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SURFACES OF REVOLUTION WITH LIGHT-LIKE AXIS

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Abstract. In this paper, we investigate the surfaces of revolution with light-like axis satisfying some equation in terms of a position vector field and the Laplacian with respect to the non-degenerate third fundamental form in Minkowski 3-space. As a result, we give some special example of the surfaces of revolution with light-like axis.

1. Introduction

Let $x : M \longrightarrow \mathbb{E}^m$ be an isometric immersion of a connected n-dimensional manifold in Euclidean m-space $\mathbb{E}^m$. Denote by Δ the Laplacian of M with respect to the Riemannian metric on M induced from that of $\mathbb{E}^m$. Relative to Takahashi's theorem ([13]) for minimal submanifolds, the idea of submanifolds of finite type in Euclidean space was introduced by Chen ([4]). As a generalization of Takahashi's theorem for the case of hypersurfaces, Garay ([8]) considered the hypersurfaces in $\mathbb{E}^m$ whose coordinate functions are eigenfunctions of the Laplacian, but not necessarily associated to the same eigenvalues, that is, he investigated the hypersurfaces satisfying the condition

$$\Delta x = Ax, \tag{1.1}$$

where $A \in \mathrm{Mat}(m, \mathbb{R})$ is an $m \times m$- diagonal matrix.
On the other hand, the study of an isometric immersion satisfying (1.1) can be extended to Gauss map on a hypersurface of Euclidean space. The Gauss map is a useful tool to examine the character of the hypersurfaces in Euclidean space. In [7], Dillen, Pas and Verstraelen studied the surfaces of revolution in Euclidean 3-space $\mathbb{E}^3$ such that its Gauss

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map G satisfies the condition

$$\Delta G = AG, \tag{1.2}$$

where $A \in \mathrm{Mat}(3, \mathbb{R})$ is a 3×3-real matrix. Baikoussis and Blair ([1]) investigated the ruled surfaces in $\mathbb{E}^3$ satisfying the condition (1.2). Baikoussis and Verstraelen ([2, 3]) studied the helicoidal surfaces and the spiral surfaces in $\mathbb{E}^3$ satisfying the condition (1.2). Also, for the Lorentz version, Choi ([5, 6]) completely classified the surfaces of revolution and the ruled surfaces with non-null(light-like) base curve satisfying the condition (1.2) in Minkowski 3-space $\mathbb{E}^3_1$. On the other hand, the conditions (1.1) and (1.2) are special cases of a finite type immersion and an immersion with finite type Gauss map, respectively.

If a surface M has the non-degenerate second fundamental form II or the non-degenerate third fundamental form III, then it is regarded as a new (pseudo-)Riemannian metric on M. So, considering the conditions (1.1) and (1.2), we may have a natural question as follows: *What the surfaces in $\mathbb{E}^3$ satisfying the conditions*

$$\Delta^\alpha x = Ax, \tag{1.3}$$

$$\Delta^\alpha G = AG, \tag{1.4}$$

where Δ^α is the Laplacian with respect to α of M and $\alpha = II$ or III.

For the above question, in [9] Kaimakamis and Papantoniou proved that the surfaces of revolution with space-like or time-like axis in Minkowski 3-space $\mathbb{E}^3_1$ satisfying $\Delta^{II} x = Ax$ are minimal or Lorentz hyperbolic cylinders or pseudospheres of real or imaginary radius. On the other hand, in [11], the authors and Kim studied the surfaces of revolution satisfying $\Delta^{II} G = AG$, proving that the only such surfaces are the catenoid or the sphere. The authors and Kim ([12]) also investigated ruled surfaces in $\mathbb{E}^3_1$ satisfying

$$\Delta^{III} x = Ax, \tag{1.5}$$

proved that such surfaces are either minimal or quadric null scroll.
For the surfaces satisfying (1.5), Kaimakamis, Papantoniou and Petoumenos ([10]) proved the following theorem.

