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A GENERALIZATION OF CARLESON INEQUALITY IN THE CONTEXT OF SPACES OF HOMOGENEOUS TYPE

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Abstract. In this paper we first introduce a Carleson inequality and study the generalized form of Carleson inequality in the context of spaces of homogeneous type. The previous inequality is known to play important roles in harmonic analysis.

1. Preliminaries and notations

We begin by introducing the notion of a space of homogeneous type [2]: Let X be a topological space endowed with Borel measure μ. Assume that d is a pseudo-metric on X, that is, a nonnegative function defined on $X \times X$ satisfying

(i) $d(x,x) = 0; d(x,y) > 0$ if $x \neq y$,
(ii) $d(x,y) = d(y,x)$, and
(iii) $d(x,z) \leq K(d(x,y) + d(y,z))$, where K is some fixed constant.

Assume further that

(a) the balls $B(x,\rho) = \{y \in X : d(x,y) < \rho\}$, $\rho > 0$, form a basis of open neighborhoods at $x \in X$,

and that μ is a Borel measure on X satisfying the doubling property:

(b) $0 < \mu(B(x,2\rho)) \leq A\mu(B(x,\rho)) < \infty$, where A is some fixed constant.

Then we call (X,d,μ) a *space of homogeneous type.*

Note that the volume of balls will be proportional to a fixed power of the radius. Thus assume there exist a $\sigma \in R$ and constants C_1 and

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C_2 such that

$$C_1\rho^\sigma \le \mu(B(x,\rho)) \le C_2\rho^\sigma. \tag{1.1}$$

We will denote $\mu(B(x,\rho)) \approx \rho^\sigma$ for the simplicity of the notation.

Now consider the space $X\times(0,\infty)$, which is a kind of generalized upper half-space over X. We then introduce the analogue of nontangential or conical regions as follows. For $x \in X$, set

$$\Gamma(x) = \{(y,t) \in X \times (0,\infty) : x \in B(y,t)\}.$$

For an open set $E \subset X$, the *tent* over E is the set

$$T(E) = \{(y,t) \in X \times (0,\infty) : B(y,t) \subset E\}.$$

It is then very easy to check that

$$T(E) = (X \times (0,\infty)) \setminus \bigcup_{x\notin E} \Gamma(x).$$

For a measurable function f defined on $X \times (0,\infty)$, and $\sigma \in R$, we define an area function $A_p(f)$, for $x \in X$, by

$$A_p(f)(x) = \left(\int_{\Gamma(x)} |f(y,t)|^p \frac{d\mu(y)dt}{t^{\sigma+1}}\right)^{1/p} \text{ if } 1 \le p < \infty, \tag{1.2}$$

and

$$A_\infty(f)(x) = \sup_{(y,t)\in\Gamma(x)} |f(y,t)| \text{ if } p = \infty.$$

We now introduce certain maximal operators acting on functions on $X\times(0,\infty)$ as follows. For a measurable function f defined on $X\times(0,\infty)$, we define a maximal function $M_p(f)$, for $x \in X$, by

$$\begin{aligned} &M_p(f)(x) \\ &= \sup_{x\in B}\left(\frac{1}{\mu(B)}\int_{T(B)} |f(y,t)|^p \frac{d\mu(y)dt}{t}\right)^{1/p},\ 1 \le p < \infty, \end{aligned} \tag{1.3}$$

where the supremum is taken over all balls B containing x.

Let $f \in L^1(d\mu)$ and $x \in X$. Then we define

$$M_{\mathrm{HL}}(f)(x) = \sup_{x\in B} \frac{1}{\mu(B)} \int_B |f(x)| d\mu(x),$$

where the supremum is taken over all balls B containing x. Then we call M_{HL} the *Hardy-Littlewood maximal operator* on X.

The inequality

$$\left| \int_{X\times(0,\infty)} f(y,t)g(y,t)\frac{d\mu(y)dt}{t} \right| \le C \int_X A_\infty(f)(x)d\mu(x) \tag{1.4}$$

is called the *Carleson inequality*, where $g(y,t)\frac{d\mu(y)dt}{t}$ is a *Carleson measure* on $X \times (0,\infty)$, that is, $g(y,t) \ge 0$ and $\sup_{x\in B} \int_{T(B)} g(y,t)\frac{d\mu(y)dt}{t} \le C$, where the supremum is taken over all balls B containing x.

