JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **25**, No. 4, November 2012

THE ITÔ INTEGRAL WITH RESPECT TO ANALOGUE OF WIENER PROCESS

Kun Sik Ryu*

Abstract. In this note, we define the Itô integral with respect to analogue of Wiener process and investigate its various properties and examples.

1. Introduction

The stochastic integral was first defined by Itô [3]. It was based on the standard Brownian motion or the Wiener process. Later, it was extended to arbitrary local martingales and semimartingales by the work of Doob, Motoo, Watanabe and Meyer, among others. Many physics, genetics and economics models take the form of stochastic differential equations associated with the Itô integral. In 2002, the author and Dr. Im presented the definition of analogue of Wiener space, a kind of the generalization of the concrete Wiener space, and its properties [6].
In this note, we will give the definition of the Itô integral with respect to analogue of Wiener process, associated with probability measure ϕ and will search various properties of it.

2. Analogue of Wiener process and martingales of analogue of Wiener process

In this section, we introduce the definition of the analogue of Wiener process and investigate the martingale properties of it.

Received July 03, 2012; Accepted October 10, 2012.
2010 Mathematics Subject Classification: Primary 28C20, 65G20; Secondary 28C35.
Key words and phrases: analogue of Wiener measure space, Itô integral, Itô formula, martingale.
This paper was supported by Hannam University in 2012.

Throughout this note, let T be a positive real number, let $C[0,T]$ be the space of all continuous functions on a closed interval $[0,T]$ with the supremum norm $\|x\|_\infty = \sup_{t\in[0,T]}|x(t)|$, let ϕ be a probability Borel measure on $\mathbb{R}$ and let m_ϕ be the analogue of Wiener measure on $C[0,T]$ [6].

Remark 2.1. When ϕ is a Dirac measure δ_0 at the origin in $\mathbb{R}$, m_ϕ is the concrete Wiener measure, this measure is denoted by m_w.

Definition 2.2. The analogue of Wiener process AW_ϕ with respect to a probability Borel measure ϕ is the stochastic process on $(C[0,T], \mathcal{B}(C[0,T]), m_\phi)$ and $[0,T]$ defined by $AW_\phi(t,x) = x(t)$ for $(t,x) \in [0,T] \times C[0,T]$.

Remark 2.3. (1) When ϕ is a Dirac measure δ_0 at the origin in $\mathbb{R}$, $AW_{\delta_0} = W$ is a Wiener process. When ϕ is a Dirac measure δ_p at the point p in $\mathbb{R}$, AW_{δ_p} is a Brownian motion started at p.
(2)(Normal increments) For $0 \le s < t \le T$, $AW_\phi(t) - AW_\phi(s)$ has a normal distribution with mean 0 and variance $t - s$.
(3)(Independence of increments) For $0 \le u < v \le s < t \le T$, $AW_\phi(t) - AW_\phi(s)$ and $AW_\phi(v) - AW_\phi(u)$ are independent. In Wiener process case, for $0 \le s < t \le T$, $W(t) - W(s)$ is independent of the past, that is, $W(u)$, $0 \le u \le s$, but in the analogue of Wiener process, we can not say that for $0 \le u \le s < t \le T$, $AW_\phi(t) - AW_\phi(s)$ and $AW_\phi(u)$ are independent.
(4) From the definition of analogue of Wiener measure, $E((AW_\phi(t) - AW_\phi(s))^2) = t-s$, $E((AW_\phi(t)-AW_\phi(s))^3) = 0$, $E((AW_\phi(t)-AW_\phi(s))^4) = 3(t-s)^2$ for $0 \le s \le t \le T$.
(5) (Continuity of paths) $AW_\phi(t)$, $0 \le t \le T$ are continuous functions of t.
(6) The covariance function $Cov(AW_\phi(t), AW_\phi(s))$ of $AW_\phi(t)$ is $\min\{s,t\} + \int_{\mathbb{R}} u^2 d\phi(u) - [\int_{\mathbb{R}} u d\phi(u)]^2$ if $\int_{\mathbb{R}} u^2 d\phi(u)$ is finite.

