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GENERALIZED WEYL'S THEOREM FOR ALGEBRAICALLY k-QUASI-PARANORMAL OPERATORS

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Abstract. An operator $T \in B(\mathcal{H})$ is said to be k-quasi-paranormal operator if $\|T^{k+1}x\|^2 \leq \|T^{k+2}x\|\|T^k x\|$ for every $x \in \mathcal{H}$, k is a natural number. This class of operators contains the class of paranormal operators and the class of quasi - class A operators. In this paper, using the operator matrix representation of k-quasi-paranormal operators which is related to the paranormal operators, we show that every algebraically k-quasi-paranormal operator has Bishop's property (β), which is an extension of the result proved for paranormal operators in [32]. Also we prove that (i) generalized Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$; (ii) generalized a - Browder's theorem holds for $f(S)$ for every $S \prec T$ and $f \in H(\sigma(S))$; (iii) the spectral mapping theorem holds for the B - Weyl spectrum of T.

1. Introduction

Let $B(\mathcal{H})$ and $B_0(\mathcal{H})$ denotes the algebra of all bounded linear operators and the ideal of compact operaors acting on an infinite dimensional separable Hilbert space $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is positive, $T \geq 0$, if $(Tx, x) \geq 0$ for all $x \in \mathcal{H}$, and posinormal if there exists a positive $\lambda \in B(\mathcal{H})$ such that $TT^* = T^*\lambda T$. Here λ is called interrupter of T. In other words, an operator T is called posinormal if $TT^* \leq c^2 T^*T$, where T^* is the adjoint of T and $c > 0$ [15]. An operator T is said to be heminormal if T is hyponormal and T^*T commutes with TT^*. An operator

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T is said to be p-hyponormal, for $p \in (0,1)$, if $(T^*T)^p \geq (TT^*)^p$. An 1-hyponormal operator is hyponormal which has been studied by many authors and it is known that hyponormal operators have many interesting properties similar to those of normal operators [34]. Furuta et al [19], have characterized class A operator as follows. An operator T belongs to class A if and only if $(T^*|T|T)^{\frac{1}{2}} \geq T^*T$.

An operator T is said to be p-posinormal if $(TT^*)^p \leq c^2(T^*T)^p$ for some $c > 0$. An operator T is called normal if $T^*T = TT^*$ and (p,k)-quasihyponormal if $T^{*^k}((T^*T)^p - (TT^*)^p)T^k \geq 0$ $(0 < p \leq 1, k \in \mathbb{N})$. A. Aluthge [3], B.C. Gupta [12], S.C. Arora and P. Arora [5] introduced p-hyponormal, p-quasihyponormal and k-quasihyponormal operators, respectively.

$$p\text{-hyponormal} \subset p\text{-posinormal} \subset (p,k)\text{-quasiposinormal},$$

$$p\text{-hyponormal} \subset p\text{-quasihyponormal} \subset (p,k)\text{-quasihyponormal} \subset (p,k)\text{-quasiposinormal}$$

and

$$\text{hyponormal} \subset k\text{-quasihyponormal} \subset (p,k)\text{-quasihyponormal} \subset (p,k)\text{-quasiposinormal}$$

for a positive integer k and a positive number $0 < p \leq 1$.

In [31], the class of log-hyponormal operators is defined as follows: T is called log - hyponormal if it is invertible and satisfies log $(T^*T)^p \geq$ log $(TT^*)^p$. Class of p-hyponormal operators and class of log hyponormal operators were defined as extension class of hyponormal operators, i.e., $T^*T \geq TT^*$. It is well known that every p-hyponormal operator is a q - hyponormal operator for $p \geq q > 0$, by the Löwner-Heinz theorem "$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$", and every invertible p - hyponormal operator is a log-hyponormal operator since $\log(\cdot)$ is an operator monotone function. An operator T is called paranormal if $||Tx||^2 \leq ||T^2x||||x||$ for all $x \in H$. It is also well known that there exists a hyponormal operator T such that T^2 is not hyponormal (see [23]).

