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A FOURTH-ORDER FAMILY OF TRIPARAMETRIC EXTENSIONS OF JARRATT'S METHOD

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Abstract. A fourth-order family of triparametric extensions of Jarratt's method are proposed in this paper to find a simple root of nonlinear algebraic equations. Convergence analysis including numerical experiments for various test functions apparently verifies the fourth-order convergence and asymptotic error constants.

1. Introduction

Fourth-order iterative methods have been introduced by many researchers such as Argyros-Chen-Qian[1], Chun-Ham[2,3], Jarratt[4], King [5], Kou-Li-Wang[6,7], Noor-Ahmad[8], and Traub[9]. Especially, classical Jarratt's method[4] free from second derivatives has been widely used to numerically find a simple root of a nonlinear algebraic equation $f(x) = 0$. In this paper, a parametrically extended family of Jarratt's methods are proposed with their convergence results. Let $f : \mathbb{C} \to \mathbb{C}$ have a simple root α and be analytic in a small region containing α. A parametric family of two-step iterative methods are considered below: for $n = 0, 1, \cdots$,

$$\begin{cases} y_n = x_n - \gamma \cdot \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - K_f(x_n) \frac{f(x_n)}{\xi f'(x_n) + (1-\xi) f'(y_n)}, \end{cases} \tag{1.1}$$

where

$$K_f(x_n) = \frac{a f'(x_n)^2 + b f'(x_n) f'(y_n) + c f'(y_n)^2}{\theta f'(x_n)^2 + \omega f'(x_n) f'(y_n) + (1 - \theta - \omega) f'(y_n)^2}, \tag{1.2}$$

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with $a = \frac{1}{24}\{4\theta(4\xi+7)+10\omega+2\xi(4\omega+5)-1\}, b = -\frac{1}{6}\{3(\xi-1)+2\theta(4\xi+5)+(4\xi+3)\omega\}, c = \frac{1}{24}\{4\theta(3+4\xi)+2(1+4\xi)\omega+2\xi-3\}$ by means of three independent parameters ξ, θ and ω in $\mathbb{C}$.

A special case with a triple of parameters $(\xi, \theta, \omega) = (1, 0, -1/2)$ or $(1, -1/2, 3/2)$ yields classical Jarratt's method. Other interesting choices of pairs (ξ, θ, ω) will be discussed in Section 2. Observe that (1.1) has only three function evaluations per iteration and is free from second derivatives. The main aim of this paper is to show iteration scheme (1.1) has fourth-order convergence as well as to express the asymptotic error constant in terms of f, α and a triple of parameters (ξ, θ, ω).

2. Method development and convergence analysis

The convergence property of iterative method (1.1) is best illustrated in Theorem 2.1 stated below:

THEOREM 2.1. *Let f and α be described as in Section 1. Let $c_j = \frac{f^{(j)}(\alpha)}{j!f'(\alpha)}$ for $j = 2, 3, \cdots$. Assume that all three values c_2, c_3 and c_4 are not vanishing simultaneously. Let x_0 be an initial guess chosen in a sufficiently small region containing α. Let (ξ, θ, ω) be a pair of independent parameters to be freely chosen. Then iteration scheme (1.1) is of fourth-order and its asymptotic error constant η is given by*

$$\eta = \frac{1}{81}\left|-81c_2c_3+9c_4+c_2^3\{32\theta(7\xi+2)+8\omega(10\xi-1)-8\xi+125\}\right|. \quad (2.1)$$

Proof. Taylor series expansion of $f(x_n)$ about α up to fifth-order terms yields with $f(\alpha) = 0$:

$$f(x_n) = f'(\alpha)(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + O(e_n^6)), \quad (2.2)$$

where $e_n = x_n - \alpha$ for $n = 0, 1, 2, \cdots$. For ease of notation, e_n will be denoted by e (not to be confused with Napier's base for natural logarithms) for the time being. A lengthy algebraic computation induces relations (2.3)-(2.6) below:

$$f'(x_n) = f'(\alpha)(1 + 2c_2e + 3c_3e^2 + 4c_4e^3 + 5c_5e^4 + O(e^5)), \quad (2.3)$$

$$\frac{f(x_n)}{f'(x_n)} = e - c_2e^2 + 2(c_2^2 - c_3)e^3 + (-4c_2^3 + 7c_2c_3 - 3c4)e^4 + (8c_2^4 - 20c_2^2c_3 + 6c_3^2 + 10c_2c_4 - 4c_5)e^5 + O(e^6), \quad (2.4)$$

$$y_n = x_n - \gamma\frac{f(x_n)}{f'(x_n)} = \alpha + (1-\gamma)e + \gamma c_2 e^2$$
$$-2(c_2^2 - c_3)\gamma e^3 + \gamma(4c_2^3 - 7c_2c_3 + 3c4)e^4$$
$$+2\gamma(4c_2^4 - 10c_2^2c_3 + 3c_3^2 + 5c_2c_4 - 2c_5)e^5 + O(e^6), \quad (2.5)$$

$$f'(y_n) = f'(\alpha)(1 - 2(\gamma-1)c_2e + (3(\gamma-1)^2c_3 + 2\gamma c_2^2)e^2 + 2(-2(\gamma-1)^3c_4$$
$$+\gamma c_2(-2c_2^2 + c_3(5-3\gamma)))e^3 + (5(\gamma-1)^4c_5 + \gamma(8c_2^4 - 12(\gamma-1)c_3^2$$
$$+(15\gamma - 26)c_2^2c_3 + 6(2\gamma^2 - 4\gamma + 3)c_2c_4))e^4 + O(e^5)). \quad (2.6)$$