Theorem 1.1. *([10]) Let M be a surface of revolution with space-like or time-like axis in Minkowski 3-space. Then M satisfies the condition (1.5) if and only if it is an open part of one of the pseudocatenoid, the Lorentz cylinder $S^1_1 \times R$, the pseudosphere S^2_1 or the pseudohyperbolic space H^2_0.*

The main purpose of this paper is to complete Kaimakamis, Papantoniou and Petoumenos' classification of surfaces of revolution in $\mathbb{E}^3_1$ by dealing with the surfaces with light-like axis.

2. Preliminaries

Let $\mathbb{E}^3_1$ be Minkowski 3-space with the scalar product of index 1 given by $\langle \cdot, \cdot \rangle = -dx_1^2 + dx_2^2 + dx_3^2$, where (x_1, x_2, x_3) is a standard rectangular coordinate system of $\mathbb{E}^3_1$. A vector x of $\mathbb{E}^3_1$ is said to be space-like if $\langle x, x \rangle > 0$ or $x = 0$, time-like if $\langle x, x \rangle < 0$ and light-like or null if $\langle x, x \rangle = 0$ and $x \neq 0$.

We denote a surface M in $\mathbb{E}^3_1$ by

$$x(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v)).$$

Let N be the standard unit normal vector field on M defined by $N = \frac{x_u \times x_v}{||x_u \times x_v||}$, where $x_u = \frac{\partial x(u,v)}{\partial u}$ and $x_v = \frac{\partial x(u,v)}{\partial v}$. Then the first fundamental form I of M is defined by

$$I = g_{11} du^2 + 2 g_{12} du dv + g_{22} dv^2,$$

where $g_{11} = \langle x_u, x_u \rangle, \quad g_{12} = \langle x_u, x_v \rangle, \quad g_{22} = \langle x_v, x_v \rangle$. We define the second fundamental form II and the third fundamental form III of M by, respectively

$$II = h_{11} du^2 + 2 h_{12} du dv + h_{22} dv^2,$$
$$III = t_{11} du^2 + 2 t_{12} du dv + t_{22} dv^2,$$

where

$$h_{11} = \langle x_{uu}, N \rangle, \quad h_{12} = \langle x_{uv}, N \rangle, \quad h_{22} = \langle x_{vv}, N \rangle,$$
$$t_{11} = \langle N_u, N_u \rangle, \quad t_{12} = \langle N_u, N_v \rangle, \quad t_{22} = \langle N_v, N_v \rangle.$$

If the third fundamental form III is non-degenerate, then the Laplacian Δ^{III} with respect to III can be defined formally on the (pseudo-) Riemannian manifold (M, III). Using classical notation, we define the Laplacian Δ^{III} by

$$\Delta^{III} = -\frac{1}{\sqrt{|\mathcal{T}|}} \sum_{i,j}^{2} \frac{\partial}{\partial x^i} \left(\sqrt{|\mathcal{T}|} t^{ij} \frac{\partial}{\partial x^j} \right), \tag{2.1}$$

where $\mathcal{T} = \det(t_{ij})$ and $(t^{ij}) = (t_{ij})^{-1}$.

Now, we define a surface of revolution M in Minkowski 3-space $\mathbb{E}^3_1$.

Let $\gamma : I = (a, b) \subset \mathbb{R} \to \Pi$ be a curve in a plane Π of $\mathbb{E}^3_1$ and l a straight line which does not intersect the curve γ. A surface of revolution

M is a non-degenerate surface revolving the curve γ around the axis l. Depending on the axis being space-like, time-like or light-like, there are three types of motions. If the axis l is space-like (resp. time-like), then l is transformed to the x_2-axis or x_3-axis (resp. x_1-axis) by the Lorentz transformation. Therefore, we may consider x_3-axis (resp. x_1-axis) as the axis if l is space-like (resp. time-like). If the axis is light-like, we may assume that the axis is the line spanned by the vector $(1,1,0)$. Thus we consider the surfaces of revolution in $\mathbb{E}^3_1$ with space-like, time-like or light-like axis, respectively.

Case 1. The axis l is space-like.

