JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **25**, No. 1, February 2012

ANALOGUE OF WIENER INTEGRAL IN THE SPACE OF SEQUENCES OF REAL NUMBERS

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Abstract. Let $T > 0$ be given. Let $(C[0,T], m_\varphi)$ be the analogue of Wiener measure space, associated with the Borel probability measure φ on $\mathbb{R}$, let $(L_2[0,T], \widetilde{\omega})$ be the centered Gaussian measure space with the correlation operator $(-\frac{d^2}{dx^2})^{-1}$ and $(\ell_2, \widetilde{m})$ be the abstract Wiener measure space.

Let $\mathcal{U}$ be the space of all sequence $\langle c_n \rangle$ in ℓ_2 such that the limit $\lim_{m\to\infty} \frac{1}{m+1} \sum_{n=0}^{m} \sum_{k=0}^{n} c_k \cos \frac{k\pi t}{T}$ converges uniformly on $[0,T]$ and give a set function m such that for any Borel subset G of ℓ_2, $m(\mathcal{U} \cap P_0^{-1} \circ P_0(G)) = \widetilde{m}(P_0^{-1} \circ P_0(G))$.

The goal of this note is to study the relationship among the measures $m_\varphi, \widetilde{\omega}, \widetilde{m}$ and m.

1. Preliminaries

In 1965, Gross present the theory of the abstract Wiener measure ω on $\mathbb{B}$, the infinite dimensional real separable Banach space [3]. In 2002, the author and Dr. Im defined the analogue of Wiener measure m_φ on $C[0,T]$, the space of all real-valued continuous functions on $[0,T]$ [6]. This measure is a kind of the generalization of the concrete Wiener measure m_φ on $C_0[0,T]$, the space of all real-valued continuous functions on $[0,T]$ that vanish at 0.

Let $\mathcal{U}$ be the space of all sequences which consists of Fourier coefficients, related with analogue of Wiener paths and let m be a set function such that for any Borel subset G of ℓ_2, $m(\mathcal{U} \cap P_0^{-1} P_0 G) = \widetilde{m}(P_0^{-1} P_0 G)$. The purpose of this article is to study the relationship between the measures $m_\varphi, \widetilde{\omega}, \widetilde{m}$ and a set function m.

Received November 02, 2011; Accepted January 20, 2012.

2010 Mathematics Subject Classification: Primary 28C20; Secondary 28C35.

Key words and phrases: analogue of Wiener measure, Fourier cosine series, Fejér's theorem, Hilbert-Schmidt operator.

This paper was supported by the Research Fund in Hannam University in 2011.

We will introduce some notations, definitions and known facts which are need in the next sections.

(A) Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Throughout in this note, every sequences are functions on $\mathbb{N}_0$. Let $\mathbb{R}$ be the set of all real numbers and let m_L be the Lebesgue measure on $\mathbb{R}$. For a metric space X, let $\mathcal{B}(X)$ be the set of all Borel subset of X.

(B)(The analogue of Wiener measure m_φ)

Let T be a positive real number and let $C[0,T]$ be the space of all real-valued continuous functions on $[0,T]$ with the supremum norm $||\cdot||_\infty$. Let φ be the Borel probability measure on $\mathbb{R}$. For $\vec{t} = (t_0, t_1, \cdots, t_n)$ with $0 = t_0 < t_1 < t_2 < \cdots < t_n \leq T$, let $J_{\vec{t}} : C[0,T] \to \mathbb{R}^{n+1}$ be the function given by $J_{\vec{t}}(x) = (x(t_0), x(t_1), \cdots, x(t_n))$.
For B_j $(j = 0, 1, 2, \cdots, n)$ in $\mathcal{B}(\mathbb{R})$, we let

$$\begin{aligned}
&m_\varphi(J_{\vec{t}}^{-1}(\prod_{j=0}^{n} B_j)) \\
&= \prod_{j=1}^{n}[(2\pi)(t_j - t_{j-1})]^{-\frac{1}{2}} \int_{B_0} [\int_{\prod_{j=1}^{n} B_j} \exp\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}}\} \\
&\quad d(\prod_{j=1}^{n} m_L)(u_1, u_2, \cdots, u_n)] d\varphi(u_0).
\end{aligned}$$

Then m_φ can be uniquely extended onto $\mathcal{B}(C[0,T])$. We shall denote it by m_φ, again. This measure m_φ is called the analogue of Wiener measure associated with φ [6].