The purpose of this paper is to give a more general form of inequality (1.4).

2. Main result

We begin with a lemma which is of the type due to Whitney.

Lemma 2.1 ([2]). *Let O be an open subset of X. Then there exist a positive constant $N, h_1 > 1, h_2 > 1$ and $h_3 < 1$ which depend only on the space X, and a sequence $\{B(x_i,\rho_i)\}$ of balls such that*

(i) $\cup_i B(x_i,\rho_i) = O$,
(ii) $B(x_i,h_2\rho_i) \subset O$ *and* $B(x_i,h_1\rho_i) \cap (X \setminus O) \ne \emptyset$,
(iii) *the balls* $B(x_i,h_3\rho_i)$ *are pairwise disjoint, and*
(iv) *no point in O lies in more than N of the balls* $B(x_i,h_2\rho_i)$.

Lemma 2.2 ([4]). *Let M_p be defined as in* (1.3) *and* $1 \le p < \infty$. *Then M_p belongs to the Muckenhoupt's class A_1* [3]*, that is, there exists a constant C such that*

$$\frac{1}{\mu(B)} \int_B M_p(f)(x)d\mu(x) \le C \inf_{x\in B} M_p(f)(x),$$

where the infimum is taken over all balls B containing x.

We now need the notions of some sets to get main results. For each positive integer k, set

$$O_k = \{x \in X : A_p(f)(x) > 2^k\},\ 1 \le p < \infty, \tag{2.1}$$

and

$$O_k^* = \{x \in X : M_{\mathrm{HL}}(\chi_{O_k})(x) > \frac{1}{2}\}, \tag{2.2}$$

where M_{HL} is the Hardy-Littlewood maximal operator on X, and χ_{O_k} is the characteristic function of the set O_k. Then observe that for $k =$

$1,2,3,\cdots, O_{k+1} \subset O_k, O_k \subset O_k^*, T(O_k) \subset T(O_k^*)$, and $\cup_{k=-\infty}^{\infty} T(O_k^*)$ contains the support of f. Let

$$O_k^* = \bigcup_{j=1}^{\infty} B(x_{k,j}, \rho_{k,j}) \equiv \bigcup_{j=1}^{\infty} B_{k,j}$$

and

$$O_k = \bigcup_{j=1}^{\infty} B(x_{k,j}, Ch_1\rho_{k,j}) \equiv \bigcup_{j=1}^{\infty} \tilde{B}_{k,j}$$

be Whitney decompositions of the open sets O_k^* and O_k respectively, where h_1 is the given in (ii) of Lemma 2.1, and C will be chosen sufficiently large in a moment. Let now

$$V_{k,j} = T(B_{k,j}) \setminus \bigcup_{l=1}^{\infty} T(B_{k+1,l})$$

and

$$W_{k,j} = B_{k,j} \setminus \bigcup_{l=1}^{\infty} \tilde{B}_{k+1,l}. \tag{2.3}$$

We then have:

Lemma 2.3. *Let A_p be defined as in* (1.2) *and $1 \le p < \infty$. Then there exists a constant C such that*

$$\int_{V_{k,j}} |f(y,t)|^p \frac{d\mu(y)dt}{t} \le C \int_{W_{k,j}} [A_p(f)(x)]^p d\mu(x), \tag{2.4}$$

where $V_{k,j}$ and $W_{k,j}$ are defined as in (2.2) *and* (2.3) *respectively.*

Proof. Proof Assume that $1 \le p < \infty$. Then we have

$$\int_{W_{k,j}} [A_p(f)(x)]^p d\mu(x) \tag{2.5}$$

$$= \int_{W_{k,j}} d\mu(x) \int_{\Gamma(x)} |f(y,t)|^p \frac{d\mu(y)dt}{t^{\sigma+1}} \tag{2.6}$$

$$= \int_{W_{k,j}\times(X\times(0,\infty))} |f(y,t)|^p \chi_{\Gamma(x)}(y,t) \frac{d\mu(y)dt}{t^{\sigma+1}} d\mu(x) \tag{2.7}$$

$$\ge \int_{V_{k,j}} |f(y,t)|^p \frac{d\mu(y)dt}{t^{\sigma+1}} \int_{W_{k,j}} \chi_{\Gamma(x)}(y,t) d\mu(x). \tag{2.8}$$

