JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **25**, No. 4, November 2012

MODULE-THEORETIC CHARACTERIZATIONS OF GENERALIZED GCD DOMAINS, III

Hwankoo Kim* and Seungkook Park**

Abstract. In terms of divisible-like modules, some equivalent conditions for an integral domain R to be a generalized GCD domain are given.

1. Introduction

Generalized GCD domains (for short, GGCD domains) were first introduced in [5], further investigated in [6], and since then, have played important roles in multiplicative ideal theory. Several ring- or ideal-theoretic characterizations of GGCD domains were given in the literature ([1, 2, 3, 4, 7]). The purpose of this note is to give another module-theoretic characterizations of GGCD domains, as a continuation of the study of module-theoretic characterizations of certain integral domains ([9, 10, 11, 12]).

We first introduce some definitions and notations. Let R be an integral domain with quotient field K. Let I be a nonzero fractional ideal of R. Then $I^{-1} := \{x \in K \mid xI \subseteq R\}$, $I_v := (I^{-1})^{-1}$, and $I_t := \bigcup\{J_v \mid J \subseteq I$ a nonzero finitely generated (f.g.) subideal of $I\}$. An ideal J of R is called a *GV-ideal*, denoted by $J \in GV(R)$, if J is a f.g. ideal of R with $J^{-1} = R$. A fractional ideal I of R is said to be *invertible* (resp., *t-invertible*) if $II^{-1} = R$ (resp., $(II^{-1})_t = R$).

For a torsion-free R-module M, Wang and McCasland defined the w-envelope of M as $M_w := \{x \in M \otimes_R K \mid Jx \subseteq M$ for some $J \in GV(R)\}$ ([17], cf., [9]). A torsion-free R-module is called a *w-module* (or

---

Received July 12, 2012; Accepted October 10, 2012.
2010 Mathematics Subject Classification: Primary 13A15; Secondary 13E99, 13F05.
Key words and phrases: generalized GCD domain, divisible module, injective module, pseudo-Dedekind domain.
Correspondence should be addressed to Hwankoo Kim, hkkim@hoseo.edu.

semidivisorial) if $M_w = M$. We say that a torsion-free R-module M is w*-finite* if $M = N_w$, for some f.g. submodule N of M.

Following [5], an integral domain R is called a *generalized GCD domain* (*GGCD domain*) if the intersection of two (integral) invertible ideals is invertible. It is well known that R is a GGCD domain if and only if I_v (equivalently I^{-1}) is invertible for every nonzero f.g. ideal I of R ([6, Theorem 1]). Recall that an integral domain R is called a *Prüfer* v*-multiplication domain* (for short, *P*v*MD*) if I_v (equivalently I^{-1}) is t-invertible for every nonzero f.g. ideal I of R. Thus the class of GGCD domains is contained in the class of PvMDs. It is also well known that in a PvMD, $t = w$. Therefore, R is a GGCD domain if and only if every w-finite w-ideal is invertible. For any undefined terminologies, we refer to [8] or [16].

2. Main results

Let $\mathscr{F}$ be a set of ideals of the integral domain R. An R-module M is said to be $\mathscr{F}$*-injective* if for every ideal $I \in \mathscr{F}$, every R-homomorphism from I into M can be extended to an R-homomorphism from R into M. Denote by $\mathscr{F}_w(R)$ (resp., $\mathscr{F}_{w,f}(R)$) the set of all w-ideals (resp., w-finite w-ideals) of R.

In [14], Matlis introduced the notion of h-divisible modules. Recall that an R-module is said to be h*-divisible* if it is a homomorphic image of an injective R-module. In [13], Lee defined the notion of weak-injective modules: An R-module M is called *weak-injective* if $\operatorname{Ext}_R^1(N, M) = 0$ for all R-modules N of weak dimension ≤ 1. In [15], Nikandish introduced the concept of hw-divisible modules. Recall that an R-module is hw*-divisible* if it is a homomorphic image of a weak-injective R-module. In [18, Theorem 4], it was shown that an integral domain R is Prüfer if and only if for any divisible R-module M and any f.g. ideal I of R, $\operatorname{Ext}_R^1(R/I, M) = 0$. We generalize this result to GGCD domains using the technique in the proof of [18, Theorem 4] in the following:

THEOREM 2.1. *Let R be an integral domain. Then the following are equivalent:*

(1) *R is a GGCD domain;*
(2) *every divisible R-module is $\mathscr{F}_{w,f}(R)$-injective;*
(3) *every h-divisible R-module is $\mathscr{F}_{w,f}(R)$-injective;*
(4) *every hw-divisible R-module is $\mathscr{F}_{w,f}(R)$-injective.*

Proof. (1) $\Rightarrow$ (2). If R is a GGCD domain, M a divisible R-module and I a w-finite type ideal of R, then I is invertible. Thus there exist $q_1, \dots, q_n \in K$, the quotient field of R, and $a_1, \dots, a_n \in I$ such that $\sum_{i=1}^n q_i a_i = 1$ and $q_i I \subseteq R$ for $i = 1, \dots, n$. For any $f \in \mathrm{Hom}(I, M)$, since M is a divisible R-module, there exist $x_i \in M$ $(i = 1, \dots, n)$ such that $a_i x_i = f(a_i)$. Thus for any $\beta \in I$, $\beta = \sum_{i=1}^n \beta q_i a_i$. Hence $f(\beta) = \sum_{i=1}^n \beta q_i f(a_i) = \beta \sum_{i=1}^n q_i a_i x_i$. Set $x = \sum_{i=1}^n q_i a_i x_i$. Define $g : R \to M$ to be $g(r) = rx$ for every $r \in R$. Then $g \in \mathrm{Hom}_R(R, M)$ and $g|_I = f$. Thus, $\mathrm{Ext}^1_R(R/I, M) = 0$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4). These are clear.