Theorem 2.4. *The quadratic variation* $[AW_\phi,\ AW_\phi](t)$ *of analogue of Wiener process is* t m_ϕ*-a.s. for* $0 \le t \le T$.

Proof. It suffices to show that $[AW_\phi, AW_\phi](t) = \lim\limits_{n\to\infty} \sum_{i=1}^{2^n}(AW_\phi(t_i^n) - AW_\phi(t_{i-1}^n))^2 = t$ m_ϕ-a.s. where for each n, $t_i^n = \frac{it}{2^n}$. Let $T_n = \sum_{i=1}^{2^n}(AW_\phi(t_i^n) - AW_\phi(t_{i-1}^n))^2$ for each n. By Remark 2.3 (4) in above,

$E(T_n) = \sum_{i=1}^{2^n}(t_i^n - t_{i-1}^n) = t$ and

$$\begin{aligned}
&E(T_n^2)\\
&= E(\sum_{i=1}^{2^n}(AW_\phi(t_i^n) - AW_\phi(t_{i-1}^n))^4)\\
&\quad +2E(\sum_{i,j=1(i\neq j)}^{2^n}(AW_\phi(t_i^n) - AW_\phi(t_{i-1}^n))^2(AW_\phi(t_j^n) - AW_\phi(t_{j-1}^n))^2)\\
&= 3\sum_{i=1}^{2^n}(t_i^n - t_{i-1}^n)^2 + \sum_{i,j=1(i\neq j)}^{2^n}(t_i^n - t_{i-1}^n)(t_j^n - t_{j-1}^n)\\
&= 2\sum_{i=1}^{2^n}(t_i^n - t_{i-1}^n)^2 + \sum_{i=1}^{2^n}\sum_{j=1}^{2^n}(t_i^n - t_{i-1}^n)(t_j^n - t_{j-1}^n)\\
&= \frac{t^2}{2^{n-1}} + t^2,
\end{aligned}$$

so $Var(T_n) = \frac{t^2}{2^{n-1}}$. Hence $\sum_{n=1}^\infty Var(T_n) = 2t^2 < \infty$. Using the monotonic convergence theorem, we have $E(\sum_{n=1}^\infty(T_n - E(T_n))^2) < \infty$. Hence, $T_n - t \to 0$ m_ϕ-*a.s.* as $n \to \infty$, as desired. □

THEOREM 2.5. *If $\int_{\mathbb{R}} |u| d\phi(u)$ is finite then $AW_\phi(t)$ is a martingale, if $\int_{\mathbb{R}} u^2 d\phi\ (u)$ is finite then $AW_\phi(t)^2 - t$ is a martingale and if $\int_{\mathbb{R}} e^{ru} d\phi(u)$ is finite then $e^{rAW_\phi(t) - \frac{r^2t}{2}}$ is a martingale.*

Proof. Let $\mathcal{F}_t$ be the information about the process AW_ϕ up to time t, that is, $\mathcal{F}_t$ is the smallest σ-algebra that contains sets of the form $\{x | \alpha \leq AW_\phi(s,x) \leq \beta\}$ for $0 \leq s \leq t$ and $\alpha, \beta \in \mathbb{R}$. By the definition, $E(|AW_\phi(t)|) \leq 2\sqrt{2\pi} + \int_{\mathbb{R}} |u| d\phi(u) < \infty$. We claim that $E(AW_\phi(p + t) - AW_\phi(t) | \mathcal{F}_t) = 0$ for any $p \geq 0$. For $\vec{t} = (t_0, t_1, t_2, \cdots, t_n)$ with $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$, let $J_{\vec{t}} : C[0,T] \longrightarrow \mathbb{R}^{n+1}$ be a function with $J_{\vec{t}}(x) = (x(t_0), x(t_1), x(t_2), \cdots, x(t_n))$ and let $B_j (j = 0, 1, 2, \cdots, n)$ be in $\mathcal{B}(\mathbb{R})$. Then letting $\Gamma = J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)$ and let $t_n = t + s$,

$$\begin{aligned}
&\int_\Gamma (AW_\phi(s+t,x) - AW_\phi(t,x)) dm_\phi(x)\\
&= \int_{B_0}\int_{B_1}\cdots\int_{B_n}\int_{\mathbb{R}}(\prod_{j=1}^n \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}})\frac{u_{n+1} - u_n}{\sqrt{2\pi s}}
\end{aligned}$$