For any fixed $(y,t) \in V_{k,j}$, we have

$$B(x,t) \cap {O_{k+1}^*}^c \ne \emptyset.$$

It means that there exists a point $x \in B(x,t)$ such that

$$M_{\mathrm{HL}}(\chi_{O_{k+1}})(x) \le \frac{1}{2},$$

which means

$$\frac{1}{\mu(B(x,t))}\int_{B(x,t)} \chi_{O_{k+1}}(x)d\mu(x) \le \frac{1}{2}.$$

Thus

$$\frac{1}{\mu(B(x,t))}\int_{W_{k,j}} \chi_{\Gamma(x)}(y,t)d\mu(x) \tag{2.9}$$

$$=\frac{1}{\mu(B(x,t))}\int_{B(x,t)} [1-\chi_{O_{k+1}\cap B(x,t)}(x)]d\mu(x) \tag{2.10}$$

$$\ge\frac{1}{2}, \tag{2.11}$$

that is,

$$\int_{W_{k,j}} \chi_{\Gamma(x)}(y,t)d\mu(x) \ge C\mu(B(x,t)) \tag{2.12}$$

$$\approx Ct^{\sigma}. \tag{2.13}$$

Substituting (2.12) into (2.5), we prove (2.4). Thus the proof is complete. □

THEOREM 2.4. *Let M_p be defined as in* (1.3) *and* $1 \le p < \infty$. *Then there exists a constant C such that*

$$\int_{O_k^*} M_p(f)(x)d\mu(x) \le C\int_{O_k} M_p(f)(x)d\mu(x),$$

where O_k and O_k^* are defined as in (2.1) *and* (2.2) *respectively.*

Proof. Proof Since M_p satisfies the A_1 condition by Lemma 2.2, it follows from [1] that

$$\begin{aligned}&\int_X [M_{\mathrm{HL}}(\chi_{O_k})(x)]^2 M_p(f)(x)d\mu(x)\\ &\qquad\le C\int_X [\chi_{O_k}(x)]^2 M_p(f)(x)d\mu(x),\end{aligned} \tag{2.14}$$

where χ_{O_k} is the characteristic function of O_k. Thus it follows from (2.8) that

$$\begin{aligned}
&\int_{O_k^*} M_p(f)(x)d\mu(x)\\
&\leq 4\int_X [M_{\mathrm{HL}}(\chi_{O_k})(x)]^2 M_p(f)(x)d\mu(x)\\
&\leq C\int_X [\chi_{O_k}(x)]^2 M_p(f)(x)d\mu(x)\\
&= C\int_{O_k} M_p(f)(x)d\mu(x)
\end{aligned}$$

for some constant C. The proof is therefore complete. $\square$

The main result of this paper is now the following.

THEOREM 2.5. *Let $1/p+1/q=1$, $1\leq p\leq\infty$. Then there exists a constant C such that*

$$\left|\int_{X\times(0,\infty)} f(y,t)g(y,t)\frac{d\mu(y)dt}{t}\right| \leq C\int_X A_p(f)(x)M_q(g)(x)d\mu(x),$$

where A_p and M_q are defined as in (1.2) and (1.3) respectively.