Furuta, Ito and Yamazaki [21] introduced class $A(k)$ and absolute-k-paranormal operators for $k > 0$ as generalizations of class A and paranormal operators, respectively. An operator T belongs to class $A(k)$ if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2$ and T is said to be absolute-k-paranormal operator if $\||T|^kTx\| \geq \|Tx\|^{k+1}$ for every unit vector x. An operator T is called quasi class A if $T^*|T|^2T \geq T^*|T|^2T$. Fuji, Izumino and Nakamoto

[19] introduced p-paranormal operators for $p > 0$ as a generalization of paranormal operators.

Fujii, Jung, S. H. Lee, M. Y. Lee and Nakamoto [22] introduced class $A(p, r)$ as a further generalization of class $A(k)$. An operator $T \in$ class$A(p, r)$ for $p > 0$ and $r > 0$ if $(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r}$ and class $AI(p, r)$ is class of all invertible operators which belong to class $A(p, r)$. Yamazaki and Yanagida [35] introduced absolute-(p, r)-paranormal operator. It is a further generalization of the classes of both absolute-k-paranormal operators and p - paranormal operators as a parallel concept of class $A(p, r)$. An operator T is said to be paranormal operator if $\|T^2x\| \geq \|Tx\|^2$ for every unit vector x. Paranormal operators have been studied by many authors [4], [20] and [26]

In [4], Ando showed that T is paranormal if and only if

$$T^{*2}T^2 - 2\lambda T^*T + \lambda^2 \geq 0 \text{ for all } \lambda > 0.$$

In order to extend the class of paranormal operators and class of quasi-class A operators, Mecheri [29] introduced a new class of operators called k-quasi-paranormal operators. An operator T is called k-quasi-paranormal if $\|T^{k+1}x\|^2 \leq \|T^{k+2}x\|\|T^kx\|$ for all $x \in H$ where k is a natural number. A 1-quasi-paranormal operator is quasi paranormal. The following implication gives us relations among the classes of operators.

$$\text{Hyponormal} \Rightarrow p\text{-hyponormal} \Rightarrow \text{class } A \Rightarrow \text{paranormal}$$
$$\Rightarrow \text{quasi-paranormal} \Rightarrow k\text{-quasi-paranormal}$$

$$\text{Hyponormal} \Rightarrow \text{class } A \Rightarrow \text{quasi-class } A \Rightarrow \text{quasi-paranormal}$$
$$\Rightarrow k\text{-quasi-paranormal}$$

An operator T is called algebraically k-quasi-paranormal if there exists a nonconstant complex polynomial s such that $s(T)$ belongs to k-quasi-paranormal.

The following facts follows from some well known facts about paranormal operators.
(i) If T is paranormal and $M \subseteq \mathcal{H}$ is invariant under T then $T|_M$ is paranormal.
(ii) Every quasinilpotent paranormal operator is a zero operator.
(iii) T is paranormal if and only if $T^{2^*}T^2 - 2\lambda T^*T + \lambda^2 \geq 0$ for all $\lambda > 0$.

(iv) If T is paranormal and invertible, then T^{-1} is paranormal.

If $T \in B(\mathcal{H})$, we shall write $N(T)$ and $R(T)$ for the null space and the range of T, respectively. Also, let $\sigma(T)$ and $\sigma_a(T)$ denote the spectrum and the approximate point spectrum of T, respectively. An operator T is called Fredholm if $R(T)$ is closed, $\alpha(T) = \dim N(T) < \infty$ and $\beta(T) = \dim \mathcal{H}/R(T) < \infty$. Moreover if $i(T) = \alpha(T) - \beta(T) = 0$, then T is called Weyl. The essential spectrum $\sigma_e(T)$ and the Weyl $\sigma_W(T)$ are defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{is not Fredholm}\}$$

and

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{is not Weyl}\},$$

respectively. It is known that $\sigma_e(T) \subset \sigma_W(T) \subset \sigma_e(T) \cup$ acc $\sigma(T)$ where we write acc K for the set of all accumulation points of $K \subset \mathbb{C}$. If we write iso $K = K \setminus$ acc K, then we let

$$\pi_{00}(T) = \{\lambda \in \text{iso } \sigma(T) : 0 < \alpha(T - \lambda) < \infty\}.$$

