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A NOTE ON COMPLETE MOMENT CONVERGENCE FOR ARRAYS OF ROWWISE EXTENDED NEGATIVELY ORTHANT DEPENDENT RANDOM VARIABLES

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Abstract. In this paper we obtain the complete moment convergence for an array of rowwise extended negative orthant dependent random variables. By using the result we can prove the complete moment convergence for some positively orthant dependent sequence satisfying the extended negative orthant dependence.

1. Introduction

Ebrahimi and Ghosh(1981) and Joag-Dev and Proschan(1983) introduced the concept of negative orthant dependent random variables:

A sequence $\{X_i, 1 \le i \le n\}$ of random variables is said to be negatively upper orthant dependent(NUOD) if for all real numbers $x_1, \cdots, x_n$,

$$P(X_1 > x_1, \cdots, X_n > x_n) \le \prod_{i=1}^{n} P(X_i > x_i) \tag{1.1}$$

and it is said to be negatively lower orthant dependent(NLOD) if for all real numbers $x_1, \cdots, x_n$,

$$P(X_1 \le x_1, \cdots, X_n \le x_n) \le \prod_{i=1}^{n} P(X_i \le x_i). \tag{1.2}$$

A sequence $\{X_i, 1 \le i \le n\}$ of random variables is said to be negatively orthant dependent(NOD) if it is both NUOD and NLOD.

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Recently, Liu(2009) introduced the concept of extended negative dependence in the multivariate case. A sequence $\{X_i, 1 \le i \le n\}$ of random variables is said to be extended negatively upper orthant dependent(ENUOD) if for all real numbers $x_1, \cdots, x_n$, there exists a constant $M > 0$ such that

$$P(X_1 > x_1, \cdots, X_n > x_n) \le M \prod_{i=1}^{n} P(X_i > x_i) \tag{1.3}$$

and it is said to be extended negatively lower orthant dependent(ENLOD) if for all real numbers $x_1, \cdots, x_n$, there exists a constant $M > 0$ such that

$$P(X_1 \le x_1, \cdots, X_n \le x_n) \le M \prod_{i=1}^{n} P(X_i \le x_i). \tag{1.4}$$

A sequence $\{X_i, 1 \le i \le n\}$ of random variables is said to be extended negatively orthant dependent(ENOD) if it is both ENUOD and ENLOD.

It is clear that a sequence $\{X_i, 1 \le i \le n\}$ of random variables is called NOD if (1.3) and (1.4) hold when $M = 1$, the sequence is called positively orthant dependent(POD) if the inequalities (1.3) and (1.4) hold both in the reverse direction when $M = 1$. Obviously, an NOD sequence must be an ENOD sequence. On the other hand, for some POD sequences, it is possible to find a corresponding positive constant M such that (1.3) and (1.4) hold.

Therefore, the ENOD structure is substantially more comprehensive than the NOD structure in which it can reflect not only a negative dependence structure also positive one to some extent. For instance, the ENOD sequence $\{X_i, i \ge 1\}$ in the following example can be taken as NOD or POD since the are no restrictions on the dependence between X_1 and X_2.

Example 1.1 (Liu(2009)). If $\{X_i, i = 1, 2\}$ and $\{X_i, i \ge 3\}$ are independent of each other, where X_1 is possibly valued at $x_{11} \le x_{12} \le \cdots \le x_{1N}$ and $\{X_i, i \ge 3\}$ is a sequence of mutually independent random variables. Then the sequence $\{X_i, i \ge 1\}$ is ENOD. In fact, for any x_1 and x_2 such that

$$P(X_1 \le x_1)P(X_2 \le x_2) = 0 \text{ or } P(X_1 > x_1)P(X_2 > x_2) = 0$$

both (1.3) and (1.4) hold trivially. Additionally, for any x_1 and x_2 such that

$$P(X_1 \le x_1)P(X_2 \le x_2) \ne 0 \text{ and } P(X_1 > x_1)P(X_2 > x_2) \ne 0,$$

take

$$M = 1/\min\{P(X_1 = x_1), P(X_1 = x_{1N})\},$$

then both (1.3) and (1.4) still hold. Notice that there are no dependence restrictions between random variables X_1 and X_2.