(C) (The centered Gaussian measure $\widetilde{\omega}$ with the correlation operator $(-\frac{d^2}{dx^2})^{-1}$ on $L_2[0,T]$)

For n in $\mathbb{N}$, let $e_n(t) = \cos\frac{n\pi t}{T}$ and let $\lambda_n = \frac{n^2\pi^2}{T^2}$. Then λ_n is an eigenvalue of $-\frac{d^2}{dt^2}$ with respect to e_n. For n in $\mathbb{N}$ and for F in $\mathcal{B}(\mathbb{R}^n)$, let $M_F = \{f \text{ in } L_2[0,T] \mid (\int_0^T f(t)e_1(t)dm_L(t), \int_0^T f(t)e_2(t)dm_L(t), \cdots, \int_0^T f(t) e_n(t)dm_L(t)) \text{ is in } F\}$. Let $\tilde{\omega}(M_F) = (2\pi)^{-\frac{n}{2}}(\prod_{j=1}^{n}\lambda_j^{\frac{1}{2}}) \int_F exp\{-\frac{1}{2}\sum_{j=1}^{n} \lambda_j u_j^2\} d(\prod_{j=1}^{n} m_L)$ $(u_1, u_2, \cdots, u_n)$. Then $\tilde{\omega}$ can be uniquely extended onto $\mathcal{B}(L_2[0,T])$. We shall denote it by $\tilde{\omega}$, again.

Remark that if f is in $L_2[0,T]$ and α is any real number then $\int_0^T (f(t)+\alpha)e_n(t)dm_L(t) = \int_0^T f(t)e_n(t)dm_L(t)$ for all n in $\mathbb{N}$, so if f is in M_F as in above then $f+\alpha$ is in M_F for all real number α. Hence, by the essentially similar method as in [7], we have the following lemma.

LEMMA 1.1. *For any A in $\mathcal{B}(L_2[0,T])$, $A \cap C[0,T]$ is in $\mathcal{B}(C[0,T])$ and $\widetilde{\omega}(A) = m_\varphi(A \cap C[0,T])$.*

(D)(The abstract Wiener measure)

Let H be an infinite dimensional real Hilbert space with norm $||\cdot|| = \sqrt{\langle\cdot,\cdot\rangle}$. For a finite dimensional orthogonal projection P of H with $dimP(H) = n$ and for F in $\mathcal{B}(\mathbb{R}^n)$, we let

$$\begin{aligned} & v(E) \\ & = (2\pi)^{-\frac{n}{2}} \int_F \exp\{-\frac{1}{2}\sum_{j=1}^n u_i^2\}\ d(\prod_{j=1}^n m_L)(u_1,u_2,\cdots,u_n) \end{aligned}$$

where $E = \{x \text{ in } H | P(x) \text{ is in } F\ \}$. Then v is finitely additive but not countably-additive. Let $\{h_k | k \text{ is in } \mathbb{N}\}$ be an orthonormal basis of H. For F in $\mathcal{B}(\mathbb{R}^n)$, we let $v_{h_1,h_2,\cdots,h_n}(F) = v(\{x \text{ in } H | (\langle x,h_1\rangle, \langle x,h_2\rangle, \cdots, \langle x,h_n\rangle) \text{ is in } F\})$. Then $\{v_{h_1,h_2,\cdots,h_n} | n \text{ is in } \mathbb{N}\}$ is a consistent family of probability measures. By Kolmogorov's theorem, there exists a probability measure space (Ω, m) and random variables X_k (k is in $\mathbb{N}$) on Ω such that for F in $\mathcal{B}(\mathbb{R}^n)$, $m(\{\omega \text{ in } \Omega | (X_1(\omega), X_2(\omega), \cdots, X_n(\omega)) \text{ is in } F\}) = v_{h_1,h_2,\cdots,h_n}(F)$.