$$
\begin{aligned}
&\cdot \exp\{-\frac{1}{2}\sum_{j=1}^{n+1}\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\}du_{n+1}du_n\cdots du_1 d\phi(u_0)\\
&= 0\\
&= \int_\Gamma 0 dm_\phi(x),
\end{aligned}
$$

so putting $\mathcal{M} = \{\Gamma | \Gamma$ is in $\mathcal{F}_t$ and $\int_\Gamma (AW_\phi(s+t,x) - AW_\phi(t,x))dm_\phi(x) = 0 = \int_\Gamma 0 dm_\phi(x)\}$, $\mathcal{M}$ is a σ-algebra contains all subsets of the form $J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)$. Hence, $E(AW_\phi(s+t) - AW_\phi(t)|\mathcal{F}_t) = 0$ for any $s \geq 0$. Therefore, we have $E(AW_\phi(s+t)|\mathcal{F}_t) = E(AW_\phi(t)|\mathcal{F}_t) + E(AW_\phi(s+t) - AW_\phi(t)|\mathcal{F}_t) = AW_\phi(t)$. Thus, we obtain $AW_\phi(t)$ is a martingale. Since $E(|AW_\phi(t)^2 - t|) \leq E(AW_\phi(t)^2) + t = 2t + \int_{\mathbb{R}} u^2 d\phi(u) < \infty$ because $\int_{\mathbb{R}} u^2 d\phi(u)$ is finite, it remains to show that $E(AW_\phi(s+t)^2 - (s+t)|\mathcal{F}_t) = AW_\phi(t)^2 - t$. By the essentially same method as in the proof of a martingale of $AW_\phi(t)$, we have $E((AW_\phi(s+t) - AW_\phi(t))^2|\mathcal{F}_t) = s$. So

$$
\begin{aligned}
&E(AW_\phi(s+t)^2|\mathcal{F}_t)\\
&= E(AW_\phi(t)^2|\mathcal{F}_t) + 2E(AW_\phi(t)(AW_\phi(s+t) - AW_\phi(t))|\mathcal{F}_t)\\
&\quad + E((AW_\phi(s+t) - AW_\phi(t))^2|\mathcal{F}_t)\\
&= AW_\phi(t)^2 + 2AW_\phi(t)E(AW_\phi(s+t) - AW_\phi(t)|\mathcal{F}_t) + s\\
&= AW_\phi(t)^2 + s.
\end{aligned}
$$

By subtracting $t+s$ from both sides in above, we obtain $E(AW_\phi(s+t)^2 - t|\mathcal{F}_t) = AW_\phi(t)^2 - t$. Lastly, we prove that $e^{rAW_\phi(t) - \frac{r^2 t}{2}}$ is a martingale. From the elementary calculus, $E(e^{rAW_\phi(t) - \frac{r^2 t}{2}}) = \int_{\mathbb{R}} e^{ru} d\phi(u) < \infty$ and for $\Gamma = J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)$,

$$
\begin{aligned}
&\int_\Gamma e^{r(AW_\phi(s+t) - AW_\phi(t))} dm_\phi(t)\\
&= \int_{B_0}\int_{B_1}\cdots\int_{B_n}\int_{\mathbb{R}} (\prod_{j=1}^n \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}}) \frac{1}{\sqrt{2\pi s}} e^{r(u_{n+1} - u_n)}\\
&\quad \cdot \exp\{-\frac{1}{2}\sum_{j=1}^{n+1}\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\}du_{n+1}du_n\cdots du_1 d\phi(u_0)
\end{aligned}
$$

$$= \int_{B_0}\int_{B_1}\cdots\int_{B_n}(\prod_{j=1}^{n}\frac{1}{\sqrt{2\pi(t_j-t_{j-1})}})e^{\frac{sr^2}{2}}$$

$$\cdot \exp\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\}du_n\cdots du_1 d\phi(u_0)$$