Let $K_f(x_n)$ in (1.2) be written in a more generalized form as

$$K_f(x_n) = \frac{\lambda f'(x_n)^2 + \delta f'(x_n)f'(y_n) + \mu f'(y_n)^2}{\beta f'(x_n)^2 + \rho f'(x_n)f'(y_n) + \sigma f'(y_n)^2}, \quad (2.7)$$

where $\gamma, \lambda, \delta, \mu, \beta, \rho$ and σ are constant parameters to be determined later. Substituting relations (2.1) - (2.7) in (1.1) by the aid of symbolic computation of Mathematica, x_{n+1} can be written as with $\beta+\rho+\sigma \neq 0$:

$$x_{n+1} = \alpha + \frac{A_1}{(\beta+\rho+\sigma)}e + \frac{A_2}{(\beta+\rho+\sigma)^2}e^2 + \frac{A_3}{(\beta+\rho+\sigma)^3}e^3$$
$$+\frac{A_4}{(\beta+\rho+\sigma)^4}e^4 + O(e^5), \quad (2.8)$$

where $A_i = A_i(\xi, \lambda, \delta, \mu, \beta, \rho, \sigma, \gamma)(i = 1, 2, 3, 4)$ are multivariate polynomials in $\xi, \lambda, \delta, \mu, \beta, \rho, \sigma$ and γ, for instance, $A_1 = (\gamma-1)(\beta+\rho+\sigma) + \lambda + \delta + \mu$. We impose conditions $A_1 = A_2 = A_3 = 0$ and $A_4 \neq 0$ for iteration scheme (1.1) to have fourth-order convergence. Solving $A_1 = 0$ for μ yields

$$\mu = -\lambda - \delta + (1-\gamma)(\beta+\rho+\sigma). \quad (2.9)$$

Substituting this μ into $A_2 = 0$ after simplification with $\beta+\rho+\sigma \neq 0$ yields

$$c_2\{\beta(1 + 2\gamma(1+\xi) - 2\gamma^2(1+\xi)) + (1 + 2\gamma\xi - 2\gamma^2\xi)\rho$$
$$+(1 + 2\gamma(-1+\xi) - 2\gamma^2(-1+\xi))\sigma - 4\gamma\lambda\} = 0,$$

from which it follows that

$$\delta = -\frac{\beta(4\gamma^2 - 4\gamma - 1) + \rho(2\gamma^2 - 2\gamma - 1) + 4\gamma\lambda - \sigma}{2\gamma}, \ \gamma \neq 0, \quad (2.10)$$

being independent of c_2. Substituting μ and δ found by (2.9) and (2.10) into $A_2 = 0$ after simplification with $\beta + \rho + \sigma \neq 0$ yields

$$c_3 \frac{(3\gamma - 2)}{2} + 2c_2^2 \frac{\beta(1 + \gamma\xi + 2\gamma^2\xi - 2\gamma^3\xi) + (1 - \gamma + \gamma\xi)\rho + \kappa\sigma - 2\gamma^2\lambda}{(\beta + \rho + \sigma)} = 0, \tag{2.11}$$

where $\kappa = 1 - 2\gamma + 2\gamma^2 - 2\gamma^3 + \gamma(1 - 2\gamma + 2\gamma^2)\xi$. To determine $\xi, \beta, \gamma, \rho, \sigma, \lambda$ independently of c_2 and c_3, we set $(3\gamma - 2) = 0$ and $\beta(1 + \gamma\xi + 2\gamma^2\xi - 2\gamma^3\xi) + (1 - \gamma + \gamma\xi)\rho + \kappa\sigma - 2\gamma^2\lambda = 0$ and get

$$\lambda = \frac{1}{24}\{\beta(26\xi + 27) + 9(1 + 2\xi)\rho + (10\xi - 1)\sigma\},\ \gamma = \frac{2}{3}. \tag{2.12}$$

Substituting these λ and γ into (2.9) and (2.10) also yields

$$\mu = \frac{1}{24}\{9\beta(1 + 2\xi) + (10\xi - 1)\rho + (2\xi - 3)\sigma\},$$
$$\delta = -\frac{1}{6}\{\beta(11\xi + 7) + 7\xi\rho + 3(\xi - 1)\sigma\}. \tag{2.13}$$