For a measurable semi-norm $||\cdot||$ in H, let $\mathbb{B} = \overline{H}^{||\cdot||}$, the closure of H with respect to $||\cdot||$. In case $(i, H, \mathbb{B})$ is called an abstract Wiener space where $i : H \to \mathbb{B}$ is the inclusion map.

Given an element $h \in H$, the Wiener integral $\langle h, \cdot\rangle^\wedge$ of h is defined on $\mathbb{B}$ by

$$\langle h, \cdot\rangle^\wedge = L_2 - \lim_{n\to\infty} \langle h, \sum_{j=1}^n h_j(\cdot)\rangle$$

if the L_2-limit exists. (Compare with [2] and [5])

(E) (Spaces ; $\mathcal{H}, \ell_1, \mathcal{U}$ and ℓ_2)

Let ℓ_1 be the space of all sequences $\langle c_n\rangle$ in $\mathbb{R}$ with a norm $||\langle c_n\rangle||_1 = \sum_{n=0}^\infty |c_n| < +\infty$. Let ℓ_2 be the space of all sequences $\langle c_n\rangle$ in $\mathbb{R}$ with a norm $||\langle c_n\rangle||_2 = \sqrt{\sum_{n=0}^\infty c_n^2} < +\infty$. Let $\mathcal{H}$ be the space of all sequences $\langle c_n\rangle$ in $\mathbb{R}$ with an inner product $\langle\langle c_n\rangle, \langle d_n\rangle\rangle = \sum_{n=0}^\infty (n+1)^2 c_n d_n$

for $\langle c_n\rangle$ and $\langle d_n\rangle$ in $\mathcal{H}$. For $\langle c_n\rangle$ in $\mathcal{H}$, let $||\langle c_n\rangle||_{\mathcal{H}} = \sqrt{\langle\langle c_n\rangle, \langle c_n\rangle\rangle}$. Let $\mathcal{U}$ be the space of all sequences $\langle c_n\rangle$ in ℓ_2 such that the limit $\lim_{m\to\infty} \frac{1}{m+1}\sum_{n=0}^{m}\sum_{k=0}^{n} c_k cos\frac{k\pi t}{T}$ converges uniformly on $[0,T]$.

(F) (The Fourier cosine series)

For x in $L_2[0,T]$, we let $x^*(t) = x(|t|)$ on $[-T,T]$ and $\hat{x}(t+2kT) = x^*(t)$ for all t in $[-T,T]$ and for all integer k. Then $\hat{x}$ is an even periodic function on $\mathbb{R}$ having the period $2T$. For x in $L_2[0,T]$, we let $a_{x,0} = \frac{1}{2T}\int_{-T}^{T}\hat{x}(t)dm_L(t)$ and $a_{x,n} = \frac{1}{T}\int_{-T}^{T}\hat{x}(t)cos\frac{n\pi t}{T}dm_L(t)$. Then $\sum_{n=0}^{\infty} a_{x,n}cos\frac{n\pi t}{T}$ is the Fourier cosine series of $\hat{x}$.

2. Relationship between the spaces $\mathcal{H}, \ell_1, \mathcal{U}$ and ℓ_2

In this section, we will treat the relationship between the spaces $\mathcal{H}, \ell_1, \mathcal{U}$ and ℓ_2.

THEOREM 2.1. $\mathcal{H} \underset{\neq}{\subset} \ell_1 \underset{\neq}{\subset} \mathcal{U} \underset{\neq}{\subset} \ell_2$.