We say that Weyl's theorem holds for T if

$$\sigma(T) \backslash \sigma_W(T) = \pi_{00}(T).$$

Let $\sigma_p(T), \pi(T), E(T)$ denotes the point spectrum of T, the set of poles of the resolvent of T, the set of all eigenvalues of T which are isolated in $\sigma(T)$, respectively. An operator $T \in B(\mathcal{H})$ is called upper semi-Fredholm if it has closed range and finite dimensional null space and is called lower semi - Fredholm if it has closed range and its range has finite co-dimension. If $T \in B(\mathcal{H})$ is either upper or lower semi - Fredholm, then T is called semi-Fredholm. For $T \in B(\mathcal{H})$ and a non negative integer n define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ to $R(T^n)$ (in particular $T_0 = T$). If for some integer n the range $R(T^n)$ is closed and T_n is upper (resp. lower) semi-Fredholm, then T is called upper (resp. lower) semi-B-Fredholm.

Moreover, if T_n is Fredholm, then T is called B - Fredholm. An operator T is called semi-B-Fredholm if it is upper or lower semi-B-Fredholm. Let T be semi-B-Fredholm and let d be the degree of stable iteration of T. It follows from [10, Proposition 2.1] that T_m is semi-Fredholm and $i(T_m) = i(T_d)$ for each $m \geq d$. This enables us to define the index of semi-B-Fredholm T as the index of semi-Fredholm T_d. Let $BF(\mathcal{H})$ be the class of all B-Fredholm operators. In [6], they studied this class of operators and they proved [6, Theorem 2.7] that an operator $T \in B(\mathcal{H})$ is B-Fredholm if and only if $T = T_1 \oplus T_2$, where T_1 is Fredholm and T_2 is nilpotent. It appears that the concept of Drazin invertibility plays an important role for the class of B-Fredholm operators. Let $\mathcal{A}$ be a unital algebra. We say that an element $x \in A$ is Drazin invertible of degree k if there exists an element $a \in \mathcal{A}$ such that

$$x^k ax = x^k, \ axa = a, \text{ and } xa = ax$$

Let $a \in \mathcal{A}$. Then the Drazin spectrum is defined by

$$\sigma_D(a) = \{\lambda \in \mathbb{C} : a - \lambda \text{ is not Drazin invertible}\}.$$

For $T \in B(\mathcal{H})$, the smallest nonnegative integer p such that $N(T^p) = N(T^{p+1})$ is called the ascent of T and denoted by $p(T)$. If no such integer exists, we set $p(T) = 1$. The smallest nonnegative integer q such that $R(T^q) = R(T^{q+1})$ is called the descent of T and denoted by $q(T)$. If no such integer exists, we set $q(T) = 1$. It is well known that T is Drazin invertible if and only if it has finite ascent and descent, which is also equivalent to the fact that

$$T = T_1 \oplus T_2, \text{ where } T_1 \text{ is invertible and } T_2 \text{ is nilpotent.}$$

An operator $T \in B(\mathcal{H})$ is called B - Weyl if it is B-Fredholm of index 0. The B-Fredholm spectrum $\sigma_{BF}(T)$ and B-Weyl spectrum $\sigma_{BW}(T)$ of T are defined by

$$\sigma_{BF}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not B-Fredholm } \},$$

and

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not B-Weyl } \}.$$

Now, we consider the following sets:

$$BF_+(\mathcal{H}) = \{T \in B(\mathcal{H}) : T \text{ is upper semi-B-Fredholm}\},$$
$$BF_+^-(\mathcal{H}) = \{T \in B(\mathcal{H}) : T \in BF_+(\mathcal{H}) \text{ and } i(T) \leq 0\},$$
$$LD(\mathcal{H}) = \{T \in B(\mathcal{H}) : p(T) < \infty \text{ and } R(T^{p(T)+1}) \text{ is closed } \}.$$