A sequence of random variables $\{U_n, n \geq 1\}$ is said to converge completely to a constant c if for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P(|U_n - c| > \epsilon) < \infty. \tag{1.5}$$

This notion was given by Hsu and Robbins(1947). Let $\{Z_n, n \geq 1\}$ be a sequence of random variables and $a_n, b_n > 0, q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|Z_n| - \epsilon\}_+^q < \infty \text{ for all } \epsilon > 0, \tag{1.6}$$

then (1.6) was called the complete moment convergence by Chow(1988). For the complete moment convergence, Chow(1988) obtained for independent random variables, Wang and Zhao(2006) investigated for negatively associated random variables and Zhu(2007) studied for array of rowwise ρ^*-mixing random variables.

In this paper we study the complete moment convergence for partial sums of rowwise ENOD random variables.

2. Some lemmas

In this section we introduce some lemmas which will be used to prove the main result.

LEMMA 2.1 (Liu(2009)). *Let $\{X_i, i \geq 1\}$ be a sequence of ENOD random variables.*

(1) If $\{g_i(\cdot), i \geq 1\}$ is a sequence of monotone increasing(decreasing) functions, then $\{g_i(X_i), i \geq 1\}$ is still a sequence of END random variables.

(2) If X_i's are nonnegative random variables, then there exists a constant $M > 0$ such that

$$E(\prod_{i=1}^{n} X_i) \leq M \prod_{i=1}^{n}(EX_i).$$

From (1) and (2) in Lemma 2.1 we obtain the following result.

Lemma 2.2. *Let $\{X_i, i \geq 1\}$ be a sequence of ENOD random variables. Then for any $h > 0$ there exists a constant $M > 0$ such that*

$$E[\exp(h\sum_{i=1}^{n} X_i)] \leq M \prod_{i=1}^{n} E[\exp(hX_i)]. \tag{2.1}$$

Proof. Note that $\{\exp(hX_i), i \geq 1\}$ is ENOD sequence by Lemma 2.1 (1). Hence by Lemma 2.1 (2) we have

$$E[\exp(h\sum_{i=1}^{n} X_i)] = E[\prod_{i=1}^{n} \exp(hX_i)] \leq M \prod_{i=1}^{n} E[\exp(hX_i)].$$

□

Lemma 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of ENOD random variables with mean zero and $0 < B_n = \sum_{i=1}^{n} EX_i^2 < \infty$. Then there exists an $M > 0$ such that for all $x > 0, y > 0$,*

$$P(|S_n| \geq x) \leq \sum_{i=1}^{n} P(|X_i| \geq y) + 2M \exp(\frac{x}{y} - \frac{x}{y} \log(1 + \frac{xy}{B_n})), \tag{2.2}$$

where $S_n = \sum_{i=1}^{n} X_i$.

Proof. The proof is similar to that of Theorem 2 in Fuk and Nagev (1971). Let $Y_i = X_i I(X_i \leq y) + yI(X_i > y)$ and $T_n = \sum_{i=1}^{n} Y_i$ and note that $Y_i \leq X_i$. It is easy to show that $EY_i \leq 0$ and $EY_i^2 \leq EX_i^2$. By Lemma 2.1 (1) for $h > 0$ $\{e^{hY_i}, 1 \leq i \leq n\}$ is a sequence of nonnegative ENOD random variables. Thus, by Lemma 2.2 there exists a constant $M > 0$ such that

$$E\exp(hT_n) = E\prod_{i=1}^{n} \exp(hY_i) \leq M \prod_{i=1}^{n} E\exp(hY_i). \tag{2.3}$$

Let $F_i(x) = P(X_i < x_i)$. Then, we have for $h > 0$

$$\begin{aligned} E\exp(hY_i) &= \int_{-\infty}^{y} \exp(h(x))dF_i(x) + e^{hy}P(X_i \geq y) \\ &= 1 + hEY_i + \int_{-\infty}^{y} (e^{hx} - 1 - hx)dF_i(x) \\ &\qquad + (e^{hy} - 1 - hy)P(X_i \geq y) \\ &\leq 1 + \int_{-\infty}^{y} (e^{hx} - 1 - hx)dF_i(x) + (e^{hy} - 1 - hy)P(X_i \geq y). \end{aligned} \tag{2.4}$$

