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LARGE DEVIATIONS FOR THE BUSY PERIOD IN THE M/G/1 QUEUE

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Abstract. When the service time distribution has a finite exponential moment, we present a large deviations result for the busy period distribution in the M/G/1 queue without the assumption of Abate and Whitt (1997) and Kyprianou (1971).

1. Introduction

We are interested in the tail asymptotics of the busy period distribution. De Meyer and Teugels [6] showed that if the service time distribution has a regularly varying tail then the busy period distribution in the M/G/1 queue has also regularly varying tail of the same index, and vice versa. Zwart [7] generalized a result of de Meyer and Teugels [6] and characterized the tail behaviour of the busy period distribution in the GI/G/1 queue under the assumption that the tail of the service time distribution is of intermediate regular variation. In [2] and [4], the tail asymptotics of the busy period distribution was investigated for the GI/G/1 queue when the service time distribution belongs to another subclass of heavy tailed distributions.

On the other hand, this paper presents a large deviations analysis for the busy period distribution in the M/G/1 queue under the light tailed assumption of the service time distribution, i.e., the service time distribution has a finite exponential moment. Abate and Whitt [1] and Kyprianou [5] gave the exact tail asymptotics of the busy period distribution, under a technical assumption. The contribution of this paper

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is to obtain the large deviations result for the busy period distribution without the assumption of Abate and Whitt [1] and Kyprianou [5].

2. Main result

We consider the M/G/1 queue where customers arrive according to a Poisson process with rate λ and service times are independent and identically distributed. Let B denote a generic random variable representing the service time, $g(s) = \mathbb{E}[e^{sB}]$ be the moment generating function of the service time distribution, and $\frac{1}{\mu}$ be the mean service time of a customer. The traffic load ρ is defined as $\rho = \frac{\lambda}{\mu}$. We assume that $\rho < 1$ for stability of the system. Moreover, it is assumed that the service time distribution has a finite exponential moment, i.e.,

$$s^* = \sup\{s \geq 0 : \mathbb{E}[e^{sB}] < \infty\} > 0.$$

We define

$$\zeta = \sup\{s \geq 0 : \mathbb{E}[Be^{sB}] \leq \frac{1}{\lambda}\}.$$

Remark 2.1. Let $g_1(s) = \mathbb{E}[Be^{sB}] = g'(s)$. If $g_1(s^*) = \mathbb{E}[Be^{s^*B}] > \frac{1}{\lambda}$, then ζ is the unique solution of $g_1(s) = \frac{1}{\lambda}$ in $(0, s^*)$. If $g_1(s^*) \leq \frac{1}{\lambda}$, then $\zeta = s^*$. See Figure 1.

Before presenting our main result, we introduce the following lemma; its proof can be found in Theorem 7.1 of Abate and Whitt [1], the appendix of Cox and Smith [3] (Equation (46) on page 154) and Theorem 1 of Kyprianou [5]. Let G be a generic random variable representing the busy period and $b(t)$ its density. We will write $f(t) \sim h(t)$ as $t \to \infty$ to mean that $\lim_{t\to\infty} \frac{f(t)}{h(t)} = 1$.

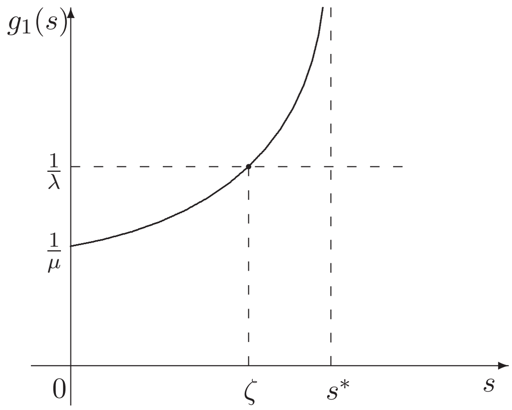
Lemma 2.2. *(Abate and Whitt* [1], *Cox and Smith* [3], *and Kyprianou* [5]*) If $\zeta < s^*$, then*

$$b(t) \sim ct^{-\frac{3}{2}} e^{-\sigma t} \quad \text{as } t \to \infty,$$

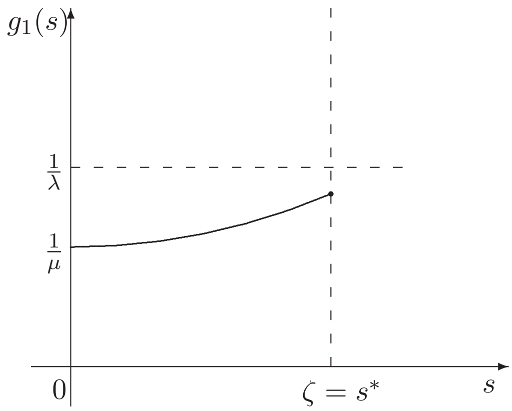
where

$$\begin{aligned} c &= (2\pi\lambda^3 g''(\zeta))^{-\frac{1}{2}}, \\ \sigma &= \lambda + \zeta - \lambda g(\zeta). \end{aligned} \tag{2.1}$$

The constant σ is called the asymptotic decay rate. The next corollary is immediate from Lemma 2.2.