(4) $\Rightarrow$ (1). Let M be any R-module and E its injective hull. For a w-finite type ideal I of R, the exact sequence $0 \to M \to E \to C \to 0$ induces the exact sequence $0 = \mathrm{Ext}^1_R(R/I, C) \to \mathrm{Ext}^2_R(R/I, M) \to \mathrm{Ext}^2_R(R/I, E) = 0$, where the first Ext term vanishes by assumption. Hence $\mathrm{Ext}^2_R(R/I, M) = 0$. Since M was arbitrary, we conclude $pd(R/I) \leq 1$. Hence $pd(I) \leq 1$, that is, I is projective. Therefore R is a GGCD domain. □

Recall that an integral domain R is a *pseudo-Dedekind domain* if every v-ideal of R is invertible, equivalently every w-ideal of R is invertible. With a similar proof as in that of Theorem 2.1, we have the following:

Theorem 2.2. *Let R be an integral domain. Then the following are equivalent:*

(1) *R is a pseudo-Dedekind domain;*
(2) *every divisible R-module is $\mathscr{F}_w(R)$-injective;*
(3) *every h-divisible R-module is $\mathscr{F}_w(R)$-injective;*
(4) *every hw-divisible R-module is $\mathscr{F}_w(R)$-injective.*

Finally we provide an example of an $\mathscr{F}_w(R)$-injective (and hence $\mathscr{F}_{w,f}(R)$-injective) module but not an injective module in the following:

Example 2.3. Let $R := k[x, y]$ be the polynomial ring in two variables over a field k and let $Q := k(x, y)$ be its field of quotients. We consider the module $M := Q/R$. It is easy to see that M is a divisible R-module. Note that R is a factorial domain, and hence a pseudo-Dedekind domain. Thus by Theorem 2.2, M is $\mathscr{F}_w(R)$-injective. Now we show that M is not an injective R-module as follows: Consider the ideal $I = \{f(x, y) \in R \mid f(0, 0) = 0\}$ and the homomorphism $\varphi : I \to M$ defined by $\varphi(f(x, y)) = \overline{f(x, 0)/xy}$. Since there is no extension of φ to a homomorphism $\bar{\varphi} : R \to M$, M is not an injective R-module.

Acknowledgements

We would like to thank the referee for very helpful comments. This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2010-0011996).

References

[1] M. M. Ali, *Some remarks on generalized GCD domains*, Comm. Algebra **36** (2008), 142-164.
[2] M. M. Ali, *Generalized GCD rings III*, Beitrage zur Algebra und Geom. **51** (2010), 531.546.
[3] M. M. Ali and D. J. Smith, *Generalized GCD rings*, Beitrage zur Algebra und Geom. **42** (2001), 219-233.
[4] M. M. Ali and D. J. Smith, *Generalized GCD rings II*, Beitrage zur Algebra und Geom. **44** (2003), 75-98.
[5] D. D. Anderson, *π-domains, divisorial ideals, overrings*, Glasgow Math. J. **19** (1978), 199-203.
[6] D. D. Anderson and D. F. Anderson, *Generalized GCD-domains*, Comment. Math. Univ. St. Pauli, **23** (1979), 213-221.
[7] S. El Baghdadi, *Semistar GCD domains*, Comm. Algebra 38 (2010), 3029-3044.
[8] R. Gilmer, *Multiplicative Ideal Theory*, Queen's Papers in Pure and Applied Mathematics, vol 90, Queen's University, Kingston, Ontario, 1992.
[9] H. Kim, *Module-theoretic characterizations of t-linkative domains*, Comm. Algebra **36** (2008), 1649-1670.
[10] H. Kim, *Module-theoretic characterizations of generalized GCD domains*, Comm. Algebra **38** (2010), 759-772.
[11] H. Kim, *Module-theoretic characterizations of generalized GCD domains, II*, J. Chungcheong Math. Soc. **24** (2011), 131-135.
[12] H. Kim, E. S. Kim and Y. S. Park, *Injective modules over strong Mori domains*, Houston J. Math. **34** (2008), 349-360.
[13] S. B. Lee, *Weak-injective modules*, Comm. Algebra **34** (2006), 361-370.
[14] E. Matlis, *Cotorsion modules*, Mem. Amer. Math. Soc. **49** (1964).
[15] R. Nikandish, *A generalization of h-divisible modules*, Tarbiat Moallem University, 20th Seminar on Algebra, 2-3 Ordibehesht, 1388 (April 22-23, 2009) pp. 159-161.
[16] J. J. Rotman, *An Introduction to Homological Algebra*, 2nd ed., Springer, New York, 2009.
[17] F. Wang and R. L. McCasland, *On w-modules over strong Mori domains*, Comm. Algebra **25** (1997), 1285-1306.
[18] J. H. Zeng, *A characterization of Prüfer rings (In Chinese)*, J. Math. Res. Exposition **18** (1998), 145-146.

*
Department of Information Security
Hoseo University
Asan 336-795, Republic of Korea
E-mail: hkkim@hoseo.edu

**
Department of Mathematics
Sookmyung Women's University
Seoul 140-742, Republic of Korea
E-mail: skpark@sookmyung.ac.kr