Suppose that the profile curve γ lies in the x_2x_3-plane or x_1x_3-plane. Then the curve γ can be represented by $\gamma(u) = (0, f(u), g(u))$ or $\gamma(u) = (f(u), 0, g(u))$ for some smooth functions f and g on an open interval $I = (a, b)$. It can be seen that the rotation matrix which fixes the space-like axis $x_3 = \mathbb{R}(0,0,1)$ is the set of 3×3-matrices defined by

$$\begin{pmatrix} \cosh v & \sinh v & 0 \\ \sinh v & \cosh v & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for any $v \in \mathbb{R}$. Hence the surface M can be parameterized by

$$x(u,v) = (f(u)\sinh v, f(u)\cosh v, g(u)), \quad f(u) > 0 \tag{2.2}$$

or

$$x(u,v) = (f(u)\cosh v, f(u)\sinh v, g(u)), \quad f(u) > 0. \tag{2.3}$$

Case 2. The axis l is time-like.

Without loss of generality, we may assume that the profile curve γ lies in the x_1x_2-plane. Then one of its parametrization is $\gamma(u) = (g(u), f(u), 0)$ for some positive function $f = f(u)$ on an open interval $I = (a, b)$. The rotation matrix which fixes the time-like axis $x_1 = \mathbb{R}(1,0,0)$ is the set of 3×3-matrices given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos v & -\sin v \\ 0 & \sin v & \cos v \end{pmatrix}$$

for any $v \in \mathbb{R}$. Hence the surface of revolution M revolving γ around the axis Ox_1 can be written as

$$x(u,v) = (g(u), f(u)\cos v, f(u)\sin v). \tag{2.4}$$

Case 3. The axis l is light-like.

Suppose that the axis of revolution is light-like line on x_1x_2-plane spanned by the vector $(1,1,0)$. Then the rotation matrix which the light-like axis $\mathbb{R}(1,1,0)$ is the set of 3×3-matrices given by

$$\begin{pmatrix} 1+\frac{v^2}{2} & -\frac{v^2}{2} & v \\ \frac{v^2}{2} & 1-\frac{v^2}{2} & v \\ v & -v & 1 \end{pmatrix}$$

for any $v\in\mathbb{R}$. Thus, if the axis of revolution is the line spanned by the vector $(1,1,0)$ and the curve $\gamma(u)=(f(u),g(u),0)$ lies in the x_1x_2-plane, the surface of revolution M can be parametrized as

$$x(u,v)=(f(u)+\frac{v^2}{2}p(u),g(u)+\frac{v^2}{2}p(u),p(u)v), \tag{2.5}$$

where $p(u)=f(u)-g(u)\neq 0$.

Consider $\gamma(u)$ a curve in the plane spanned by the two vectors $(1,1,0)$ and $(-1,1,0)$ given as a graph on the straight line spanned by the vector $(-1,1,0)$, that is, $\gamma(u)=(-u+k(u),u+k(u),0)$, where $k(u)$ is a smooth function. Then (2.4) can be rewritten as the form:

$$x(u,v)=(k(u)-u-uv^2,k(u)+u-uv^2,-2uv). \tag{2.6}$$

3. Main results

In this section, we investigate the surfaces of revolution with light-like axis in $\mathbb{E}^3_1$ satisfying the condition (1.5).

Let M be a surface of revolution with light-like axis in $\mathbb{E}^3_1$ parameterized by

$$x(u,v)=(k(u)-u-uv^2,k(u)+u-uv^2,-2uv). \tag{3.1}$$

Then, the components of the first fundamental form of the surface are

$$g_{11}=4k'(u),\quad g_{12}=0,\quad g_{22}=4u^2. \tag{3.2}$$

From this, $k'(u)\neq 0$ and $u\neq 0$ because M is non-degenerate. On the other hand, the unit normal vector field N of M is given by

$$N=\frac{1}{2\sqrt{u^2|k'(u)|}}(uk'(u)+u+uv^2,uk'(u)-u+uv^2,2uv).$$

Suppose that M is a space-like surface, that is, $k'(u)>0$. Then, the components of the third fundamental form III are

$$t_{11}=\frac{k''^2}{4k'^2},\quad t_{12}=0,\quad t_{22}=\frac{1}{k'}. \tag{3.3}$$