(a) $\zeta < s^*$



(b) $\zeta = s^*$

FIGURE 1. The graph of $g_1(s)$.

COROLLARY 2.3. *If $\zeta < s^*$, then*

$$\lim_{t\to\infty} -\frac{1}{t}\log \mathbb{P}(G > t) = \sigma,$$

where σ is given in (2.1).

Now, we present the following large deviations result for the busy period distribution without the assumption of Corollary 2.3. The proof is given in the next section.

THEOREM 2.4. *We have*

$$\lim_{t\to\infty} -\frac{1}{t}\log \mathbb{P}(G > t) = \sigma.$$

3. Proof of main result

In this section we prove our main result, Theorem 2.4. If $g_1(s^*) > \frac{1}{\lambda}$, then $\zeta < s^*$ and so Theorem 2.4 follows from Corollary 2.3. Now, suppose that $g_1(s^*) \leq \frac{1}{\lambda}$. To prove Theorem 2.4, we have to show the following:

$$\limsup_{t\to\infty} -\frac{1}{t}\log \mathbb{P}(G > t) \leq \sigma, \tag{3.1}$$

$$\liminf_{t\to\infty} -\frac{1}{t}\log \mathbb{P}(G > t) \geq \sigma, \tag{3.2}$$

whose proofs are carried out in the following two subsections.

3.1. Upper bound

To prove (3.1), we consider the M/G/1 queue with arrival rate λ and service times $B \wedge M \equiv \min(B, M)$ for $M > 0$. Let G^M be a generic random variable for the busy period in the M/G/1 queue with arrival rate λ and service times $B \wedge M$. Since $G \geq G^M$ stochastically, we have

$$\limsup_{t\to\infty} -\frac{1}{t}\log \mathbb{P}(G > t) \leq \lim_{t\to\infty} -\frac{1}{t}\log \mathbb{P}(G^M > t). \tag{3.3}$$

Let $g^M(s) = \mathbb{E}[e^{s(B\wedge M)}]$ and $g_1^M(s) = \mathbb{E}[(B\wedge M)e^{s(B\wedge M)}]$. Then $g_1^M(s) < \infty$ for all $s > 0$, $g_1^M(s)$ is increasing in s, and $\lim_{s\to\infty} g_1^M(s) = \infty$. Therefore, by Corollary 2.3,

$$\lim_{t\to\infty} -\frac{1}{t}\log \mathbb{P}(G^M > t) = \lambda + \zeta^M - \lambda g^M(\zeta^M), \tag{3.4}$$

where ζ^M is the unique positive solution of $\mathbb{E}[(B \wedge M)e^{\zeta^M(B\wedge M)}] = \frac{1}{\lambda}$. It can be easily seen that $\lim_{M\to\infty} \zeta^M = \zeta$. Now we will prove

$$\lim_{M\to\infty} g^M(\zeta^M) = \lim_{M\to\infty} \mathbb{E}[e^{\zeta^M(B\wedge M)}] = g(\zeta). \tag{3.5}$$

By Fatou's lemma,

$$\liminf_{M\to\infty} \mathbb{E}[e^{\zeta^M(B\wedge M)}] \geq \mathbb{E}[e^{\zeta B}] = g(\zeta). \tag{3.6}$$

To show $\limsup_{M\to\infty} \mathbb{E}[e^{\zeta^M(B\wedge M)}] \leq g(\zeta)$, we note that

$$\mathbb{E}[e^{\zeta^M(B\wedge M)}\mathbb{1}_{\{B\wedge M > k\}}] \leq \frac{1}{k}\mathbb{E}[(B \wedge M)e^{\zeta^M(B\wedge M)}] = \frac{1}{k\lambda}.$$

Therefore, for any $\epsilon > 0$, there exists k such that $\mathbb{E}[e^{\zeta^M(B\wedge M)}\mathbb{1}_{\{B\wedge M>k\}}] < \epsilon$ for all $M > 0$ and so

$$\begin{aligned}\limsup_{M\to\infty}\mathbb{E}[e^{\zeta^M(B\wedge M)}] &\leq \lim_{M\to\infty}\mathbb{E}[e^{\zeta^M(B\wedge M)}\mathbb{1}_{\{B\wedge M\leq k\}}] + \epsilon\\ &= \mathbb{E}[e^{\zeta B}\mathbb{1}_{\{B\leq k\}}] + \epsilon\\ &\leq g(\zeta) + \epsilon.\end{aligned}$$

Since ϵ is arbitrary,

$$\limsup_{M\to\infty}\mathbb{E}[e^{\zeta^M(B\wedge M)}] \leq g(\zeta),$$

from which and (3.6) we have (3.5). Therefore, letting $M \to \infty$ in (3.4) and using (3.3) yields (3.1).