Proof. If $\langle c_n\rangle$ is in $\mathcal{H}$ then from Schwarz's inequality,

$$\begin{aligned}
&\sum_{n=0}^{\infty}|c_n| \\
&= \sum_{n=0}^{\infty}(\frac{1}{n+1})((n+1)|c_n|) \\
&\leq \sqrt{\sum_{n=0}^{\infty}\frac{1}{(n+1)^2}}\sqrt{\sum_{n=0}^{\infty}(n+1)^2c_n^2} \\
&= \frac{\pi}{\sqrt{6}}||\langle c_n\rangle||_{\mathcal{H}} < +\infty,
\end{aligned}$$

so we have $\mathcal{H} \subset \ell_1$. Putting $c_n = \frac{1}{(n+1)^{\frac{3}{2}}}$ for n in $\mathbb{N}_0$, by the p-series $\sum_{n=0}^{\infty}|c_n| < +\infty$, that is, $\langle c_n\rangle$ is in ℓ_1 but $||\langle c_n\rangle||_{\mathcal{H}} = \sqrt{\sum_{n=0}^{\infty}\frac{1}{n+1}}$, which implies that $\mathcal{H} \underset{\neq}{\subset} \ell_1$.

Suppose $\langle c_n\rangle$ is in ℓ_1. From ℓ_1 is a subset of ℓ_2, $\langle c_n\rangle$ is in ℓ_2. Using Weierstrass's M-test, the series $\sum_{k=0}^{\infty} c_k cos\frac{k\pi t}{T}$ converges uniformly on $[0,T]$, so a function $x(t) \equiv \sum_{k=0}^{\infty} c_k cos\frac{k\pi t}{T}$ is continuous on $[0,T]$. From the dominated convergence theorem, the Fourier series of $\hat{x}$ is $\sum_{k=0}^{\infty} c_k cos\frac{k\pi t}{T}$ on $[-T,T]$. By Fejér's theorem, the limit $\lim_{m\to\infty}\frac{1}{m+1}\sum_{n=0}^{m}\sum_{k=0}^{n} c_k$

$\cos\frac{k\pi t}{T}$ converges uniformly to $\hat{x}$ on $[-T,T]$, so $\langle c_n\rangle$ is in $\mathcal{U}$, that is, $\ell_1 \subset \mathcal{U}$. Fejér gave an example of an even continuous function f, having a period 2π, whose Fourier series diverges at origin [1], so $\sum_{n=0}^{\infty} a_{f,n}$ diverges, that is, $\sum_{n=0}^{\infty} |a_{f,n}|$ diverges. Hence, we obtain $\ell_1 \underset{\neq}{\subset} \mathcal{U}$.

Lastly, we must show that $\mathcal{U} \underset{\neq}{\subset} \ell_2$. For $n \in \mathbb{N}_0$, we let $c_n = \frac{1}{n+1}$. Then $\langle c_n\rangle$ is in ℓ_2. We assume that $\langle \frac{1}{m+1}\sum_{n=0}^{m}\sum_{k=0}^{n} c_k \cos\frac{k\pi}{T}t\rangle$ converges uniformly of t. Then putting $t=0$, $\langle \frac{1}{m+1}\sum_{n=0}^{m}\sum_{k=0}^{n} c_k cos\frac{k\pi}{T}t\rangle$ converges. Then from Tauberian's theorem [2], $\sum_{k=0}^{\infty}\frac{1}{k+1}$ converges, a contradiction. Thus, we have $\mathcal{U} \underset{\neq}{\subset} \ell_2$. □

THEOREM 2.2. *Let $J_1 : L_2[0,T] \to \ell_2$ be a function with $J_1(x) = \langle a_{x,n}\rangle$. Then J_1 is an isometric isomorphism and $J_1(C[0,T]) = \mathcal{U}$.*