By definition,

$$\sigma_{B_{ea}}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin BF_+^-(\mathcal{H})\},$$

is the upper semi-B-essential approximate point spectrum and

$$\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin LD(\mathcal{H})\}$$

is the left Drazin spectrum. It is well known that

$$\sigma_{B_{ea}}(T) = \sigma_{LD}(T) = \sigma_{B_{ea}}(T) \cup \text{ acc } \sigma_a(T) \subseteq \sigma_D(T),$$

where we write acc K for the accumulation points of $K \subseteq \mathbb{C}$. If we write iso $K = K\setminus$ acc K then we let

$$p_0^a(T) = \{\lambda \in \sigma_a(T) : T - \lambda \in LD(\mathcal{H})\},$$
$$\pi_0^a(T) = \{\lambda \in \text{ iso } \sigma_a(T) : \lambda \in \sigma_p(T)\}.$$

We say that an operator T has the single valued extension property at λ (abbreviated SVEP atλ) if for every open set U containing λ the only analytic function $f : U \to \mathcal{H}$ which satisfies the equation

$$(T - \lambda)f(\lambda) = 0$$

is the constant function $f \equiv 0$ on U. An operator T has SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

We say that Generalized Weyl's theorem holds for T if (in symbols, $T \in g\mathcal{W}$) if

$$\sigma(T)\backslash\sigma_{BW}(T) = E(T).$$

We say that Generalized Browder's theorem holds for T if (in symbols, $T \in g\mathcal{B}$) if

$$\sigma(T)\backslash\sigma_{BW}(T) = \pi(T).$$

We say that Generalized a - Weyl's theorem holds for T if (in symbols, $T \in ga\mathcal{W}$) if

$$\sigma_a(T)\backslash\sigma_{B_{ea}}(T) = \pi_0^a(T).$$

We say that Generalized a - Browder's theorem holds for T if (in symbols, $T \in ga\mathcal{B}$) if

$$\sigma_a(T)\backslash\sigma_{B_{ea}}(T) = p_0^a(T).$$

In local spectral theory, the quasi-nilpotent part $H_0(T)$ of an operator T is defined by

$$H_0(T) = \{x \in \mathcal{H} : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}$$

and the analytic core $K(T)$ is defined as

$$\begin{aligned} K(T) = \{x \in \mathcal{H}&\text{: there exists a sequence } \{x_n\} \subset \mathcal{H} \\ &\text{and } \delta > 0 \text{ for which } x = x_0, T(x_{n+1}) = x_n \\ &\text{and } \|x_n\| \le \delta^n \|x\| \text{ for all } n = 1, 2, 3, ...\} \end{aligned}$$

Let $\mathcal{P}(\mathcal{H})$ denotes the class of all operators for which there exists $p = p(\lambda) \in \mathbb{N}$ for which

$$H_0(T - \lambda) = N(T - \lambda)^p \text{ for all } \lambda \in E(T).$$

Evidently, $\mathcal{P}(\mathcal{H}) \subseteq \mathcal{P}_1(\mathcal{H})$. Now we give a characterization of $\mathcal{P}_1(\mathcal{H})$.

THEOREM 1.1. *$T \in \mathcal{P}_1(\mathcal{H})$ if and only if $\pi(T) = E(T)$.*

Proof. Suppose $T \in \mathcal{P}_1(\mathcal{H})$ and let $\lambda \in E(T)$. Then there exists $p \in \mathbb{N}$ such that $H_0(T - \lambda) = N(T - \lambda)^p$. Since λ is an isolated point of $\sigma(T)$, it follows from [1, Theorem 3.74] that

$$\mathcal{H} = H_0(T - \lambda) \oplus K(T - \lambda) = N(T - \lambda)^p \oplus K(T - \lambda).$$

Therefore, we have

$$(T - \lambda)^p(\mathcal{H}) = (T - \lambda)^p(K(T - \lambda)) = K(T - \lambda),$$

and hence $\mathcal{H} = N(T - \lambda)^p \oplus (T - \lambda)^p(\mathcal{H})$, which implies, by [1, theorem 3.6], that $p(T - \lambda) = q(T - \lambda) \le p$. But $\alpha(T - \lambda) > 0$, hence $\lambda \in \pi(T)$. Therefore $E(T) \subseteq \pi(T)$. Since the opposite inclusion holds for every operator T, we then conclude that $\pi(T) = E(T)$.