3.2. Lower bound

To prove (3.2), for $\epsilon \in (0, (1-\rho)\frac{s^*}{\lambda})$, let $\tilde{B}^\epsilon$ be a nonnegative random variable with the complementary distribution function

$$\mathbb{P}(\tilde{B}^\epsilon > t) = \epsilon e^{-s^*t},\ t \geq 0,$$

and assume that it is independent of B. Consider the M/G/1 queue with arrival rate λ and service times $B + \tilde{B}^\epsilon$. Note that $\lambda\mathbb{E}[B + \tilde{B}^\epsilon] = \rho + \frac{\lambda\epsilon}{s^*} < 1$. Let $\tilde{G}^\epsilon$ be a generic random variable for the busy period in the M/G/1 queue with arrival rate λ and service times $B + \tilde{B}^\epsilon$. Since $G \leq \tilde{G}^\epsilon$ stochastically, we have

$$\liminf_{t\to\infty} -\frac{1}{t}\log\mathbb{P}(G > t) \geq \lim_{t\to\infty} -\frac{1}{t}\log\mathbb{P}(\tilde{G}^\epsilon > t). \tag{3.7}$$

Let $\tilde{g}^\epsilon(s) = \mathbb{E}[e^{s(B+\tilde{B}^\epsilon)}]$ and $\tilde{g}_1^\epsilon(s) = \mathbb{E}[(B + \tilde{B}^\epsilon)e^{s(B+\tilde{B}^\epsilon)}]$. Then $\tilde{g}_1^\epsilon(s) < \infty$ for $s \in (0, s^*)$, $\tilde{g}_1^\epsilon(s)$ is increasing in s on $(0, s^*)$ and $\lim_{s\to s^*-}\tilde{g}_1^\epsilon(s) = \infty$. Therefore, by Corollary 2.3,

$$\lim_{t\to\infty} -\frac{1}{t}\log\mathbb{P}(\tilde{G}^\epsilon > t) = \lambda + \tilde{\zeta}^\epsilon - \lambda\tilde{g}^\epsilon(\tilde{\zeta}^\epsilon), \tag{3.8}$$

where $\tilde{\zeta}^\epsilon$ is the unique positive solution of $\mathbb{E}[(B + \tilde{B}^\epsilon)e^{\tilde{\zeta}^\epsilon(B+\tilde{B}^\epsilon)}] = \frac{1}{\lambda}$. It can be easily seen that $\lim_{\epsilon\to 0+}\tilde{\zeta}^\epsilon = \zeta$. In what follows we will prove

$$\lim_{\epsilon\to 0+}\tilde{g}^\epsilon(\tilde{\zeta}^\epsilon) = g(\zeta). \tag{3.9}$$

Since $\mathbb{E}[(B+\tilde{B}^\epsilon)e^{\tilde{\zeta}^\epsilon(B+\tilde{B}^\epsilon)}] = \frac{1}{\lambda}$, i.e., $\mathbb{E}[Be^{\tilde{\zeta}^\epsilon B}e^{\tilde{\zeta}^\epsilon\tilde{B}^\epsilon}] + \mathbb{E}[\tilde{B}^\epsilon e^{\tilde{\zeta}^\epsilon B}e^{\tilde{\zeta}^\epsilon\tilde{B}^\epsilon}] = \frac{1}{\lambda}$, we have

$$\mathbb{E}[Be^{\tilde{\zeta}^\epsilon B}]\mathbb{E}[e^{\tilde{\zeta}^\epsilon\tilde{B}^\epsilon}] \leq \frac{1}{\lambda}.$$

We note that $\lim_{\epsilon\to 0+}\mathbb{E}[Be^{\tilde\zeta^\epsilon B}]=\frac{1}{\lambda}$. Therefore, $\limsup_{\epsilon\to 0+}\mathbb{E}[e^{\tilde\zeta^\epsilon\tilde B^\epsilon}]\le 1$. Moreover, $\mathbb{E}[e^{\tilde\zeta^\epsilon\tilde B^\epsilon}]\ge 1$ trivially. Thus $\lim_{\epsilon\to 0+}\mathbb{E}[e^{\tilde\zeta^\epsilon\tilde B^\epsilon}]=1$ and so

$$\lim_{\epsilon\to 0+}\tilde g^\epsilon(\tilde\zeta^\epsilon)=\lim_{\epsilon\to 0+}\mathbb{E}[e^{\tilde\zeta^\epsilon(B+\tilde B^\epsilon)}]=\lim_{\epsilon\to 0+}\mathbb{E}[e^{\tilde\zeta^\epsilon B}]\mathbb{E}[e^{\tilde\zeta^\epsilon\tilde B^\epsilon}]=g(\zeta),$$

from which (3.9) follows. Finally, letting $\epsilon\to 0+$ in (3.8) and using (3.7) yields (3.2).

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