Proof. By the uniqueness theorem for Fourier series of L_2-function, J_1 is injective and by Bessel's inequality and monotonic convergence theorem, J_1 is isometric. From Parserval's identity, we have J_1 is isometric. By Fejér's theorem, we obtain $J_1(C[0,T]) \subset \mathcal{U}$. Now, we assume that $\langle c_n\rangle$ is in $\mathcal{U}$. Put $x(t) = lim_{m\to\infty}\frac{1}{m+1}\sum_{n=0}^{m}\sum_{k=0}^{n} c_k \cos\frac{k\pi t}{T}$ for t in $[0,T]$. Then $a_{x,0} = \frac{1}{2T} lim_{m\to\infty}\frac{1}{m+1}\sum_{n=0}^{m}\sum_{k=0}^{n} c_k \int_{-T}^{T}\cos\frac{k\pi t}{T}\, dm_L(t) = c_0$ and for p in $\mathbb{N}$,

$$\begin{aligned}
a_{x,p} \quad &= \frac{1}{T}\int_{-T}^{T}\hat{x}\cos\frac{p\pi t}{T}\, dm_L(t)\\
&= \frac{1}{T}\lim_{m\to\infty}\frac{1}{m+1}\sum_{n=0}^{m}\sum_{k=0}^{n} c_k\int_{-T}^{T}\cos\frac{p\pi t}{T}\cos\frac{k\pi t}{T}\, dm_L(t)\\
&= \frac{1}{T}\lim_{m\to\infty}\frac{1}{m+1}\sum_{n=0}^{m} T\ c_p\chi_{A_p}(n)\\
&= \lim_{m\to\infty}\frac{m-p+1}{m+1}c_p\\
&= c_p,
\end{aligned}$$

as desired. Here $A_p = \{n \text{ in } \mathbb{N}_0 | n \geq p\}$ and χ_{A_p} is a characteristic function, associated with A_p. □

REMARK 2.3. (1) Let $I_1 : (C[0,T], ||\cdot||_\infty) \to L_2[0,T]$ be an inclusion map. Then I_1 is an injective continuous function. Since $C[0,T]$ is a dense subset of $L_2[0,T]$, $(J_1 \circ I_1)(C[0,T]) = \mathcal{U}$ is a dense subset of ℓ_2. Let $I_2 : \mathcal{U} \to \ell_2$ be an inclusion map. Then I_2 is an injective continuous function. Let $J_2 : C[0,T] \to \mathcal{U}$ be a function with $J_2(x) = \langle a_{x,n} \rangle$. Then J_2 is a bijective continuous function and $J_1 \circ I_1 = I_2 \circ J_2$.

(2) Let $\langle c_n \rangle$ be in ℓ_2. For m in $\mathbb{N}_0$, put

$$d_n^{(m)} = \begin{cases} c_n \,, & \text{if } \ n \leq m \\ 0 \,, & \text{otherwise} \end{cases} .$$

Then $\langle d_n^{(m)} \rangle$ is in $\mathcal{H}$ and $\langle \langle d_n^{(m)} \rangle \rangle$ converges to $\langle c_n \rangle$ as $m \to \infty$ with respect to $|| \cdot ||_2$. Hence, we have $\overline{\mathcal{H}}^{|| \cdot ||_2} = \overline{\mathcal{U}}^{|| \cdot ||_2} = \ell_2$ where $\overline{X}^{|| \cdot ||_2}$ means the closure of X with respect to $|| \cdot ||_2$.

We give a topology on $\mathcal{U}$ such that J_2 is a homeomorphism.

3. The analogue of Wiener measure and a measure on $\mathcal{U}$

In this section, we will derive a measure m on $\mathcal{U}$ and investigate the properties of it.

THEOREM 3.1. $|| \cdot ||_2$ *is a measurable norm on* $\mathcal{H}$.