Conversely, suppose $\pi(T) = E(T)$. Let $\lambda \in E(T)$. Then $p = p(T - \lambda) = q(T - \lambda) < \infty$. By [1, Theorem 3.74], $H_0(T - \lambda) = N(T - \lambda)^p$. Therefore $T \in \mathcal{P}_1(\mathcal{H})$. □

In [33], H. Weyl proved that Weyl's theorem holds for hermitian operators. Weyl's theorem has been extended from hermitian operators to hyponormal operators [14], algebraically hyponormal operators [24], p-hyponormal operators [13] and algebraically p-hyponormal operators [18]. More generally, M. Berkani investigated generalized Weyl's theorem which extends Weyl's theorem, and proved that generalized Weyl's theorem holds for hyponormal operators [[8, 9, 10]]. In a recent paper [28] the author showed that generalized Weyl's theorem holds for (p, k)-quasihyponormal operators. Recently, X. Cao, M. Guo and B. Meng [11] proved Weyl type theorems for p - hyponormal operators.

In this paper, we prove some basic structural properties of k-quasi-paranormal operators and also using the operator matrix representation of k-quasi-paranormal operators which is related to the paranormal operators, we show that every algebraically k-quasi-paranormal operator has Bishop's property (β), which is an extension of the result proved for paranormal operators in [32]. We also prove that (i) generalized Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$; (ii) generalized a-Browder's theorem holds for $f(S)$ for every $S \prec T$ and $f \in H(\sigma(S))$; (iii) the spectral mapping theorem holds for the B-Weyl spectrum of T.

2. On k - quasi - paranormal operators

Salah Mecheri [29] has introduced k-quasi-paranormal operators and has proved many interesting properties of it.

Lemma 2.1. ([29]) *(1) Let $T \in B(\mathcal{H})$ be a k-quasi-paranormal, the range of T^k be not dense and*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

on $\mathcal{H} = \overline{ran(T^k)} \oplus ker(T^{*k})$. Then T_1 is paranormal, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.
(2) Let M be a closed T-invariant subspace of $\mathcal{H}$. Then the restriction $T|_M$ of a k-quasi-paranormal operator T to M is a k-quasi-paranormal.

Lemma 2.2. ([29]) *Let $T \in B(\mathcal{H})$ be a k-quasi-paranormal operator. Then T has Bishop's property (β), i.e., if $f_n(z)$ is analytic on D and*

$(T-z)f_n(z) \to 0$ uniformly on each compact subset of D, then $f_n(z) \to 0$ uniformly on each compact subset of D. Hence T has the single valued extension property.

Corollary 2.3. *Suppose that $T \in k$-quasi-paranormal has dense range. Then T is paranormal.*

Proof. Since T has dense range, $\overline{T(\mathcal{H})} = \mathcal{H}$. Let $y \in \mathcal{H}$. Then there exists a sequence $\{x_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ such that $T(x_k) \to y$ as $k \to \infty$. Since T is k-quasi-paranormal,$\langle (T^{*^2}T^2 - 2\lambda T^*T + \lambda^2)T^k x_k, T^k x_k\rangle \geq 0$ for all $k \in \mathbb{N}$ and all $\lambda > 0$. By the continuity of the inner product, we have $\langle (T^{*^2}T^2 - 2\lambda T^*T + \lambda^2)y, y\rangle \geq 0$ for all $\lambda > 0$, and hence $T^{*^2}T^2 - 2\lambda T^*T + \lambda^2 \geq 0$ for all $\lambda > 0$. Therefore T is paranormal. □

Corollary 2.4. *Suppose that T is an invertible k-quasi-paranormal. Then T and T^{-1} are paranormal.*

Proof. Suppose that $T \in k$-quasi-paranormal is invertible. Then it has dense range, and so it is paranormal by Corollary 2.3. Hence T^{-1} is also paranormal. □

Corollary 2.5. *Suppose that $T \in k$-quasi-paranormal is not paranormal. Then T is not invertible..*

Corollary 2.6. *Suppose that $T \in k$-quasi-paranormal is nonzero and suppose that T has no nontrivial T-invariant subspace. Then T is paranormal.*

Proof. Suppose that $T \in k$-quasi-paranormal. If an operator has no nontrivial invariant subspace, then it is injective and has dense range. It follows from Corollary 2.3 that T is paranormal. □

