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COMPUTATIONS OF BASES FOR THE SPACES OF CUSPFORMS OF WEIGHT 2

Daeyeol Jeon*

Abstract. In this paper, we present a explicit procedure to compute a basis for the spaces of cuspforms of weight 2 on $X_0(N)$ which consists of eigenforms for the Atkin-Lehner involutions.

1. Introduction

For any positive integer N, let $\Gamma_0(N)$ be a congruence subgroup of $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ consisting of the matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ congruent modulo N to $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$. We let $X_0(N)$ be the modular curve associated to $\Gamma_0(N)$.

For each divisor $d|N$ with $(d, N/d) = 1$ (we write $d||N$), consider the matrices of the form $\left(\begin{smallmatrix} dx & y \\ Nz & dw \end{smallmatrix}\right)$ with $x, y, z, w \in \mathbb{Z}$ and determinant d. Then these matrices define a unique involution on $X_0(N)$ which is called the *Atkin-Lehner involution* and denoted by $W_d = W_d^{(N)}$. In particular, if $d = N$, then W_N is called the *full Atkin-Lehner involution.*

It is known that the group $\mathrm{Aut}_{\mathbb{Q}}(X_0(N))$ of automorphisms of $X_0(N)$ over $\mathbb{Q}$ contains the group $\mathcal{W} = \{W_d\}_{d||N}$ of Atkin-Lehner involutions. Let $\mathcal{W}'$ be a subgroup of $\mathcal{W}$, and consider the quotient curve $X_0(N)/\mathcal{W}'$, which is denoted by $X_0^{+d}(N)$(resp. $X_0^*(N)$) when $\mathcal{W}' = \langle W_d \rangle$(resp. $\mathcal{W}' = \mathcal{W}$). For the case $\langle W_N \rangle$, this curve is denoted by $X_0^+(N)$.

It is a well known fact that there exists a basis of the spaces of cuspforms of weight 2 on $X_0(N)$ which consist of eigenforms for W_d. Such a basis gives bases for the spaces of cuspforms of weight 2 on the quotient spaces $X_0(N)/\mathcal{W}'$ from which one can obtain canonical embeddings of that spaces. In fact, eigenforms on $X_0(N)$ for W_d with eigenvalue $+1$ for all $W_d \in \mathcal{W}'$ are cuspforms $X_0(N)/\mathcal{W}'$. One can easily

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get linearly independent eigenforms from newforms in Stein's tables [3], but it is complicated to compute linearly independent eigenforms from oldforms. Many literatures use a basis for the spaces of cuspforms of weight 2 on $X_0(N)$ which consists of eigenforms for W_d without sufficient explanations how to find them.

In this paper, we will give full details of a method dealing with especially eigenforms form oldforms by an example.

2. Preliminary

The paper of Atkin and Lehner[1] gives some information about the behaviour of the newforms. We quote what we need from their main theorem.

Theorem 2.1. *The vector space, of cuspforms of even weight* 2 *on* $\Gamma_0(N)$, *has a basis consisting of oldcalsses and newclasses. All forms in a class have the same eigenvalues with respect to the Hecke operators* T_p $(p \nmid N)$. *Each newclass consists of a single form* f *which is also an eigenform for the* W_l $(l|N)$. *We choose* f *to be normalized (i.e.* $a_1 = 1$ *in the* q*-expansion). Then* f *satisfies*

$$f|T_p = a_p f,\ \ f|W_l = \lambda_l f$$

where, if $l||N$ *we have* $a_l = -\lambda_l$, *and if* $l^2|N$ *then* $a_l = 0$. *Further, each oldclass is of the forms* $\{g(d\tau)\,|\,g$ *is a newform of some level* M, *and* d *runs through all divisors of* $N/M\}$. *The old classes may be given a different basis consisting of forms which are eigenforms for all the* W_l.