Since $f(x) = (e^{hx}-1-hx)/x^2$ is increasing for all x, $h > 0$ and $1+u \le e^u$ for all real number u it follows from (2.4) that

$$\begin{aligned}(2.5)\ E\exp(hY_i) &\le 1 + \frac{e^{hy}-1-hy}{y^2}\Big(\int_{-\infty}^{y} x^2 dF_i(x) + y^2 P(X_i \ge y)\Big)\\ &\le 1 + \frac{e^{hy}-1-hy}{y^2} EX_i^2\\ &\le \exp\Big(\frac{e^{hy}-1-hy}{y^2} EX_i^2\Big).\end{aligned}$$

Therefore by (2.3) and (2.5) we obtain, for all $x > 0$ and all $h > 0$

$$(2.6)\qquad \exp(-hx)E\exp(hT_n) \le M\exp\Big(-hx + B_n\frac{e^{hy}-1-hy}{y^2}\Big).$$

Letting $h = \log(1 + \frac{xy}{B_n})/y$, we have

$$\begin{aligned}(2.7)\qquad &\exp(-hx)E\exp(hT_n)\\ &\le M\exp\Big[\frac{x}{y} - \frac{x}{y}\log\Big(1+\frac{xy}{B_n}\Big) - \frac{B_n}{y^2}\log\Big(1+\frac{xy}{B_n}\Big)\Big]\\ &\le M\exp\Big[\frac{x}{y} - \frac{x}{y}\log\Big(1+\frac{xy}{B_n}\Big)\Big],\end{aligned}$$

which yields

$$\begin{aligned}(2.8)\ P(S_n \ge x) &\le P(S_n \ne T_n) + P(T_n \ge x)\\ &\le \sum_{i=1}^{n} P(X_i \ge y) + \exp(hx)E\exp(hT_n)\\ &\le \sum_{i=1}^{n} P(X_i \ge y) + M\exp\Big[\frac{x}{y} - \frac{x}{y}\log\Big(1+\frac{xy}{B_n}\Big)\Big].\end{aligned}$$

Similarly, we have

$$(2.9)\qquad P(-S_n \ge x) \le \sum_{i=1}^{n} P(-X_i \ge y) + M\exp\Big[\frac{x}{y} - \frac{x}{y}\log\Big(1+\frac{xy}{B_n}\Big)\Big]$$

since $\{-X_n, n \ge 1\}$ is a sequence of ENOD by Lemma 2.1 (1).

From (2.8) and (2.9) we obtain

$$\begin{aligned}P(|S_n| \ge x) &\le P(S_n \ge x) + P(-S_n \ge x)\\ &\le \sum_{i=1}^{n} P(|X_i| \ge y) + 2M\exp\Big[\frac{x}{y} - \frac{x}{y}\log\Big(1+\frac{xy}{B_n}\Big)\Big].\end{aligned}$$

□

3. Complete moment convergence

THEOREM 3.1. *Let* $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ *be an array of ENOD random variables with* $EX_{ni} = 0$ *and* $EX_{ni}^2 < \infty$, *for* $1 \le i \le n, n \ge 1$ *and* $\{a_n, n \ge 1\}$ *be a sequence of positive real numbers with* $a_n \uparrow$. *If*

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{EX_{ni}^2}{a_n^2} < \infty, \tag{3.1}$$

then we obtain

$$\sum_{n=1}^{\infty} a_n^{-1} E\{|\sum_{i=1}^{n} X_{ni}| - \epsilon a_n\}^+ < \infty \text{ for all } \epsilon > 0.$$

Proof.