3. Generalized Weyl's Theorem for algebraically k-quasi-paranormal operators

The following facts follows from the definition and some well known facts about k-quasi-paranormal operators [30, 29]:
(i) If $T \in B(\mathcal{H})$ is algebraically k-quasi-paranormal, then so is $T - \lambda$ for each $\lambda \in \mathbb{C}$.
(ii) If $T \in B(\mathcal{H})$ is algebraically k-quasi-paranormal and M is a closed T-invariant subspace of $\mathcal{H}$, then $T|_M$ is algebraically k-quasi-paranormal.
(iii) If T is algebraically k-quasi-paranormal, then T has SVEP.
(iv) Suppose T does not have dense range. Then we have:

$$T \text{ is } k\text{-quasi-paranormal} \Leftrightarrow T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

on $\mathcal{H} = \overline{ran(T^k)} \oplus ker(T^{*k})$ where T_1 is paranormal operator.

In general, the following implications hold:

paranormal $\Rightarrow$ k-quasi-paranormal $\Rightarrow$ algebraically k-quasi-paranormal.

PROPOSITION 3.1. *Suppose that T is algebraically k-quasi-paranormal. Then T has Bishop's property (β).*

Proof. We first suppose that $T \in k$-quasi-paranormal. We consider two cases:

Case I: Suppose T has dense range. Then T is paranormal by Corollary 2.3, and so it has Bishop's property (β) by [32, Corollary 3.6].

Case II: Suppose T does not have dense range. It follows from Lemma 2.1 that

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

on $\mathcal{H} = \overline{ran(T^k)} \oplus ker(T^{*k})$ where T_1 is paranormal operator. Since T_1 is paranormal, it follows from [36, Theorem 2.12] that T has Bishop's property (β).

Now suppose that T is algebraically k-quasi-paranormal. Then $s(T) \in k$-quasi-paranormal for some nonconstant polynomial s, and so it follows from the first part of the proof that $s(T)$ has Bishop's property (β). Therefore T has Bishop's property (β) [27, Theorem 3.3.9]. □

COROLLARY 3.2. *Suppose T is algebraically k-quasi-paranormal. Then T has SVEP.*

LEMMA 3.3. *Let $T \in B(\mathcal{H})$ be a quasinilpotent algebraically k-quasi-paranormal operator. Then T is nilpotent.*

Proof. We first assume that T is k-quasi-paranormal. We consider two cases:

Case I: Suppose T has dense range. Then clearly, it is paranormal. Therefore T is nilpotent by [16, Lemma 2.2].

Case II: Suppose T does not have dense range. Then we can represent T as the upper triangular matrix

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

on $\mathcal{H} = \overline{ran(T^k)} \oplus ker(T^{*k})$ where T_1 is paranormal operator. Since T is quasinilpotent, $\sigma(T) = \{0\}$. But $\sigma(T) = \sigma(T_1) \cup \{0\}$, hence $\sigma(T_1) = \{0\}$: Since T_1 is paranormal, $T_1 = 0$ and therefore T is nilpotent. Thus if T is a quasinilpotent k-quasi-paranormal operator, then it is nilpotent. Now, we suppose T is algebraically k-quasi-paranormal. Then there exists a nonconstant polynomial s such that $s(T)$ is k-quasi-paranormal. If $s(T)$ has dense range, then $s(T)$ is paranormal. So T is algebraically paranormal, and hence T is nilpotent by [16, Lemma 2.2]. If $s(T)$ does not have dense range, we can represent $s(T)$ as the upper triangular matrix

$$s(T) = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

on $\mathcal{H} = \overline{ran(s(T^k))} \oplus ker(s(T^{*k}))$ where $T_1 = s(T)|_{\overline{s(T)(\mathcal{H})}}$ is paranormal operator. Since T is quasinilpotent, $\sigma(s(T)) = s(\sigma(T)) = \{s(0)\}$. But $\sigma(s(T)) = \sigma(T_1) \cup \{0\}$ by [25, Corollary 8], hence $\sigma(T_1) \cup \{0\} = \{s(0)\}$. So $s(0) = 0$, and hence $s(T)$ is quasinilpotent. Since $s(T)$ is k-quasi-paranormal, by the previous argument $s(T)$ is nilpotent. On the other hand, since $s(0) = 0$, $s(z) = cz^m(z - \lambda_1)(z - \lambda_2)....(z - \lambda_n)$ for some natural number m. Therefore $s(T) = cT^m(T - \lambda_1)(T - \lambda_2)....(T - \lambda_n)$. Since $s(T)$ is nilpotent and $T - \lambda_i$ is invertible for every $\lambda_i \neq 0$, T is nilpotent. Hence the proof. □

