigcup S)$ will be denoted by simply $l(a, S)$ and $u(a, S)$ means $u(\{a\} \bigcup S)$.

For the terminology not introduced in the paper, we refer to [1] and [3] for the lattice theory and order structure. Throughout this paper, we mean that an ordered set is a partially ordered set.

2. Modular ordered sets

We first recall that a lattice L is called modular if it satisfies : for any $a, b, c \in L$ with $b \leq c$, $c \wedge (a \vee b) = (c \wedge a) \vee b$.

It is straighforward that every distributive lattice is modular. Further, a lattice L is modular iff for any $a, b, c \in L$, $(c \wedge b) \vee (a \wedge b) = [(c \wedge b) \vee a] \wedge b$. Thus the class of modular lattices is an equational class.

The following definition is due to Chajda and Rachunek([2]).

Definition 2.1. An ordered set $(P, \leq)$ is called *modular* if for any $a, b, c \in P$ with $b \leq c$, $l(u(b, a), c) = l(u(b, l(a, c)))$.

We note that for any a, b, c in a lattice L, $l(a, c) = \downarrow (a \wedge c)$ and $u(a, b) = \uparrow (a \vee b)$ and hence $l(u(b, a), c) = \downarrow [c \wedge (a \vee b)]$ and $l(u(b, l(a, c))) = \downarrow [(c \wedge a) \vee b]$. Thus every modular lattice is a modular ordered set. This amounts to saying that the concept of modular ordered sets is a generalization of modular lattices. We give some examples of modular ordered sets which need not be a lattice.

Example 2.2. In Figure 1.1, consider the ordered set $\underline{N_3} = \{a, b, c, p\}$ such that $p < a$ and $p < b < c$. It is not a modular ordered set since

$l(u(b,a),c) = \{b,c,p\}$ and $l(u(b,l(a,c))) = \{b,p\}$. Similarly N_1, $\overline{N_3}$ and N_5 are not modular, whereas N_0, $\overline{N_2}$, $\underline{N_2}$ and N_2 are modular.

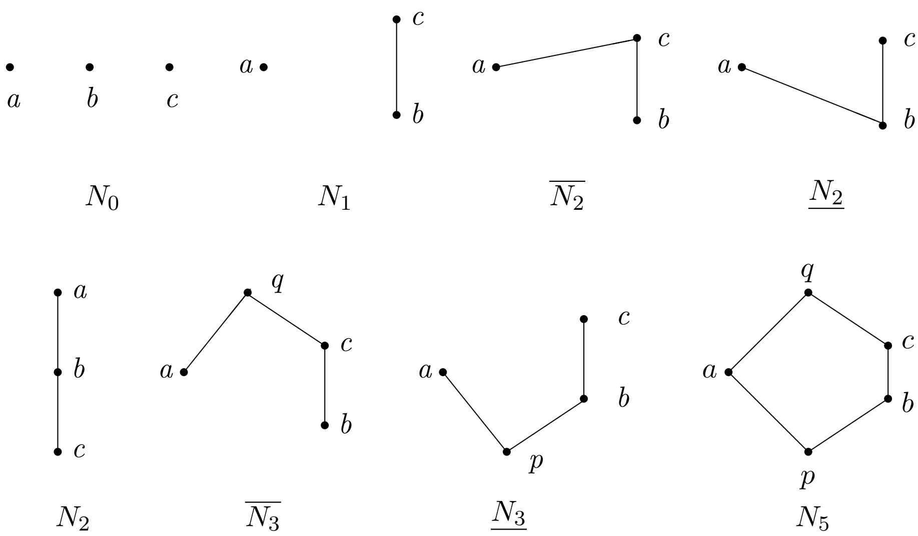


Figure 1.1

It is well known that a lattice L is modular iff for any a, b, c in L with $b \leq c$, $a \wedge b = a \wedge c$ and $a \vee b = a \vee c$ implies $b = c$.

Extending the above, we have a characterization of modular ordered sets as follows:

THEOREM 2.3. *For an ordered set $(P, \leq)$, the following are equivalent:*
(1) P is a modular ordered set.
(2) For $a, b, c \in P$ with $b \leq c$, if $l(a,b) = l(a,c)$ and $u(a,b) = u(a,c)$, then $b = c$.