$$= \int_{\Gamma} e^{\frac{sr^2}{2}} dm_{\phi}(x),$$

we have $E(e^{r(AW_{\phi}(t+s)-AW_{\phi}(t))}|\mathcal{F}_t) = e^{\frac{sr^2}{2}}$. Hence

$$E(e^{r(AW_{\phi}(t+s)-\frac{(t+s)r^2}{2})}|\mathcal{F}_t)$$

$$= E(e^{r\ AW_{\phi}(t+s)}|\mathcal{F}_t)e^{-\frac{(t+s)r^2}{2}}$$

$$= e^{r\ AW_{\phi}(t)}E(e^{r(AW_{\phi}(t+s)-AW_{\phi}(t))}|\mathcal{F}_t)e^{-\frac{(t+s)r^2}{2}}$$

$$= e^{r\ AW_{\phi}(t)-\frac{tr^2}{2}},$$

as desired. □

THEOREM 2.6. *The analogue of Wiener process AW_{ϕ} with respect to a probability Borel measure ϕ possesses Markov property if $\int_{\mathbb{R}} e^{Tu}d\phi(u)$ is finite.*

Proof. Let $\mathcal{F}_t$ be the information about the process AW_{ϕ} up to time t. For $\Gamma = J_{\vec{t}}^{-1}(\prod_{j=0}^{n} B_j)$,

$$\int_{\Gamma} e^{u(AW_{\phi}(s+t)-AW_{\phi}(t))}dm_{\phi}(t)$$

$$= \int_{B_0}\int_{B_1}\cdots\int_{B_n}\int_{\mathbb{R}}(\prod_{j=1}^{n}\frac{1}{\sqrt{2\pi(t_j-t_{j-1})}})\frac{1}{\sqrt{2\pi s}}e^{u(u_{n+1}-u_n)}$$

$$\cdot \exp\{-\frac{1}{2}\sum_{j=1}^{n+1}\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\}du_{n+1}du_n\cdots du_1 d\phi(u_0)$$

$$= \int_{B_0}\int_{B_1}\cdots\int_{B_n}(\prod_{j=1}^{n}\frac{1}{\sqrt{2\pi(t_j-t_{j-1})}})e^{\frac{su^2}{2}}$$

$$\cdot \exp\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\}du_n\cdots du_1 d\phi(u_0)$$

$$= \int_{\Gamma} e^{\frac{su^2}{2}} dm_{\phi}(x),$$

we have $E(e^{u(AW_\phi(t+s)-AW_\phi(t))}|\mathcal{F}_t) = e^{\frac{su^2}{2}}$. Since $AW_\phi(t+s) - AW_\phi(t)$ has a normal distribution with mean 0 and variance s, that is, the moment generating function of $AW_\phi(t+s) - AW_\phi(t)$ given $AW_\phi(t)$ is $e^{\frac{su^2}{2}}$. So

$$\begin{aligned}
&E(e^{uAW_\phi(t+s)}|\mathcal{F}_t)\\
&= e^{uAW_\phi(t)}E(e^{u(AW_\phi(t+s)-AW_\phi(t))}|\mathcal{F}_t)\\
&= e^{u\ AW_\phi(t)}E(e^{u(AW_\phi(t+s)-AW_\phi(t))}|AW_\phi(t))\\
&= E(e^{uAW_\phi(t+s)}|AW_\phi(t)),
\end{aligned}$$

which is what had to show. □

3. The Itô Integral on the Analogue of Wiener Space

In this section, we present the definition of the Itô integral with respect to analogue of Wiener process and establish the existence theorem for this integral.

Firstly, we consider the Itô integral with deterministic simple process $X(t)$, which is a function of t and does not depend on $AW_\phi(t)$.

Definition 3.1. We say that a stochastic process $X(t)$ is a deterministic simple process provided that there exists a partition $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$ and constants $c_0, c_1, c_2, \cdots, c_{n-1}$ such that $X(t) = c_0\chi_{[t_0,t_1]}(t) + \sum_{i=1}^{n-1} c_i\chi_{(t_i,t_{i+1}]}(t)$ where χ_A is the characteristic function associated with A. The Itô integral with respect to analogue of Wiener process $\int_0^T X(t)dAW_\phi(t)$ is defined as a sum $\int_0^T X(t)dAW_\phi(t) = \sum_{i=0}^{n-1} c_i(AW_\phi(t_{i+1}) - AW_\phi(t_i))$.