THEOREM 3.4. *Let $T \in B(H)$ be algebraically k-quasi-paranormal. Then $T \in \mathcal{P}_1(\mathcal{H})$.*

Proof. Suppose T is algebraically k-quasi-paranormal. Then $s(T)$ is a k-quasi-paranormal operator for some nonconstant complex polynomial s. Let $\lambda \in E(T)$. Then λ is an isolated point of $\sigma(T)$ and $\alpha(T - \lambda) > 0$. Using the spectral projection $P = \frac{1}{2\pi i}\int_{\partial D}(\mu - T)^{-1}d\mu$, where D is a closed disk of center λ which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix},$$

where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T)\backslash\{\lambda\}$.

Since T_1 is algebraically k-quasi-paranormal, so is $T_1 - \lambda$. But $\sigma(T_1 - \lambda) = \{0\}$, it follows from Lemma 3.3 that $T_1 - \lambda$ is nilpotent. Therefore $T_1 - \lambda$ has finite ascent and descent. On the other hand, since $T_2 - \lambda$ is invertible, clearly it has finite ascent and descent. Therefore λ is a pole of the resolvent of T, and hence $\lambda \in \pi(T)$. Hence $E(T) \subseteq \pi(T)$. Since $\pi(T) \subseteq E(T)$ holds for any operator T, we have $\pi(T) = E(T)$. It follows from Theorem 1.1 that $T \in \mathcal{P}_1(\mathcal{H})$. □

We now show that generalized Weyl's theorem holds for algebraically k-quasi-paranormal operators. In the following theorem, recall that $H(\sigma(T))$ is the space of functions analytic in an open neighborhood of $\sigma(T)$.

Theorem 3.5. *Suppose that T or T^* is an algebraically k-quasi-paranormal operator. Then $f(T) \in g\mathcal{W}$ for each $f \in H(\sigma(T))$.*

Proof. Suppose T is algebraically k-quasi-paranormal. We first show that $T \in g\mathcal{W}$. Suppose that $\lambda \in \sigma(T)\backslash\sigma_{BW}(T)$. Then $T-\lambda$ is B-Weyl but not invertible. It follows from [7, Lemma 4.1] that we can represent $T-\lambda$ as the direct sum

$$T-\lambda = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix},$$

where T_1 is Weyl and T_2 is nilpotent.

Since T is algebraically k-quasi-paranormal, it has SVEP. So T_1 and T_2 have both finite ascent. But T_1 is Weyl, hence T_1 has finite descent. Therefore $T-\lambda$ has finite ascent and descent, and so $\lambda \in E(T)$. Conversely, suppose that $\lambda \in E(T)$. Since T is algebraically k-quasi-paranormal, it follows from Theorem 3.4 that $T \in \mathcal{P}_1(\mathcal{H})$. Since $\pi(T) = E(T)$ by Theorem 1.1, $\lambda \in \pi(T)$. Therefore $T-\lambda$ has finite ascent and descent, and so we can represent $T-\lambda$ as the direct sum

$$T-\lambda = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix},$$

where T_1 is invertible and T_2 is nilpotent.

Therefore $T-\lambda$ is B - Weyl, and so $\lambda \in \sigma(T)\backslash\sigma_{BW}(T)$. Thus $\sigma(T)\backslash\sigma_{BW}(T) = E(T)$, and hence $T \in g\mathcal{W}$.