$$\begin{aligned}
& \sum_{n=1}^{\infty} a_n^{-1} E\{|\sum_{i=1}^{n} X_{ni}| - \epsilon a_n\}^+ \\
= & \sum_{n=1}^{\infty} a_n^{-1} \int_0^{\infty} (P\{|\sum_{i=1}^{n} X_{ni}| - \epsilon a_n\} > u) du \\
= & \sum_{n=1}^{\infty} a_n^{-1} \int_0^{a_n} P\{|\sum_{i=1}^{n} X_{ni}| > \epsilon a_n + u\} du \\
& + \sum_{n=1}^{\infty} a_n^{-1} \int_{a_n}^{\infty} P\{|\sum_{i=1}^{n} X_{ni}| > \epsilon a_n + u\} du \\
\le & \sum_{n=1}^{\infty} P\{|\sum_{i=1}^{n} X_{ni}| > \epsilon a_n\} \\
& + \sum_{n=1}^{\infty} a_n^{-1} \int_{a_n}^{\infty} P\{|\sum_{i=1}^{n} X_{ni}| > u\} du \\
= & I + II.
\end{aligned} \tag{3.2}$$

We need to prove that $I < \infty$ and $II < \infty$.

For any $1 \le i \le n, n \ge 1$, let

$$\begin{aligned}
Y_{ni} &= -a_n I(X_{ni} < -a_n) + X_{ni} I(|X_{ni}| \le a_n) + a_n I(X_{ni} > a_n), \\
Z_{ni} &= X_{ni} - Y_{ni} \\
&= (X_{ni} + a_n) I(X_{ni} < -a_n) + (X_{ni} - a_n) I(X_{ni} > a_n).
\end{aligned} \tag{3.3}$$

To prove $I < \infty$, it is enough to show that

$$\sum_{n=1}^{\infty} P(\frac{1}{a_n}|\sum_{i=1}^{n} Z_{ni}| > \epsilon) < \infty, \tag{3.4}$$

$$\sum_{n=1}^{\infty} P(\frac{1}{a_n}|\sum_{i=1}^{n}(Y_{ni} - EY_{ni})| > \epsilon) < \infty, \tag{3.5}$$

$$\frac{1}{a_n}\sum_{i=1}^{n} EY_{ni} \to 0 \ as \ n \to \infty. \tag{3.6}$$

Because the proof of (3.4) is a standard argument, we omit to prove (3.4). Now we prove (3.5). By Lemma 2.1 (1) $\{Y_{ni} - EY_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of rowwise ENOD random variables with mean zero. Let $B_n^{'} = \sum_{i=1}^{n} E(Y_{ni} - EY_{ni})^2 < \infty$. Take $x = \epsilon a_n$, $y = \epsilon a_n/2$. Then, by Lemma 2.2, for all $\epsilon > 0$

$$\begin{aligned} &\sum_{n=1}^{\infty} P(\frac{1}{a_n}|\sum_{i=1}^{n}(Y_{ni} - EY_{ni})| > \epsilon) \\ &\le \sum_{n=1}^{\infty}\sum_{i=1}^{n} P(|Y_{ni} - EY_{ni}| > \epsilon a_n/2) + 2e^2 M \sum_{n=1}^{\infty}(\frac{B_n^{'}}{B_n^{'} + \epsilon^2 a_n^2/2})^2 \\ &= I_1 + I_2. \end{aligned} \tag{3.7}$$

By (3.1), (3.3) and the Chebyshev's inequality

$$\begin{aligned} I_1 &\le \sum_{n=1}^{\infty}\sum_{i=1}^{n} P(|Y_{ni} - EY_{ni}| > \epsilon a_n/2) \\ &\le 4\epsilon^{-2}\sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{E(Y_{ni} - EY_{ni})^2}{a_n^2} \\ &\le 4\epsilon^{-2}\sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{EX_{ni}^2}{a_n^2} < \infty. \end{aligned} \tag{3.8}$$

Note that $|Y_{ni}| \le |X_{ni}|$ and $EY_{ni}^2 \le EX_{ni}^2$ for $i, 1 \le i \le n$.