Proof. Assume that P is a modular ordered set. For $a, b, c \in P$ with $b \leq c$, suppose that $l(a,b) = l(a,c)$ and $u(a,b) = u(a,c)$. Then $l(u(b,a),c) = l(u(a,c),c) = \downarrow c$ and $l(u(b,l(a,c))) = l(u(b,l(a,b))) = \downarrow b$. Since these two sets are the same in the modular ordered set P, we have $b = c$. Conversely suppose that P is not modular. Then there are a, b and c in P with $b \leq c$, $l(u(b,a),c) \neq l(u(b,l(a,c)))$. We claim that a and c are incomparable. Indeed, if $a \leq c$ or $c \leq a$ then $l(u(b,a),c) = l(u(b,l(a,c)))$, which is a contradiction. Similarly a and b are also incomparable. For such a, b and c, $l(a,b) = l(a,c)$ and $u(a,b) = u(a,c)$ but $b \neq c$. This completes the proof. □

Using the above, one can also prove that every modular lattice is a modular ordered set, because for $a, b, c \in P$ with $b \leq c$, if $l(a, b) = l(a, c)$ and $u(a, b) = u(a, c)$, then $a \wedge b = a \wedge c$ and $a \vee b = a \vee c$. Hence $b = b \vee (a \wedge b) = b \vee (a \wedge c) = (b \vee a) \wedge c = (c \vee a) \wedge c = c$.

Let P and Q be ordered sets. A function $\varphi : P \longrightarrow Q$ is called *order-preserving* if $x \leq y$ in P implies $\varphi(x) \leq \varphi(y)$ in Q. And it is called *an order-embedding* if $x \leq y$ in P is equivalent to $\varphi(x) \leq \varphi(y)$ in Q. Clearly an order-embedding $\varphi : P \longrightarrow Q$ is 1-1, for P and Q are ordered sets. In case we write $P \hookrightarrow Q$.

Moreover, if $\varphi : P \longrightarrow Q$ is an order-embedding from P *onto* Q, then φ is called *an order-isomorphism*. In case we say that P and Q are *order-isomorphic* and write $P \cong Q$.

Notation. For two ordered sets P and Q, we write $P \not\hookrightarrow Q$ to indicate that Q has no subsets which are order-isomomorphic to P.

We quote the following Theorem 2.4 obtained by Chajda and Rachunek ([2]). Before we state it, we recall the following definition introduced in [2].

A subset Q of an ordered set P is said to be *an LU subset* of P if for all $a, b \in Q$, (i) $l_Q(a, b) = \emptyset$ if and only if $l_P(a, b) = \emptyset$ and (ii) $u_Q(a, b) = \emptyset$ if and only if $u_P(a, b) = \emptyset$.

Further, the follwoing ordered sets $\overline{N_5}$ and $\underline{N_5}$ are defined as well in [2].

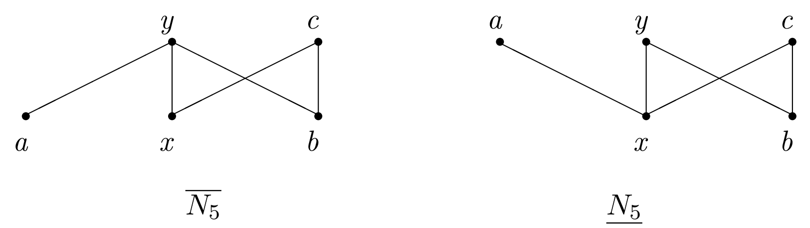


Figure 1.2

Theorem 2.4. ([2]) *If an ordered set $(P, \leq)$ is not modular, then P contains an LU subset which is order-isomorphic to one of the ordered sets $N_1, \overline{N_3}, \underline{N_3}, N_5, \overline{N_5}, \underline{N_5}$ given in the above.*

We now have a more simple forbidden configuration for a modular ordered set as follows.