Let $S_2(N)$ and $S_2^\circ(N)$ be the space of cuspforms of weight 2 on $\Gamma_0(N)$ and the space spanned by newforms of weight 2 on $\Gamma_0(N)$ respectively. For the behaviour of the oldforms, we need the following Theorem:

Theorem 2.2. *Let* N *be a positive integer. Let* N' *be a positive divisor of* N *and let* d *be a positive divisor of* N/N'. *For a prime* $p|N$, *let* $p^\alpha||N, p^{\alpha-\beta}||N', p^\gamma||d$, *so that* $\gamma \leq \beta \leq \alpha$. *Then the following holds:*

(1) *If* $f \in S_2(N')$, *then*

$$f(d\tau)|W_{p^\alpha}^{(N)} = p^{\beta-2\gamma}\left(f|W_{p^{\alpha-\beta}}^{(N')}\right)(d'\tau)$$

where $d' = p^{\beta-2\gamma}d$.

(2) *Let* $f \in S_2^\circ(N')$. *If* $f|W_{p^{\alpha-\beta}}^{(N')} = \lambda_p f$ *and* $\beta \neq 2\gamma$*(resp.* $\beta = 2\gamma$*), then*

$$f(d\tau) \pm p^{\beta-2\gamma}\lambda_p f(d'\tau)\ (resp.\ f(d\tau))$$

is an eigenform for $W_{p^\alpha}^{(N)}$ *with eigenvalue* ± 1*(resp.* λ_p*).*

Proof. See [1]. □

Note that if $\alpha = \beta$ then we consider $W_{p^{\alpha-\beta}}^{(N')}$ an identity, and hence λ_p is regarded as 1.

3. Examples

In this section, we explain how to find a basis of the spaces of cuspforms of weight 2 on $X_0(N)$ which consist of eigenforms for W_d by an example. In the Modular Forms Database of Stein [3], there are two tables as follows:

- q-expansions of eigenforms on $\Gamma_0(N)$ of weight $k \le 14$
- Eigenvalues of modular forms on $\Gamma_0(N)$ of weight ≤ 4 and high level, and of weight ≤ 100 and low level

For simplicity, we call the first table Table 1 and the second Table 2. Table 1 is a table of q-expansions of normalized newforms of even weight on $\Gamma_0(N)$, and Table 2 is a table of eigenvalues of newforms on $\Gamma_0(N)$ with the first few Hecke eigenvalues a_p of a basis of representatives for the Galois conjugacy classes of newforms.

Suppose $N = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is the standard form of prime factorization of N. Then the third column of Table 5 in [2] lists the splitting of the space of all differential forms, old and new, given by the involutions $W_i = W_{p^i}$ $(i = 1, \ldots, r)$. The dimensions of the eigenspaces corresponding to the r-tuple of eigenvalues $(\epsilon_1, \ldots, \epsilon_r)$ of $W_1, \ldots, W_r$ are listed in the order

$$\epsilon_1 = +1, \epsilon_1 = -1, \text{ if } r = 1;$$
$$(\epsilon_1, \epsilon_2) = (+1,+1), (+1,-1), (-1,+1), (-1,-1), \text{ if } r = 2.$$

Now consider $N = 108 = 2^2 \cdot 3^3$. The genus of $X_0(108)$ is equal to 10 and the dimensions of the eigenspaces corresponding to $(+1,+1)$, $(+1,-1)$, $(-1,+1)$ and $(-1,-1)$ of W_4 and W_{27} are 1,3,3 and 3 respectively. Now we compute a basis of the space of cuspforms of weight 2 on $X_0(108)$ which consist of eigenforms for W_4 and W_{27}. Table 1 shows that $S_2^\circ(108)$ is an 1 dimensional eigenspace corresponding to $(-1,+1)$ of W_4 and W_{27}, which gives one eigenform as follows:

$$f_1 = q + 5q^7 - 7q^{13} - q^{19} - 5q^{25} - 4q^{31} + \cdots .$$

Thus all the other linearly independent eigenforms should be obtained from oldforms. From Table 1 and Table 2, among $\Gamma_0(N')$ with $N'|108$ we have linearly independent four newforms as follows: Two newforms g_1 and g_2 on $\Gamma_0(54)$ are eigenforms corresponding to $(+1,-1)$ and $(-1,+1)$ respectively of $W_2^{(54)}$ and $W_{27}^{(54)}$, one newform g_3 on $\Gamma_0(27)$ is an eigenform corresponding to -1 of $W_{27}^{(27)}$, and one newform g_4 on $\Gamma_0(36)$ is an eigenform corresponding to $(-1,+1)$ of $W_4^{(36)}$ and $W_9^{(36)}$. Their q-expansions are as follows:

$$\begin{aligned}
g_1 &= q - q^2 + q^4 + 3q^5 - q^7 - q^8 - 3q^{10} + \cdots, \\
g_2 &= q + q^2 + q^4 - 3q^5 - q^7 + q^8 - 3q^{10} + \cdots, \\
g_3 &= q - 2q^4 - q^7 + 5q^{13} + 4q^{16} - 7q^{19} + \cdots, \\
g_4 &= q - 4q^7 + 2q^{13} + 8q^{19} - 5q^{25} - 4q^{31} + \cdots.
\end{aligned}$$

Firstly we will find two more linearly independent eigenforms corresponding to $(-1,+1)$ apart from f_1 by using Theorem 2.2. Consider g_2 and take $d=2$, then $\alpha=2, \beta=\gamma=1$ and $d'=1$. Thus

$$f_2 = g_2(2\tau) + \frac{g_2(\tau)}{2} = \frac{1}{2}q + \frac{3}{2}q^2 + \frac{3}{2}q^4 - \frac{3}{2}q^5 - \frac{1}{2}q^7 + \frac{3}{2}q^8 + \cdots$$

is an eigenform corresponding to $(-1,+1)$. Consider g_4 and take $d=3$, then $\alpha=3, \beta=\gamma=1$ and $d'=1$. Thus

$$f_3 = g_4(3\tau) + \frac{g_4(\tau)}{3} = \frac{1}{3}q + q^3 - \frac{4}{3}q^7 + \frac{2}{3}q^{13} + \frac{8}{3}q^{19} - 4q^{21} + \cdots$$

is an eigenform corresponding to $(-1,+1)$.

Secondly we will find three linearly independent eigenform corresponding to $(+1,-1)$. The first one can be obtained from g_1. We take $d=2$, then $\alpha=2, \beta=\gamma=1$ and $d'=1$. Thus

$$f_4 = g_1(2\tau) + \frac{g_1(\tau)}{2} = \frac{1}{2}q + \frac{1}{2}q^2 - \frac{1}{2}q^4 + \frac{3}{2}q^5 - \frac{1}{2}q^7 + \frac{1}{2}q^8 + \cdots$$

is an eigenform corresponding to $(+1,-1)$. The second and third one can be obtained from g_3. Consider the cases $d=2$ and $d=4$. If $d=2$, then $\alpha=\beta=2, \gamma=1$, and hence

$$f_5 = g_3(2\tau) = q^2 - 2q^8 - q^{14} + 5q^{26} + 4q^{32} - 7q^{38} + \cdots$$

is an eigenform corresponding to $(+1,-1)$. If $d=4$, then $\alpha=\beta=2, \gamma=2$, and hence

$$f_6 = g_3(4\tau) + \frac{g_3(\tau)}{4} = \frac{1}{4}q + \frac{1}{2}q^4 - \frac{1}{4}q^7 + \frac{5}{4}q^{13} - 2q^{16} + \cdots$$

is an eigenform corresponding to $(+1,-1)$.