By (3.1) we obtain

$$
\begin{aligned}
(3.9)\ I_2 &\le 8e^2\epsilon^{-4}M\sum_{n=1}^{\infty}(\sum_{i=1}^{n}a_n^{-2}E(Y_{ni}-EY_{ni})^2)^2 \\
&\le 8e^2\epsilon^{-4}M\sum_{n=1}^{\infty}(\sum_{i=1}^{n}\frac{EY_{ni}^2}{a_n^2})^2 \\
&\le 8e^2\epsilon^{-4}M\sum_{n=1}^{\infty}(\sum_{i=1}^{n}\frac{EX_{ni}^2}{a_n^2})^2 \\
&\le 8e^2\epsilon^{-4}M\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{EX_{ni}^2}{a_n^2} < \infty.
\end{aligned}
$$

From (3.7)-(3.9) (3.5) follows.

To prove (3.6): Since $EX_{ni}=0$ for $1\le i\le n, n\ge 1$, $EY_{ni}=-EZ_{ni}$. If $X_{ni}>a_n$, $0<Z_{ni}=X_{ni}-a_n<X_{ni}$ and if $X_{ni}<-a_n$, $X_{ni}<Z_{ni}=X_{ni}+a_n\le 0$. So $|Z_{ni}|\le|X_{ni}|I(|X_{ni}|>a_n)$. Consequently

$$
\begin{aligned}
\frac{1}{a_n}|\sum_{i=1}^{n}EY_{ni}| &= \frac{1}{a_n}|\sum_{i=1}^{n}EZ_{ni}| \\
&\le \sum_{i=1}^{n}\frac{E|Z_{ni}|}{a_n} \\
&\le \sum_{i=1}^{n}\frac{Ea_n|X_{ni}|I(|X_{ni}|>a_n)}{a_n^2} \\
&\le \sum_{i=1}^{n}\frac{EX_{ni}^2}{a_n^2} \to 0\ as\ n\to\infty \text{ by (3.1).}
\end{aligned}
$$

Hence the proof of $I<\infty$ is complete.

Next we prove that $II<\infty$.

$$
\begin{aligned}
II &\le \sum_{n=1}^{\infty}a_n^{-1}\sum_{i=1}^{n}\int_{a_n}^{\infty}P\{|X_{ni}|>u\}du \\
&\quad +\sum_{n=1}^{\infty}a_n^{-1}\sum_{i=1}^{n}\int_{a_n}^{\infty}P\{|\sum_{i=1}^{n}I(|X_{ni}|\le u)|>u\}du \\
&= II_1+II_2.
\end{aligned}
$$

Clearly, for $u \geq a_n$ we have

$$\begin{aligned} II_1 &= \sum_{n=1}^{\infty} \sum_{i=1}^{n} a_n^{-1} \int_{a_n}^{\infty} P\{|X_{ni}| > u\} du \\ &\leq \sum_{n=1}^{\infty} \sum_{i=1}^{n} a_n^{-1} \int_{0}^{\infty} P\{|X_{ni}| I(|X_{ni}| > a_n) > u\} du \\ &= \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{E|X_{ni}| I(|X_{ni}| > a_n)}{a_n} \\ &\leq \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{EX_{ni}^2}{a_n^2} < \infty \text{ by (3.1)}. \end{aligned}$$

It remains to prove $II_2 < \infty$. It follows from $EX_{ni} = 0$ and (3.1) that

$$\begin{aligned} \max_{u \geq a_n} |u^{-1} \sum_{i=1}^{n} EX_{ni} I(|X_{ni}| \leq u)| &= \max_{u \geq a_n} |u^{-1} \sum_{i=1}^{n} EX_{ni} I(|X_{ni}| > u)| \\ &\leq \sum_{i=1}^{n} \frac{E|X_{ni}| I(|X_{ni}| > a_n)}{a_n} \\ &\leq \sum_{i=1}^{n} \frac{EX_{ni}^2}{a_n^2} \to 0 \text{ by (3.1)}. \end{aligned}$$

Therefore, while n is sufficiently large, for $u \geq a_n$,

$$|\sum_{i=1}^{n} EX_{ni} I(|X_{ni}| \leq u)| \leq \frac{u}{2},$$

which yields

$$\begin{aligned} \text{(3.10)} \qquad & P\{|\sum_{i=1}^{n} X_{ni} I(|X_{ni}| \leq u)| > u\} \\ \leq\ & P\{|\sum_{i=1}^{n} (X_{ni} I(|X_{ni}| \leq u) - EX_{ni} I(|X_{ni}| \leq u))| > \frac{u}{2}\}. \end{aligned}$$