Example 3.2. 1) In the definition in above, $E(\int_0^T X(t)dAW_\phi(t)) = 0$ and $Var(\int_0^T X(t)dAW_\phi(t)) = \sum_{i=0}^{n-1} c_i^2(t_{i+1} - t_i)$.
2) Let ϕ have the standard normal distribution. Then $AW_\phi(t)$ has the normal distribution with mean 0 and variance $1 + t$. Let $X(t) = \begin{cases} -2 & , 0 \le t \le 1 \\ 0 & , 1 < t \le 2 \\ 1 & , 2 < t \le 3 \end{cases}$. Then $\int_0^T X(t)dAW_\phi(t)$ has the normal distribution with mean 0 and variance 5.

Definition 3.3. We say that a stochastic process $X(t)$ is a simple process provided that there exist a partition $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$ and random variables $\xi_0, \xi_1, \xi_2, \cdots, \xi_{n-1}$ such that $X(t) =$

$\xi_0\chi_{[t_0,t_1]}(t) + \sum_{i=1}^{n-1}\xi_i\chi_{(t_i,t_{i+1}]}(t)$ where for all i $(0 \le i \le n-1)$, $E(\xi_i^2)$ is finite. The Itô integral with respect to analogue of Wiener process $\int_0^T X(t)dAW_\phi(t)$ is defined as a sum $\int_0^T X(t)dAW_\phi(t) = \sum_{i=0}^{n-1}\xi_i(AW_\phi(t_{i+1}) - AW_\phi(t_i))$.

Using essentially the same method as in the theory in Brownian motion calculus and the proof of theorems in above, we obtain the following theorem.

THEOREM 3.4. *Under the definition in above,* $E(\int_0^T X(t)dAW_\phi(t)) = 0$ *and* $E([\int_0^T X(t)dAW_\phi(t)]^2) = \int_0^T E(X(t)^2)dt$.

DEFINITION 3.5. Let $\langle X_n\rangle$ be a sequence of simple processes convergent in the probability measure m_ϕ to the process $X(t)$. If the sequence $\langle\int_0^T X_n(t)dAW_\phi(t)\ \rangle$ converges in the probability measure m_ϕ then we say that the Itô integral with respect to analogue of Wiener process associated with ϕ exists and that limit is taken to be the integral $\int_0^T X(t)dAW_\phi(t)$.

EXAMPLE 3.6. We want to calculate $\int_0^T AW_\phi(t)dAW_\phi(t)$. For $0 = t_0^n < t_1^n < t_2^n < \cdots < t_n^n = T$, let $X_n(t) = AW_\phi(0)\chi_{[t_0^n,t_1^n]}(t) + \sum_{i=1}^{n-1} AW_\phi(t)\chi_{(t_i^n,t_{i+1}^n]}(t)$. Then for n, $X_n(t)$ is a simple predictable process. By the continuity of analogue of Wiener paths, for fixed t, $\lim_{n\to\infty} X_n(t) = AW_\phi(t)$ m_ϕ-*a.s.* as $max\{t_i^n - t_{i-1}^n | 1 \le i \le n\} \to 0$. Then the Itô integral of $X_n(t)$ with respect to $AW_\phi(t)$ is given by

$$\begin{aligned}
&\int_0^T X_n(t)dAW_\phi(t)\\
&= \sum_{i=0}^{n-1} AW_\phi(t_i^n)(AW_\phi(t_{i+1}^n) - AW_\phi(t_i^n))\\
&= \frac{1}{2}(AW_\phi(T)^2 - AW_\phi(0)^2) + \frac{1}{2}\sum_{i=0}^{n-1}(AW_\phi(t_{i+1}^n) - AW_\phi(t_i^n))^2.
\end{aligned}$$

From Theorem 2.4, we obtain

$$\begin{aligned}
&\int_0^T AW_\phi(t)dAW_\phi(t)\\
&= \lim_{n\to\infty}\int_0^T X_n(t)dAW_\phi(t)\\
&= \frac{1}{2}(AW_\phi(T)^2 - AW_\phi(0)^2 - T).
\end{aligned}$$

Here, if we take $\phi = \delta_p$ then we have $\int_0^T AW_\phi(t)dAW_\phi(t) = \frac{1}{2}(AW_\phi(T)^2 - p^2 - T)$.