THEOREM 2.5. *An ordered set $(P, \leq)$ is modular if $N_1 \not\hookrightarrow P$.*

Proof. Suppose that P is not modular, then there are $a, b, c \in P$ such that $b \leq c$, $l(a,b) = l(a,c)$ and $u(a,b) = u(a,c)$ but $b \neq c$. Then a and b are incomparable. Indeed, assume that $a \leq b$, then $u(a,b) = \uparrow b$ and $u(a,c) = \uparrow c$. Since $u(a,b) = u(a,c)$, $\uparrow b = \uparrow c$ and hence $b = c$ which is a contradiction. Also if $b \leq a$, then $l(a,b) = \downarrow b$ and $u(a,b) = \uparrow a$. Since $u(a,b) = u(a,c)$, $u(a,c) = \uparrow a$ and hence $c \leq a$. Since $l(a,b) = l(a,c)$, $\downarrow b = \downarrow c$ so that $b = c$. This is also a contradiction. Moreover suppose that $a \leq c$, then $l(a,b) = l(a,c) = \downarrow a$ which implies $a \leq b$, and hence a contradiction by the above proof. If $c \leq a$, then $u(a,b) = u(a,c) = \uparrow a$ which implies $b \leq a$, so tht one has a contradiction. Hence a and c are incomparable. Thus $\{a, b, c\}$ is a subset of P which is order-isomorphic to N_1. This completes the proof. $\square$

Sometimes it is convenient to use the contrapositive statement of Theorem 2.5, i.e., if an ordered set $(P, \leq)$ is not modular, then $N_1 \hookrightarrow P$.

The next remark says that the converse of Theorem 2.5 need not be true.

REMARK 2.6. Consider $P = \{a, b, c, x, y\}$ and $Q = \{a, b, c\} \subset P$ in Figure 1.3. Then P satisfies the condition in Theorem 2.3 for all cases and hence P is a modular ordered set but $(Q, \leq_Q)$ is not modular, where $\leq_Q$ is the induced partial order on Q. Thus a subset of a modular ordered set need not be modular again.

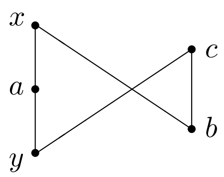


Figure 1.3

3. Dedekind-MacNeille completions of ordered sets

For an ordered set $(P, \leq)$, let $DM(P) = \{S \subseteq P \mid l(u(S)) = S\}$. Then $(DM(P), \subseteq)$ is the smallest lattice containing P and also a complete lattice. It is called the *Dedekind-MacNeille completion(simply,* *DM-completion) of* P. In fact, $DM(P)$ is a completion via the order embedding $\varphi : P \longrightarrow DM(P)$ given by $\varphi(a) = \downarrow a$ for all $a \in P$([3]).

In this section, we discuss the modularity of the Dedekind-MacNeille completions of ordered sets.

THEOREM 3.1. ([3]) *Let* P *be an ordered set and* $\varphi : P \longrightarrow DM(P)$ *be the order embedding of* P *into its Dedekind-MacNeille completion* $DM(P)$. *Then*

(1) $\varphi(P)$ *is both join-dense and meet-dense in* $DM(P)$.

(2) If Q *is an ordered set and* P *is a subset of* Q *which is both join-dense and meet-dense in* Q, *then there is an order embedding* ψ *of* Q *into* $DM(P)$. *Moreover,* ψ *agrees with* φ *on* P, *i.e.,* $\psi(a) = \varphi(a)$ *for all* $a \in P$.

(3) Let L *be a complete lattice and let* P *be a subset of* L *which is both join-dense and meet-dense in* L. *Then* $L \cong DM(P)$ *via an order isomorphism which agrees with* φ *on* P.

We recall that the DM-completion of a modular ordered set in the sense of Kolibiar([5]) need not be a modular lattice([4]).

EXAMPLE 3.2. (1) Consider the modular ordered set N_0. Then the DM-completion $DM(N_0)$ of N_0 is a modular lattice, since $N_5 \not\hookrightarrow DM(N_0)$(See Figure 1.4).