Next, we claim that $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$ for each $f \in H(\sigma(T))$. Since $T \in g\mathcal{W}, T \in g\mathcal{B}$. It follows from [17, Theorem 2.1] that $\sigma_{BW}(T) = \sigma_D(T)$. Since T is algebraically k-quasi-paranormal, $f(T)$ has SVEP for each $f \in H(\sigma(T))$. Hence $f(T) \in g\mathcal{B}$ by [17, Theorem 2.9], and so $\sigma_{BW}(f(T)) = \sigma_D(f(T))$. Therefore we have

$$\sigma_{BW}(f(T)) = \sigma_D(f(T)) = f(\sigma_D(T)) = f(\sigma_{BW}(T)).$$

Since T is algebraically k-quasi-paranormal, it follows from the proof of Theorem 3.4 that it is isoloid. Hence for any $f \in H(\sigma(T))$ we have

$$\sigma(f(T))\backslash E(f(T)) = f(\sigma(T)\backslash E(T)).$$

Since $T \in g\mathcal{W}$, we have

$$\sigma(f(T))\backslash E(f(T)) = f(\sigma(T)\backslash E(T)) = f(\sigma_{BW}(T)) = \sigma_{BW}(f(T)).$$

which implies that $f(T) \in g\mathcal{W}$.

Now suppose that T^* is algebraically k-quasi-paranormal. We first show that $T \in g\mathcal{W}$. Let $\lambda \in \sigma(T)\backslash\sigma_{BW}(T)$. Observe that $\sigma(T^*) = \overline{\sigma(T)}$ and $\sigma_{BW}(T^*) = \overline{\sigma_{BW}(T)}$. So $\overline{\lambda} \in \sigma(T^*)\backslash\sigma_{BW}(T^*)$, and so $\overline{\lambda} \in E(T^*)$ because $T^* \in g\mathcal{W}$. Since T^* is algebraically k-quasi-paranormal, it follows from Theorem 3.4 that $\overline{\lambda} \in \pi(T^*)$. Hence $T-\lambda$ has finite ascent and descent, and so $\lambda \in E(T)$.

Conversely, suppose $\lambda \in E(T)$. Then λ is an isolated point of $\sigma(T)$ and $\alpha(T-\lambda) > 0$. Since $\sigma(T^*) = \overline{\sigma(T)}$, $\overline{\lambda}$ is an isolated point of $\sigma(T^*)$. Since T^* is isoloid, $\overline{\lambda} \in E(T^*)$. But $E(T^*) = \pi(T^*)$ by Theorem 3.4, hence we have $T-\lambda$ has finite ascent and descent. Therefore we can represent $T-\lambda$ as the direct sum

$$T-\lambda = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix},$$

where T_1 is invertible and T_2 is nilpotent.

Therefore $T-\lambda$ is B - Weyl, and so $\lambda \in \sigma(T)\backslash\sigma_{BW}(T)$. Thus $\sigma(T)\backslash\sigma_{BW}(T) = E(T)$, and hence $T \in g\mathcal{W}$. If T^* is algebraically k-quasi-paranormal then T is isoloid. It follows from the first part of the proof that $f(T) \in g\mathcal{W}$. This completes the proof. □

Corollary 3.6. *Suppose T or T^* is algebraically k-quasi-paranormal. Then*

$$\sigma_{BW}(f(T)) = f(\sigma_{BW}(T)) \text{ for every } f \in H(\sigma(T)).$$

An operator $X \in B(\mathcal{H})$ is called a quasiaffinity if it has trivial kernel and dense range. An operator $S2 \in B(\mathcal{H})$ is said to be a quasiaffine transform of $T \in B(\mathcal{H})$ (notation: $S \prec T$) if there is a quasiaffinity $X \in B(\mathcal{H})$ such that $XS = TX$. If both $S \prec T$ and $T \prec S$, then we say that S and T are quasisimilar.

Corollary 3.7. *Suppose T is algebraically k-quasi-paranormal and $S \prec T$. Then$f(S) \in ga\mathcal{B}$ for each $f \in H(\sigma(S))$.*

Proof. Suppose T is algebraically k-quasi-paranormal. Then T has SVEP. Since $S \prec T$, $f(S)$ has SVEP by [16, Lemma 3.1]. It follows from [27, Theorem 3.3.6] that $f(S)$ has SVEP. Therefore $f(S) \in ga\mathcal{B}$ by [2, Corollary 2.5]. □

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