DEFINITION 3.7. Let $L^2_\phi[0,T]$ be the space of all stochastic processes $X(t,x)$ on $[0,T]\times C[0,T]$ with $\int_{C[0,T]}\{\int_0^T X(t,x)^2 dt\}dm_\phi(x) < \infty$.

From the essentially same method as in the theory in Brownian motion calculus, we obtain the following existence theorem for Itô integral with respect to analogue of Wiener process.

THEOREM 3.8. *(Existence theorem for Itô integral) If X is predictable in $L^2_\phi[0,T]$ then the Itô integral $\int_0^T X(t)dAW_\phi(t)$ with respect to analogue of Wiener process associated with ϕ, exists.*

The following theorem results from Theorem 2.4.

THEOREM 3.9. *Suppose X is predictable in $L^2_\phi[0,T]$. Then $E(\int_0^T X(t) dAW_\phi(t)) = 0$ (Zero mean property) and*

$$E([\int_0^T X(t)dAW_\phi(t)]^2) = \int_0^T E(X(t)^2)dt$$

(Isometry property).

EXAMPLE 3.10. In Example 3.6, AW_ϕ is predictable and if $\int_{\mathbb{R}} u^2 d\phi(u)$ is finite then AW_ϕ is in $L^2_\phi[0,T]$. And $E(\int_0^T AW_\phi(t)dAW_\phi(t)) = \frac{1}{2}E(AW_\phi(T)^2 - AW_\phi(0)^2 - T) = \frac{1}{2}[E(AW_\phi(T)^2) - E(AW_\phi(0)^2) - E(T)] = \frac{1}{2}[(\int_{\mathbb{R}} u^2 d\phi(u) + T) - \int_{\mathbb{R}} u^2 d\phi(u) - T] = 0$ and $E([\int_0^T AW_\phi(t)dAW_\phi(t)]^2) = \int_0^T E(AW_\phi(t)^2)dt = \frac{1}{2}T^2 + T\int_{\mathbb{R}} u^2 d\phi(u)$ if $\int_{\mathbb{R}} u^2 d\phi(u)$ is finite. Remark that from the definition of m_ϕ, $E([\int_0^T AW_\phi(t)dAW_\phi(t)]^2) = \frac{1}{4}[(\int_{\mathbb{R}} u^4 d\phi(u) + 6T\int_{\mathbb{R}} u^2 d\phi(u) + 3T^2) + \int_{\mathbb{R}} u^4 d\phi(u) + T^2 - 2T\int_{\mathbb{R}} u^2 d\phi(u) - 2T^2 - 2T\int_{\mathbb{R}} u^2 d\phi(u) - 2\int_{\mathbb{R}} u^4 d\phi(u) + 2T\int_{\mathbb{R}} u^2 d\phi(u)] = \frac{1}{2}T^2 + T\int_{\mathbb{R}} u^2 d\phi(u)$ if $\int_{\mathbb{R}} u^4 d\phi(u)$ is finite.

THEOREM 3.11. *Let X be predictable in $L^2_\phi[0,T]$ and for $0 \le t \le T$, let* $Y(t) = \begin{cases} 0 & , \ t = 0 \\ \lim_{s\downarrow t}\int_0^s X(u)dAW_\phi(u) & , \ 0 < t < T \\ \int_0^T X(u)dAW_\phi(u) & , \ t = T \end{cases}$. *Then $Y(t)$ is a continuous zero mean square integrable martingale and $[Y,Y](t) = \int_0^t X(s)^2 ds$. Here, $Y(t)$ is called the Itô integral process with respect to analogue of Wiener space, associated with ϕ.*