Since the third fundamental form III of M is non-degenerate, $k'' \neq 0$. By (2.1), the Laplacian Δ^{III} of the third fundamental form III can be expressed as follows:

$$\Delta^{III} = \left(\frac{4k'^2 k'''}{k''^3} - \frac{2k'}{k''} \right) \frac{\partial}{\partial u} - \frac{4k'^2}{k''^2} \frac{\partial^2}{\partial u^2} - k' \frac{\partial^2}{\partial v^2}. \tag{3.4}$$

By a straightforward computation, the Laplacain $\Delta^{III} x$ with the help of (3.1) and (3.4) turns out to be

$$\begin{aligned} \Delta^{III} x = (&\Phi(u)(k' - 1 - v^2) - \Psi(u), \\ &\Phi(u)(k' + 1 - v^2) + \Psi(u), \Phi(u)(-2v)), \end{aligned} \tag{3.5}$$

where $\Phi(u) = \frac{4k'^2 k'''}{k''^3} - \frac{2k'}{k''}$ and $\Psi(u) = -\frac{4k'^2}{k''} + 2uk'$.

Suppose M satisfies the condition (1.5), that is, $\Delta^{III} x = Ax$ for some matrix $A = (a_{ij})$, where $i, j = 1, 2, 3$. Then, from (3.1) and (3.5) we have the following equations:

$$\begin{aligned} \Phi(u)(k' - 1 - v^2) + \Psi(u) = &a_{11}(k(u) - u - uv^2) \\ &+ a_{12}(k(u) + u - uv^2) - 2a_{13}uv, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \Phi(u)(k' + 1 - v^2) + \Psi(u) = &a_{21}(k(u) - u - uv^2) \\ &+ a_{22}(k(u) + u - uv^2) - 2a_{23}uv, \end{aligned} \tag{3.7}$$

$$-2v\Phi(u) = a_{31}(k(u) - u - uv^2) + a_{32}(k(u) + u - uv^2) - 2a_{33}uv. \tag{3.8}$$

From (3.6) and (3.7), we have

$$\begin{aligned} -2\Phi(u) = &(a_{11} + a_{12} - a_{21} - a_{22})k(u) - (a_{11} - a_{12} - a_{21} + a_{22})u \\ &- (a_{11} + a_{12} - a_{21} - a_{22})uv^2 - 2(a_{13} - a_{23})uv. \end{aligned} \tag{3.9}$$

Hence we obtain

$$a_{11} + a_{12} - a_{21} - a_{22} = 0, \quad a_{13} - a_{23} = 0, \tag{3.10}$$

which implies that (3.9) becomes

$$2\Phi(u) = (a_{11} - a_{12} - a_{21} + a_{22})u. \tag{3.11}$$

Combining (3.8) and (3.11), we have

$$(-a_{11}+a_{12}+a_{21}-a_{22})uv = (a_{31}+a_{32})(k(u)-uv^2)+(-a_{31}+a_{32})u-2a_{33}uv, \tag{3.12}$$

from this we get

$$a_{31} = a_{32} = 0, \quad a_{11} - a_{12} - a_{21} + a_{22} = 2a_{33} \tag{3.13}$$

because of $k \neq 0$. Thus, (3.11) can be reduced as

$$\Phi(u) = a_{33}u. \tag{3.14}$$

If we add (3.6) to (3.7), then using the above results, we find

$$\begin{aligned} 2\Phi(u)(k' - v^2) + 2\Psi(u) = 2(a_{11} + a_{12})(k - uv^2) \\ - (a_{11} - a_{12} + a_{21} - a_{22})u - 4a_{13}uv, \end{aligned} \tag{3.15}$$

which implies

$$a_{13} = 0, \quad a_{33} = a_{11} + a_{12}. \tag{3.16}$$

By (3.10), (3.13) and (3.16), the matrix A is given by

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ -a_{12} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix},$$

where $a_{11} + a_{12} = -a_{12} + a_{22} = a_{33}$.
Moreover, equation (3.15) with the help of (3.14) may be rewritten as

$$(a_{33} + 2)uk' - \frac{4k'^2}{k''} = a_{33}k + 2a_{12}u.$$

Thus, from (3.14) and the above equation we have the system of differential equations as follows:

$$\begin{cases} \frac{4k'^2k'''}{k''^3} - \frac{2k'}{k''} = a_{33}u, \\ (a_{33} + 2)uk' - \frac{4k'^2}{k''} = a_{33}k + 2a_{12}u. \end{cases} \tag{3.17}$$