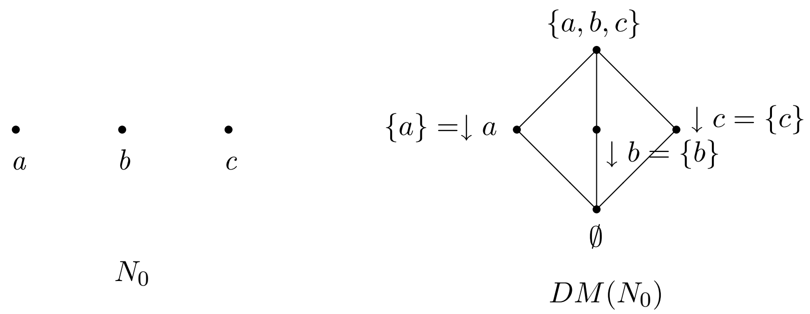
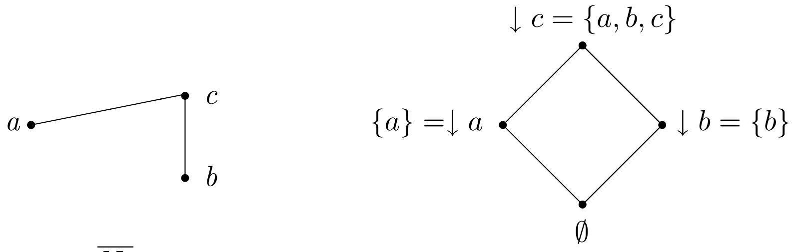


Figure 1.4

(2) Consider the modular ordered set $\overline{N_2}$. Then the DM-completion $DM(\overline{N_2})$ of $\overline{N_2}$ is a modular lattice(See Figure 1.5). Similarly, $DM(\underline{N_2})$ and $DM(N_2)$ are modular lattices.

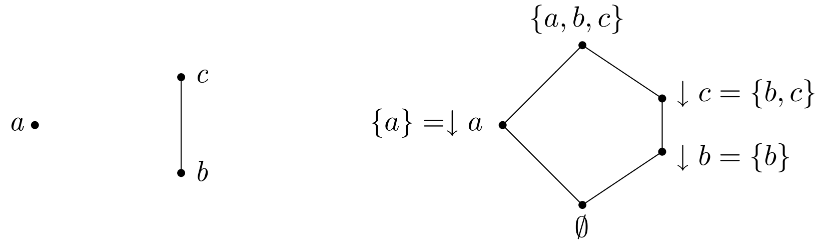


$\overline{N_2}$ $DM(\overline{N_2})$

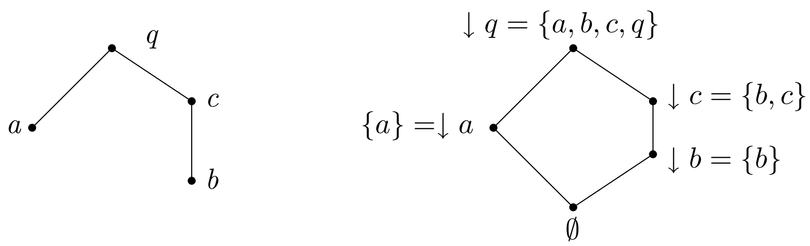
Figure 1.5

PROPOSITION 3.3. *If an ordered set P is one of the ordered sets N_1, $\underline{N_3}$, $\overline{N_3}$ or N_5 given in the previous section, then the DM-completion $(DM(P), \subseteq)$ of P is order-isomorphic to N_5. Hence the DM-completions of N_1, $\underline{N_3}$, $\overline{N_3}$ and N_5 are not modular lattices.*

Proof. We construct the DM-completion $DM(P)$ as follows.