Proof. Since X is in $L^2_\phi[0,T]$, from Hölder's inequality, Fubini's theorem and the isometry property, we have $E(|Y(t)|) \leq \sqrt{E(Y(t)^2)} = \sqrt{\int_0^t E(X(u)^2)du} = \sqrt{\int_{C[0,T]}(\int_0^t X(s,x)^2 ds)dm_\phi(x)} < \infty$. Let $\mathcal{F}_t$ be the information about the process AW_ϕ up to time t. By the zero mean property, $E(\int_s^t X(u)dAW_\phi(u)|\mathcal{F}_s) = 0$ for $s < t$, so for $s < t$, $E(Y(t)|\mathcal{F}_s)$ $= E(\int_0^s X(u)dAW_\phi(u) + \int_s^t X(u)dAW_\phi(u)|\mathcal{F}_s) = \int_0^s X(u)dAW_\phi(u) = Y(s)$ which implies that $Y(t)$ is a continuous zero mean square integrable martingale. Moreover, $[Y,Y](t) = E(Y(t)^2) - [E(Y(t))]^2 = \int_0^t X(s)^2 ds$ from the zero mean property and the isometry property. □

4. Itô formula for the Analogue of Wiener Process

In this section, we will drive the Itô's formula for the analogue of Wiener space.

From essentially the same method as in the theory in Brownian motion calculus and Remark 2.3, we obtain the following lemma.

LEMMA 4.1. *Suppose g is bounded continuous and $\langle\{t_i^n\}_{i=0}^n\rangle$ is a sequence of partitions of $[0,t]$ for $0 < t$ with $\lim\limits_{n\to\infty} \max\{t_i^n - t_{i-1}^n | i = 1,2,\cdots,n\} = 0$. Then for any θ_i^n in between $AW_\phi(t_{i-1}^n)$ and $AW_\phi(t_i^n)$, $\lim\limits_{n\to\infty} \sum_{i=0}^{n-1} g(\theta_i^n)[AW_\phi(t_{i+1}^n) - AW_\phi(t_i^n)] = \int_0^t g(AW_\phi(s))ds$ in probability measure m_ϕ.*

THEOREM 4.2. *(Itô's formula for the Analogue of Wiener Space) If f is twice continuously differentiable, then for $0 < t \leq T$, $f(AW_\phi(t))$ $= f(AW_\phi(0)) + \int_0^t f'(AW_\phi(s))dAW_\phi(s) + \frac{1}{2}\int_0^t f''(AW_\phi(s))ds$.*

Proof. Let $\{t_i^n\}_{i=0}^n$ is a partition of $[0,t]$ with $\lim\limits_{n\to\infty} \max\{t_i^n - t_{i-1}^n | i = 1,2,\cdots,n\} = 0$. Then $f(AW_\phi(t)) = f(AW_\phi(0)) + \sum_{i=0}^{n-1}[f(AW_\phi(t_{i+1}^n)) - f(AW_\phi(t_i^n))]$ and by Taylor's formula, for n and i there is θ_i^n in between $AW_\phi(t_{i-1}^n)$ and $AW_\phi(t_i^n)$ such that $f(AW_\phi(t_{i+1}^n)) - f(AW_\phi(t_i^n)) = f'(AW_\phi(t_i^n))[f(AW_\phi(t_{i+1}^n)) - f(AW_\phi(t_i^n))] + \frac{1}{2}f''(\theta_i^n)[AW_\phi(t_{i+1}^n) - AW_\phi(t_i^n)]^2$. So $f(AW_\phi(t)) = f(AW_\phi(0)) + \sum_{i=0}^{n-1} f'(AW_\phi(t_i^n))[f(AW_\phi(t_{i+1}^n)) - f(AW_\phi(t_i^n))] + \frac{1}{2}\sum_{i=0}^{n-1} f''(\theta_i^n)[AW_\phi(t_{i+1}^n) - AW_\phi(t_i^n)]^2$. Taking $n \to \infty$, by Lemma 4.1, we have $f(AW_\phi(t)) = f(AW_\phi(0)) + \int_0^t f'(AW_\phi(s))dAW_\phi(s) + \frac{1}{2}\int_0^t f''(AW_\phi(s))ds$. □