By differentiating the second equation in (3.17) with respect to u and using the first equation in (3.17) we have the following ODE

$$(a_{33} + 1)uk'' - 2k' = a_{12}. \tag{3.18}$$

Since equation (3.18) is a linear differential equation, we can easily find a general solution and its solution is given by

$$k(u) = -\frac{a_{12}}{2}u + \frac{c(a_{33} + 1)}{a_{33} + 3}u^{\frac{a_{33}+3}{a_{33}+1}} + d, \tag{3.19}$$

where $c \neq 0, d$ are constants of integration. Substituting (3.19) into the first equation in (3.17), we find $a_{12} = 0, \quad a_{33} = -2$. Thus, equation (3.19) becomes $k(u) = -\frac{c}{u}$. Furthermore, the function $k(u)$ is a solution of the second equation in (3.17) and the matrix A reduces $A = -2I$, where I is a 3×3-identity matrix.

Consequently, the surface M is parameterized by

$$x(u, v) = \Big(-\frac{c}{u} - u - uv^2, -\frac{c}{u} + u - uv^2, -2uv\Big) \tag{3.20}$$

for some non-zero constant c (see Fig. 1).
A surface M described above is called a *surface of revolution of hyperbolic type*.

Similarly, we have the same conclusion in case of time-like surface, that is, $k'(u) < 0$.

Consequently, we have

THEOREM 3.1. *Let M be a surface of revolution with light-like axis in Minkowski 3-space $\mathbb{E}^3_1$ satisfying the condition (1.5). Then M is an open part of the surface of revolution of hyperbolic type.*

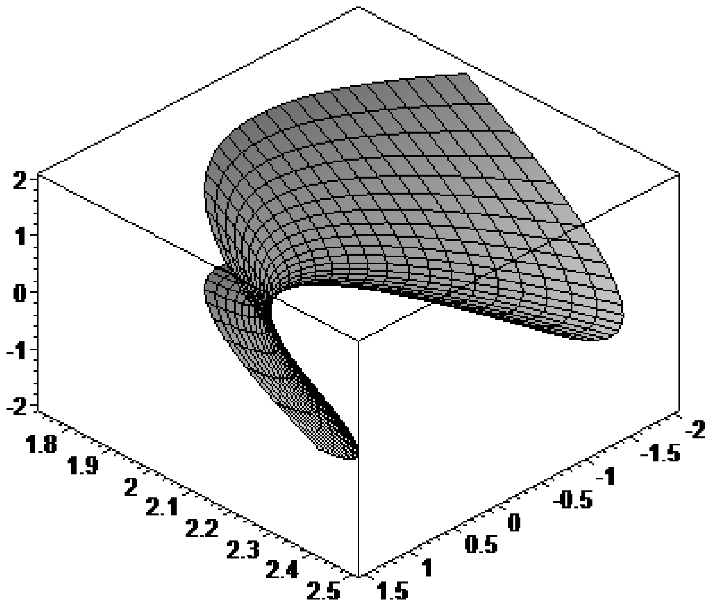

FIGURE 1

Combining Theorem 1.1 and our Theorem 3.1, we have

THEOREM 3.2. *(Classification) Let M be a surface of revolution with the non-degenerate third fundamental form III in Minkowski 3-space $\mathbb{E}^3_1$. Then, M satisfies the condition*

$$\Delta^{III} x = Ax, \quad A \in Mat(3, \mathbb{R}),$$

if and only if it is an open part of one of the pseudocatenoid, the Lorentz cylinder $S_1^1 \times R$, the pseudosphere S_1^2, the pseudohyperbolic space H_0^2 or the surface of revolution of hyperbolic type.

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