Let $B_n'' = \sum_{i=1}^n E(X_{ni} - EX_{ni})^2 I(|X_{ni}| \le u)$, $x = \frac{u}{2}, y = \frac{u}{4}$. By (3.10) and Lemma 2.3 we get

$$\begin{aligned}
II_2 &\le \sum_{n=1}^\infty a_n^{-1} \int_{a_n}^\infty P\{|\sum_{i=1}^n X_{ni} I(|X_{ni}| \le u) - EX_{ni} I(|X_{ni}| \le u)| > \frac{u}{2}\} du \\
&\le \sum_{n=1}^\infty a_n^{-1} \int_{a_n}^\infty \sum_{i=1}^n P\{|X_{ni}| I(|X_{ni}| \le u) - EX_{ni} I(|X_{ni}| \le u)| > \frac{u}{4}\} du \\
&\quad + 2e^2 M \sum_{n=1}^\infty a_n^{-1} \int_{a_n}^\infty (\frac{B_n''}{B_n'' + \frac{u^2}{8}})^2 du \\
&= II_{21} + II_{22}.
\end{aligned}$$

By (3.1) and $EX_{ni} = 0$ we also have

$$\begin{aligned}
\max_{u \ge a_n} u^{-1} |EX_{ni} I(|X_{ni}| \le u)| &= \max_{u \ge a_n} u^{-1} |EX_{ni} I(|X_{ni}| > u)| \\
&\le a_n^{-1} E|X_{ni}| I(|X_{ni}| > a_n)| \\
&\le \sum_{i=1}^n \frac{EX_{ni}^2}{a_n^2} \to 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
II_{21} &\le \sum_{n=1}^\infty \sum_{i=1}^n a_n^{-1} \int_{a_n}^\infty P\{|X_{ni}| I(|X_{ni}| \le u) > \frac{u}{8}\} du \\
&= \sum_{n=1}^\infty \sum_{i=1}^n a_n^{-1} \int_{a_n}^\infty P\{|X_{ni}| I(|X_{ni}| \le a_n) > \frac{u}{8}\} du \\
&\quad + \sum_{n=1}^\infty \sum_{i=1}^n a_n^{-1} \int_{a_n}^\infty P\{|X_{ni}| I(a_n < |X_{ni}| \le u) > \frac{u}{8}\} du \\
&= II_{211} + II_{212}.
\end{aligned}$$

By the Chebyshev's inequality and (3.1) we have

$$\begin{aligned}
I_{211} &\le 64 \sum_{n=1}^\infty \sum_{i=1}^n a_n^{-1} E|X_{ni}|^2 I(|X_{ni}| \le a_n) \int_{a_n}^\infty u^{-2} du \\
&\le 64 \sum_{n=1}^\infty \sum_{i=1}^n \frac{E|X_{ni}|^2 I(|X_{ni}| \le a_n)}{a_n^2} < \infty.
\end{aligned}$$

By the similar method to proof of II_1

$$\begin{aligned} II_{212} &= \sum_{n=1}^{\infty}\sum_{i=1}^{n} a_n^{-1}\int_{a_n}^{\infty} P\{|X_{ni}|I(a_n<|X_{ni}|\le u)|>\frac{u}{8}\}du \\ &\le \sum_{n=1}^{\infty}\sum_{i=1}^{n} a_n^{-1}\int_{0}^{\infty} P\{|X_{ni}|I(|X_{ni}|>a_n)|>\frac{u}{8}\}du \\ &= 8\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E|X_{ni}|I(|X_{ni}|>a_n)}{a_n} \\ &\le 8\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{EX_{ni}^2}{a_n^2}<\infty. \end{aligned}$$