DEFINITION 4.3. Stochastic process $Y(t)$ is called the Itô process with respect to analogue of Wiener space, associated with ϕ provided that

for any $0 \le t \le T$, it can be represented as $Y(t) = Y(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dAW_\phi(s)$, where the stochastic process μ and σ satisfy conditions:
1. $\mu(t)$ is adopted and $\int_0^T |\mu(t)|dt < \infty \quad m_\phi\text{-}a.s.$
2. $\sigma(t)$ is predictable and $\int_0^T |\sigma(t)|dt < \infty \quad m_\phi\text{-}a.s.$

EXAMPLE 4.4. 1) By taking $T = t$ in Example 3.6, we obtain $\int_0^t AW_\phi(s)\, dAW_\phi(s) = \frac{1}{2}(AW_\phi(t)^2 - AW_\phi(0)^2 - t)$ so $AW_\phi(t)^2 = AW_\phi(0)^2 + \int_0^t ds + 2\int_0^t AW_\phi(s)\ dAW_\phi(s)$. Hence, putting $Y(t) = AW_\phi(t)^2$, $\mu(s) = 1$ and $\sigma(s) = 2AW_\phi(s)$, $Y(t) = Y(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dAW_\phi(s)$.
2). If $\int_{\mathbb{R}} e^u d\phi(u)$ is finite then $\int_{C[0,T]}(\int_0^T e^{AW_\phi(t)}dt)dm_\phi(x) = 2(e^{\frac{T}{2}} - 1)\int_{\mathbb{R}} e^u\ d\phi(u) < \infty$, so $\int_0^T e^{AW_\phi(t)}dt$ is finite m_ϕ-$a.s.$ Letting $Y(t) = e^{AW_\phi(t)}$, by Itô formula for the analogue of Wiener space, we have $e^{AW_\phi(t)} = e^{AW_\phi(0)} + \frac{1}{2}\int_0^t e^{AW_\phi(s)}ds + \int_0^t e^{AW_\phi(s)}dAW_\phi(s)$.

REMARK 4.5. Along the parallel method as in the proof of the existence and uniqueness theorem for solutions of $Y(t) = Y(0) + \int_0^t \mu(s, Y(s))\, ds + \int_0^t \sigma(s, Y(s))\ dAW_\phi(s)$ in [2] we can prove easily the following fact : Suppose that
1. The functions $\mu(t, u)$ and $\sigma(t, u)$ are measurable for t in $[0, T]$ and u in $\mathbb{R}$,
2. There exists a constant K such that for all t in $[0, T]$ and u, v in $\mathbb{R}$, $|\mu(t, u) - \mu(t, v)| + |\sigma(t, u) - \sigma(t, v)| \le K|u - v|$ and $\mu(t, u)^2 + \sigma(t, u)^2 \le K^2(1 - u^2)$, and
3. $Y(0)$ is independent of $AW_\phi(t)$, for $t > 0$ and $E(Y(0)^2) < \infty$. Then there is a solution $Y(t)$ of $Y(t) = Y(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dAW_\phi(s)$, defined on $[0, T]$ which is continuous with probability 1, and such that $\sup_{t\in[0,T]} E(Y(t)^2) < \infty$. Furthermore, a solution with these properties is pathwise unique, that is, if X and Y are two such solutions, $P(\sup_{t\in[0,T]} |X(t) - Y(t)| = 0) = 1$.

References

[1] R. Durrett, *Probability : Theory and Examples*, 2nd edition, Duxbury Press, 1996.

[2] T. C. Gard, *Introduction to Stochastic Differential Equations*, MARCEL DEKKER, INC., 1988.

[3] K. Itô, *Differential equations determining Markov processes*, Zenkoku Shijo Danwakai, 1942.

[4] F. C. Klebaner, *Introduction to Stochastic Calculus with Applications*, Imperial College Press, 1998.

[5] L. C. G. Rogers and D. Williams, *Diffusions , Markov Processes and Martingales*, Volume 2 : Ito Calculus, John Wiley & Sons, 1987.
[6] K. S. Ryu and M. K. Im, *A measure-valued analogue of Wiener measure and the measure-valued Feynman-Kac formula*, Trans. Amer. Math. Soc. **354** (2002), 4921-4951.

*
Department of Mathematics Education
Hannam University
Daejeon 306-791, Republic of Korea
E-mail: ksr@hannam.ac.kr