Proof. For n in $\mathbb{N}_0$, let

$$f_n(m) = \begin{cases} \frac{1}{n+1} \,, & \text{if } \ n = m \\ 0 \,, & \text{otherwise} \end{cases} ,$$

for m in $\mathbb{N}_0$. Then $\{f_n | n \text{ is in } \mathbb{N}_0\}$ is an orthonormal basis of $\mathcal{H}$. Let $T : \mathcal{H} \to \mathcal{H}$ be an operator with $T(\langle c_n \rangle) = \langle \frac{1}{n+1} c_n \rangle$. Then $\sum_{n=0}^{\infty} ||T(f_n)||_{\mathcal{H}}^2 = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6} < +\infty$, so T is a Hilbert-Schmidt operator on $\mathcal{H}$ and $||\langle c_n \rangle||_2^2 = \sum_{n=0}^{\infty} c_n^2 = ||T(\langle c_n \rangle)||_{\mathcal{H}}^2$, so $|| \cdot ||_2$ is a measurable norm on $\mathcal{H}$ by [4]. □

From the above theorem and Remark 2.3 (2), $(i, \mathcal{H}, \ell_2)$ is an abstract Wiener space.

THEOREM 3.2. *For* $\langle v_n \rangle$ *in* $\mathcal{H}$, *the Wiener integral* $\langle \langle v_n \rangle, \cdot \rangle^{\wedge} = \ell_2$ - $\lim_{m \to \infty} \langle \langle v_n \rangle, \sum_{k=0}^{m} f_k(\cdot) \rangle$ *exists on* ℓ_2, *always where* f_n*'s are given in the proof of 3.1.*

Proof. Let $\langle c_n \rangle$ be in ℓ_2. It suffices to show that $\langle \langle \langle v_n \rangle, \sum_{k=0}^{m} f_n(\langle c_n \rangle) \rangle \rangle$ is Cauchy. Then for two natural numbers m_1, m_2 with $m_1 < m_2$, by Schwarz's inequality,

$$|\langle \langle v_n \rangle, \sum_{k=m_1+1}^{m_2} f_n(\langle c_n \rangle) \rangle|$$

$$= |\sum_{k=m_1+1}^{m_2} (k+1)\ v_k\ c_k|$$

$$\le ||\langle v_n\rangle||_{\mathcal{H}} \sqrt{\sum_{k=m_1+1}^{m_2} c_k^2},$$

$\langle\langle\langle v_n\rangle, \sum_{k=0}^{m} f_n(\langle c_n\rangle)\rangle\rangle$ is Cauchy of m in the $||\cdot||_2$-norm sense. $\square$

We can give a probability measure $\widetilde{m}$ on ℓ_2 such that for f_n in $\mathcal{H}$, $\langle f_n, \cdot\rangle^{\sim}$ is normal distributed on ℓ_2 with mean 0 and variance 1 as follows. For E in $\mathcal{B}(\mathbb{R}^m)$, letting $I_E = \{\langle c_n\rangle$ in $\ell_2|(\langle f_0, \langle c_n\rangle\rangle^{\wedge}, \langle f_1, \langle c_n\rangle\rangle^{\wedge}, \cdots, \langle f_{m-1}, \langle c_n\rangle\rangle^{\wedge})$ is in $E\} = \{\langle c_n\rangle$ in $\ell_2|(c_0, 2c_1, \cdots, mc_{m-1})$ is in $E\}$, let $\widetilde{m}(I_E) = v(E) = (2\pi)^{-\frac{m}{2}} \int_E \exp\{-\frac{1}{2}\sum_{j=1}^{m} u_j^2\} d(\prod_{j=1}^{m} m_L)\ (u_1, u_2, \cdots, u_m)$. Then $\widetilde{m}$ can be uniquely extended onto $\mathcal{B}(\ell_2)$. We shall denote it by $\widetilde{m}$, again.