Finally, we have to prove $II_{22}<\infty$. By the fact that for $x\ge 0, y\ge 0$ $(x+y)^2\le 2(x^2+y^2)$

$$\begin{aligned} II_{22} &= 2e^2M\sum_{n=1}^{\infty} a_n^{-1}\int_{a_n}^{\infty}(\frac{B_n''}{B_n''+\frac{u^2}{8}})^2du \\ &\le 128e^2M\sum_{n=1}^{\infty} a_n^{-1}\int_{a_n}^{\infty}(\frac{B_n''}{u^2})^2du \\ &\le 128e^2M\sum_{n=1}^{\infty} a_n^{-1}\int_{a_n}^{\infty}(\sum_{i=1}^{n}\frac{EX_{ni}^2I(|X_{ni}|\le a_n)}{u^2} \\ &\quad +\sum_{i=1}^{n}\frac{EX_{ni}^2I(a_n<|X_{ni}|\le u)}{u^2})^2du \\ &\le 256e^2M\sum_{n=1}^{\infty} a_n^{-1}\int_{a_n}^{\infty}(\sum_{i=1}^{n}\frac{EX_{ni}^2I(|X_{ni}|\le a_n)}{u^2})^2du \\ &\quad +256e^2M\sum_{n=1}^{\infty} a_n^{-1}\int_{a_n}^{\infty}(\frac{EX_{ni}^2I(a_n<|X_{ni}|\le u)}{u^2})^2du \\ &= II_{221}+II_{222}. \end{aligned}$$

Let $C=256e^2M$.

$$\begin{aligned} II_{221} &= C\sum_{n=1}^{\infty} a_n^{-1}(\sum_{i=1}^{n}EX_{ni}^2I(|X_{ni}|\le a_n))^2\int_{a_n}^{\infty}u^{-4}du \\ &\le \frac{C}{3}\sum_{n=1}^{\infty} a_n^{-4}(\sum_{i=1}^{n}EX_{ni}^2I(|X_{ni}|\le a_n))^2 \end{aligned}$$

$$
\begin{aligned}
&\le \frac{C}{3}\sum_{n=1}^{\infty}(\sum_{i=1}^{n}\frac{E|X_{ni}|^2 I(|X_{ni}|\le a_n)}{a_n^2})^2\\
&\le \frac{C}{3}\sum_{n=1}^{\infty}(\sum_{i=1}^{n}\frac{E(X_{ni})^2 I(|X_{ni}|\le a_n)}{a_n^2})^2<\infty.
\end{aligned}
$$

$$
\begin{aligned}
II_{222} &\le C\sum_{n=1}^{\infty}a_n^{-1}\int_{a_n}^{\infty}(\sum_{i=1}^{n}\frac{E|X_{ni}|I(a_n<|X_{ni}|\le u)}{u})^2du\\
&\le C\sum_{n=1}^{\infty}a_n^{-1}\int_{a_n}^{\infty}(\sum_{i=1}^{n}\frac{E|X_{ni}|I(|X_{ni}|>a_n)}{u})^2du\\
&\le C\sum_{n=1}^{\infty}a_n^{-1}(\sum_{i=1}^{n}E|X_{ni}|I(|X_{ni}|>a_n))^2\int_{a_n}^{\infty}u^{-2}du\\
&\le C\sum_{n=1}^{\infty}(\sum_{i=1}^{n}\frac{E|X_{ni}|I(|X_{ni}|>a_n)}{a_n})^2\\
&\le C\sum_{n=1}^{\infty}(\sum_{i=1}^{n}\frac{EX_{ni}^2 I(|X_{ni}|>a_n)}{a_n^2})^2\\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}(\frac{EX_{ni}^2}{a_n^2})^2\\
&\le C(\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{EX_{ni}^2}{a_n^2})^2<\infty.
\end{aligned}
$$

□

REMARK 3.2. In the proof of Theorem 3.1 from the fact that $I<\infty$ we obtain the complete convergence.

COROLLARY 3.3. *Let $\{X_{ni}, 1\le i\le n, n\ge 1\}$ be an array of POD random variables with $EX_{ni}=0$ and $EX_{ni}^2<\infty$ and $\{a_n, n\ge 1\}$ be a sequence positive real numbers with $a_n\uparrow\infty$. If there exists a constant $M>1$ satisfying (1.4), (1.5) and condition (3.1) then*

$$\sum_{n=1}^{\infty}a_n^{-1}E\{|\sum_{i=1}^{n}X_{ni}|-\epsilon a_n\}^+<\infty.$$

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