Thirdly we will find three linearly independent eigenform corresponding to $(-1,-1)$. The first one can be obtained from g_1. We take $d=2$, then $\alpha=2, \beta=\gamma=1$ and $d'=1$. Thus

$$f_7 = g_1(2\tau) - \frac{g_1(\tau)}{2} = -\frac{1}{2}q + \frac{3}{2}q^2 - \frac{3}{2}q^4 - \frac{3}{2}q^5 + \frac{1}{2}q^7 + \frac{3}{2}q^8 + \cdots$$

is an eigenform corresponding to $(-1,-1)$. The second one can be obtained from g_3. We take $d=4$, then $\alpha=\beta=2, \gamma=2$, and hence

$$f_8 = g_3(4\tau) - \frac{g_3(\tau)}{4} = -\frac{1}{4}q + \frac{3}{2}q^4 + \frac{1}{4}q^7 - \frac{5}{4}q^{13} - 2q^{16} + \cdots$$

is an eigenform corresponding to $(-1,-1)$. The third one can be obtained from g_4. We take $d=3$, then $\alpha=3, \beta=\gamma=1$ and $d'=1$. Thus

$$f_9 = g_4(3\tau) - \frac{g_4(\tau)}{3} = -\frac{1}{3}q + q^3 + \frac{4}{3}q^7 - \frac{2}{3}q^{13} - \frac{8}{3}q^{19} - 4q^{21} + \cdots$$

is an eigenform corresponding to $(-1,-1)$.

Lastly we will find an eigenform corresponding to $(+1,+1)$ which can be obtained from g_2. We take $d=2$, then $\alpha=2, \beta=\gamma=1$ and $d'=1$. Thus

$$f_{10} = g_2(2\tau) - \frac{g_2(\tau)}{2} = -\frac{1}{2}q + \frac{1}{2}q^2 + \frac{1}{2}q^4 + \frac{3}{2}q^5 + \frac{1}{2}q^7 + \frac{1}{2}q^8 + \cdots$$

is an eigenform corresponding to $(+1,+1)$.

Finally therefore we obtain a basis $\{f_1, f_2, \ldots, f_{10}\}$ of the spaces of cuspforms of weight 2 on $X_0(108)$ which consist of eigenforms for W_4 and W_{27}.

From this basis $\{f_1, f_2, \cdots, f_{10}\}$ we can get bases for the spaces of cuspforms of weight 2 on $X_0(N)/\mathcal{W}'$ for various $\mathcal{W}'$. If $\mathcal{W}' = \langle W_4 \rangle$, then the genus of $X_0^{+4}(108)$ is equal to 4, which is the same as the number of eigenforms f_i corresponding to $(+1,+1)$ and $(+1,-1)$. Thus $\{f_4, f_5, f_6, f_{10}\}$ forms a basis for the space of cuspforms of weight 2 on $X_0^{+4}(108)$.

If $\mathcal{W}' = \langle W_{27} \rangle$, then the genus of $X_0^{+27}(108)$ is equal to 4, which is the same as the number of eigenforms f_i corresponding to $(+1,+1)$ and $(-1,+1)$. Thus $\{f_1, f_2, f_3, f_{10}\}$ forms a basis for the space of cuspforms of weight 2 on $X_0^{+27}(108)$.

If $\mathcal{W}' = \langle W_{108} \rangle$, then the genus of $X_0^{+}(108)$ is equal to 4, which is the same as the number of eigenforms f_i corresponding to $(+1,+1)$ and $(-1,-1)$. Thus $\{f_7, f_8, f_9, f_{10}\}$ forms a basis for the space of cuspforms of weight 2 on $X_0^{+}(108)$.

Finally if $\mathcal{W}'$ is equal to the full group $\mathcal{W}$, then the genus of $X_0^*(108)$ is equal to 1, which is the same as the number of eigenforms f_i corresponding to $(+1, +1)$. Thus $\{f_{10}\}$ forms a basis for the space of cuspforms of weight 2 on $X_0^*(108)$.

References

[1] A. O. L. Atkin and J. Lehner, *Hecke operators on* $\Gamma_0(m)$*,* Math. Ann. **185** (1970), 134-160.

[2] B. J. Birch and W. Kuyk (Eds.), *Modular functions of one variable, IV,* Lecture Notes in Math. **476**, Springer-Verlag, Berlin, 1975.

[3] http://modular.math.washington.edu/Tables/tables.html

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Department of Mathematics Education
Kongju National University
Kongju 314-701, Republic of Korea
E-mail: `dyjeon@kongju.ac.kr`