N_1 $DM(N_1) \cong N_5$



$\overline{N_3}$ $DM(\overline{N_3}) \cong N_5$

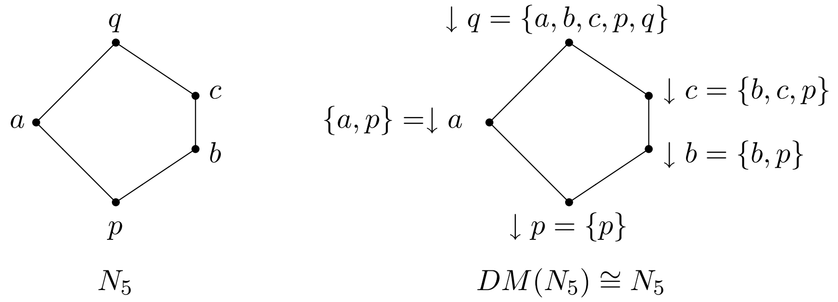


Figure 1.6

We note that $DM(\underline{N_3})$ is the dual lattice of $DM(\overline{N_3})$ which is also order-isomorphic to N_5. In all, we have the result. □

PROPOSITION 3.4. *The DM-completion $DM(\overline{N_5})$ of $\overline{N_5}$ is not modular. Further, The DM-completion $DM(\underline{N_5})$ of $\underline{N_5}$ is also not modular.*

Proof. We construct The DM-completion $DM(\overline{N_5})$ of $\overline{N_5}$ as follows.

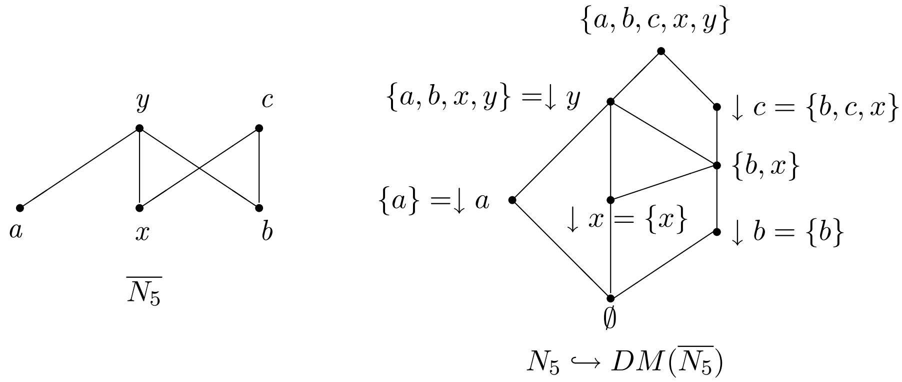


Figure 1.7

Since the subset $\{\downarrow y, \{b, x\}, \downarrow b, \downarrow a, \emptyset\}$ of $DM(\overline{N_5})$ is a sublattics of $DM(\overline{N_5})$ which is isomprhic with N_5. Thus $DM(\overline{N_5})$ is not modular. Furthermore, $\underline{N_5}$ is the dual of $\overline{N_5}$ and hence its DM-completion is dually isomorphic with $DM(\overline{N_5})$ so that it is not modular. □

REMARK 3.5. We revisit the example of a modular ordered set $P = \{a, b, c, x, y\}$ in Remark 2.6. The DM-completion $(DM(P), \subseteq)$ of P consists of 8 elements as we see in Figure 1.8. Also it does not contain N_5 as a sublattice and hence the DM-completion of P is a modular lattice.

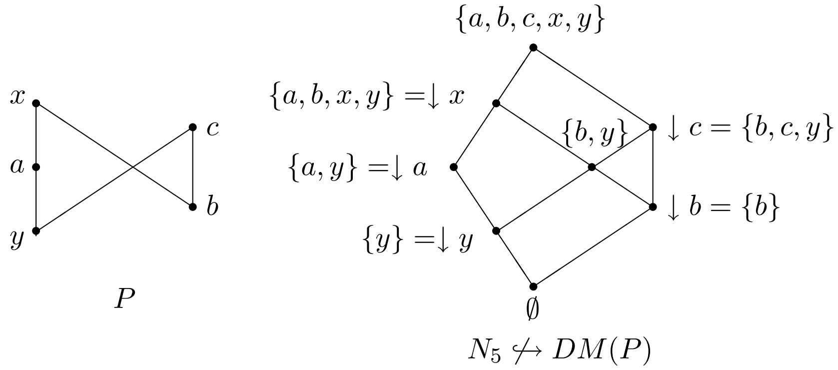


Figure 1.8

The above discussion strongly implicates a conjecture that the DM-completion of modular ordered sets would be modular but the problem is still open.

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