Remark 3.3. Let $\mathcal{S}$ be the set of all sequences. Let $P_0, L, M : \mathcal{S} \to \mathcal{S}$ be a functions with $P_0(\langle c_n\rangle) = \langle c_1, c_2, c_3, \cdots\rangle$, $L(\langle c_n\rangle) = \langle\frac{n\pi c_n}{T}\rangle$ and $M(\langle c_n\rangle) = \langle(n+1)c_n\rangle$. For m in $\mathbb{N}$, let $L_m, M_m : \mathcal{S} \to \mathbb{R}^m$ be a functions with $L_m(\langle c_n\rangle) = (\frac{\pi c_1}{T}, \frac{2\pi c_2}{T}, \cdots, \frac{m\pi c_m}{T})$ and $M_m(\langle c_n\rangle) = (2c_1, 3c_2, 4c_3, \cdots, (m+1)c_m)$. Then for E in $\mathcal{B}(\mathbb{R}^m)$, $\widetilde{\omega}(\{f$ in $L_2[0,T]|$ $L_m(J_1(f))$ is in $E\}) = \widetilde{m}(\{\langle c_n\rangle$ in $\ell_2|M_m(\langle c_n\rangle)$ is in $E\}) = (2\pi)^{-\frac{m}{2}} \int_E \exp$ $\{-\frac{1}{2}\sum_{j=1}^{m} u_j^2\} d(\prod_{j=1}^{m} m_L)\ (u_1, u_2, \cdots, u_m)$. Hence, for all cylinder sets in ℓ_2, $\widetilde{\omega} \circ (J_1^{-1} \circ L^{-1} \circ P_0^{-1}) = \widetilde{m} \circ (M^{-1} \circ P_0^{-1})$ holds. So, $\widetilde{\omega} \circ (J_1^{-1} \circ L^{-1} \circ P_0^{-1}) = \widetilde{m} \circ (M^{-1} \circ P_0^{-1})$ holds on $\mathcal{B}(\ell_2)$.

Let $\mathcal{M} = \{\mathcal{U} \cap ((P_0^{-1} \circ P_0)(B))|B$ is in $\mathcal{B}(\ell_2)\}$ and let $X : \mathcal{S} \to \mathcal{S}$ be a functions with $X(\langle c_n\rangle) = \langle\frac{2T}{\pi}c_1, \frac{3T}{2\pi}c_2, \cdots, \frac{(n+1)T}{n\pi}c_n, \cdots\rangle$. For B in $\mathcal{B}(\ell_2)$, let $m(\mathcal{U} \cap ((P_0^{-1} \circ P_0)(B))) = m_\varphi(C[0,T] \cap J_1^{-1}(X^{-1}(B)))$.

Theorem 3.4. *For B in $\mathcal{B}(\ell_2)$, $m(\mathcal{U} \cap ((P_0^{-1} \circ P_0)(B))) = \widetilde{m}((P_0^{-1} \circ P_0)(B))$.*

Proof. By Lemma 1.1, $m(\mathcal{U} \cap ((P_0^{-1} \circ P_0)(B))) = m_\varphi(C[0,T] \cap J_1^{-1}(X^{-1}(B))) = \widetilde{\omega}(J_1^{-1}(X^{-1}(B)))$. Since $M^{-1} \circ P_0^{-1} \circ P_0 \circ L \circ X^{-1}(\langle c_n\rangle) = P_0^{-1} \circ P_0(\langle c_n\rangle)$, from Remark 3.3, $\widetilde{\omega}(J_1^{-1}(X^{-1}(B))) = \widetilde{\omega}(J_1^{-1} \circ L^{-1} \circ P_0^{-1} \circ (P_0 \circ L \circ X^{-1})(B)) = \widetilde{m}(M^{-1} \circ P_0^{-1} \circ P_0 \circ L \circ X^{-1}(B)) = \widetilde{m}((P_0^{-1} \circ P_0)(B))$ as desired. $\square$

Remark 3.5. Neither $\mathcal{M}$ is a σ-algebra on $\mathcal{U}$ nor m is a measure on $\mathcal{M}$, but if we ignore the first term c_0 of $\langle c_n\rangle$ in $\mathcal{U}$ then $(\mathcal{U}, \mathcal{M}, m)$ is a measure space.

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