Proof. Assume first that $1\leq p<\infty$. Then it follows from Hölder's inequality that

$$\begin{aligned}
&\left|\int_{X\times(0,\infty)} f(y,t)g(y,t)\frac{d\mu(y)dt}{t}\right|\\
&\leq \left|\int_{T(O_k^*)\setminus T(O_{k+1}^*)} f(y,t)g(y,t)\frac{d\mu(y)dt}{t}\right|\\
&\leq \left|\sum_{k=-\infty}^{\infty}\sum_{j=1}^{\infty}\int_{V_{k,j}} f(y,t)g(y,t)\frac{d\mu(y)dt}{t}\right|\\
&\leq \sum_{k=-\infty}^{\infty}\sum_{j=1}^{\infty}\left(\int_{V_{k,j}} |f(y,t)|^p\frac{d\mu(y)dt}{t}\right)^{1/p}\left(\int_{V_{k,j}} |g(y,t)|^q\frac{d\mu(y)dt}{t}\right)^{1/q}
\end{aligned}$$

$$\leq C\sum_{k=-\infty}^{\infty}\sum_{j=1}^{\infty}\left(\int_{W_{k,j}}[A_p(f)(x)]^p d\mu(x)\right)^{1/p}\left(\int_{T(B_{k,j})}|g(y,t)|^q\frac{d\mu(y)dt}{t}\right)^{1/q}$$

$$\leq C\sum_{k=-\infty}^{\infty}\sum_{j=1}^{\infty}2^{k+1}\mu(B_{k,j})\left(\frac{1}{\mu(B_{k,j})}\int_{T(B_{k,j})}|g(y,t)|^q\frac{d\mu(y)dt}{t}\right)^{1/q}$$

$$\leq C\sum_{k=-\infty}^{\infty}2^k\sum_{j=1}^{\infty}\int_{B_{k,j}}M_q(g)(x)d\mu(x)$$

$$= C\sum_{k=-\infty}^{\infty}2^k\int_{O_k^*}M_q(g)(x)d\mu(x)$$

$$\leq C\sum_{k=-\infty}^{\infty}2^k\int_{O_k}M_q(g)(x)d\mu(x)$$

$$\leq C\int_X A_p(f)(x)M_q(g)(x)d\mu(x).$$

In the above, by Lemma 2.3, we have fourth step, and by Lemma 2.4, we have eighth step.

Second, assume $p=\infty$. Then

$$\left|\int_{X\times(0,\infty)}f(y,t)g(y,t)\frac{d\mu(y)dt}{t}\right|$$

$$\leq\sum_{k=-\infty}^{\infty}\int_{T(O_k)\setminus T(O_{k+1})}|f(y,t)g(y,t)|\frac{d\mu(y)dt}{t}$$

$$\leq\sum_{k=-\infty}^{\infty}\sum_{j=1}^{\infty}2^{k+1}\int_{T(\tilde{B}_{k,j})}g(y,t)\frac{d\mu(y)dt}{t}$$

$$\leq C\sum_{k=-\infty}^{\infty}2^k\sum_{j=1}^{\infty}\int_{\tilde{B}_{k,j}}M_1(g)(x)d\mu(x)$$

$$\leq C\sum_{k=-\infty}^{\infty}2^k\int_{O_k}M_1(g)(x)d\mu(x)$$

$$\leq C\int_X A_\infty(f)(x)M_1(g)(x)d\mu(x).$$

Thus the proof is complete. □

REMARK 2.6. In Theorem 2.5, if $M_q(g)(x) \leq C$, then

$$\left| \int_{X\times(0,\infty)} f(y,t)g(y,t)\frac{d\mu(y)dt}{t} \right| \leq C \int_X A_p(f)(x)d\mu(x).$$

In the case $q = 1$, the condition $M_q(g)(x) \leq C$ means that $g(y,t)\frac{d\mu(y)dt}{t}$ is a Carleson measure on $X \times (0,\infty)$, and the area function becomes $A_\infty(f)$, this reduces to the Carleson inequality.

References

[1] R. R. Coifman and G. Weiss, *Weighted norm inequalities for maximal functions and singular integrals,* Studia Math. **51** (1974), 141–150

[2] R. R. Coifman and G. Weiss, *Analyse Harmonique Non-commutative sur Certains Espaces Homogènes,* Lecture Notes in Math., Vol. 242, Springer-Verlag, Berlin, 1971

[3] E. M. Stein, *Singular Integrals and Differentiablity Properties of Functions,* Princeton Univ. Press, Princeton, N. J., 1970

[4] C. -S. Suh, *On maximal operators belonging to the Muckenhoupt's class A_1,* East Asian Math. J. **23** (2007), 37–43

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