Substituting these μ, σ, λ and γ into A_4 and restoring notation e back to e_n in (2.8) yields the asymptotic error constant η with convergence order 4 as follows:

$$\eta = \lim_{n\to\infty} \left|\frac{e_{n+1}}{e_n^4}\right| = \frac{1}{9}\left|c_4 - 9c_2c_3 + c_2^3 \frac{27\beta(7 + 8\xi) + 9(8\xi + 13)\rho - (8\xi - 125)\sigma}{9(\beta + \rho + \sigma)}\right|, \tag{2.14}$$

where ξ, β, ρ, σ are free parameters, but only two of β, ρ, σ can be chosen independently of each other. To see this, we let $\tau = \beta + \rho + \sigma \neq 0$, $\beta = \tau\theta$ and $\rho = \tau\omega$. Then we have $\sigma = \tau(1 - \theta - \omega)$ and

$$\frac{27\beta(7 + 8\xi) + 9(8\xi + 13)\rho - (8\xi - 125)\sigma}{\beta + \rho + \sigma} = 125 - 8\xi + 32\theta(2 + 7\xi) + 8(-1 + 10\xi)\omega \tag{2.15}$$

as well as

$$\lambda = \frac{1}{24}\tau\{4\theta(4\xi + 7) + 10\omega + 2\xi(4\omega + 5) - 1\},$$
$$\delta = -\frac{1}{6}\tau\{3(\xi - 1) + 2\theta(4\xi + 5) + (4\xi + 3)\omega\},$$
$$\mu = \frac{1}{24}\tau\{4\theta(3 + 4\xi) + 2(1 + 4\xi)\omega + 2\xi - 3\}. \tag{2.16}$$

Substituting $\lambda, \delta, \mu, \beta, \rho, \sigma$ expressed by θ and ω into (2.7) finally yields (1.2) after simplifications. Combining (2.15) with (2.14) immediately gives the desired asymptotic error constant (2.1). This completes the proof. $\square$

REMARK 2.2. (1) Although a parameter $\tau \neq 0$ was introduced, relations (1.2) and (2.1) are free of τ, acting only as a scale factor of originally given parameters $\xi, \lambda, \delta, \mu, \beta, \rho$ and σ.

(2) As mentioned in Section 1, a choice of $(\xi, \theta, \omega) = (1, 0, -1/2)$ or $(1, -1/2, 3/2)$ leads to an expression of $K_f(x_n)$ in the form:

$$K_f(x_n) = -\frac{1}{6} \cdot \frac{7f'(x_n) - 3f'(y_n)}{f'(x_n) - 3f'(y_n)}, \tag{2.17}$$

which is identical with that of classical Jarratt's method.

(3) Other interesting choices of (ξ, θ, ω) are $(1, 0, 0)$ and $(1, 0, 1)$, including a vairety of choices displayed in Table 1. Such choices define many new iterative methods of order 4, among which cases 1, 2 and 7 are paid attention to the numerical experiments to be shown in Section 3. Notaional convenience $V_f(v_n) = K_f(x_n)$ with $v_n = f'(y_n)/f'(x_n)$ plays a role in coding numerical Algorithm 3.1.

TABLE 1. Various choices of (ξ, θ, ω) for $K_f(x_n)$ or $V_f(v_n)$

Case	(ξ, θ, ω)	$K_f(x_n)$	$V_f(v_n)^*$
0	$(1, 0, -\frac{1}{2})$ or $(1, -\frac{1}{2}, \frac{3}{2})$	$-\frac{1}{6} \cdot \frac{7f'(x_n)-3f'(y_n)}{f'(x_n)-3f'(y_n)}$	$-\frac{1}{6}\left(\frac{7-3v_n}{1-3v_n}\right)$
1	$(1, 0, 0)$	$\frac{1}{24} \cdot \frac{9f'(x_n)^2-f'(y_n)^2}{f'(y_n)^2}$	$\frac{3}{8v_n^2} - \frac{1}{24}$
2	$(1, 0, 1)$	$\frac{3f'(y_n)}{8f'(x_n)} - \frac{9f'(x_n)}{8f'(y_n)} - \frac{7}{6}$	$\frac{3}{8}v_n - \frac{9}{8v_n} - \frac{7}{6}$
3	$(1, 1, 0)$	$\frac{9f'(y_n)^2}{8f'(x_n)^2} - \frac{3f'(y_n)}{8f'(x_n)} + \frac{53}{24}$	$\frac{9v_n^2}{8} - \frac{3v_n}{8} + \frac{53}{24}$
4	$(1, -\frac{9}{44}, 0)$	$-\frac{f'(y_n)}{3} \cdot \frac{81f'(x_n)-37f'(y_n)}{9f'(x_n)^2-53f'(y_n)^2}$	$\frac{1}{3}\left(\frac{37-81/v_n^2}{9/v_n^2-53}\right)$
5	$(1, 0, \frac{1}{10})$	$\frac{1}{6} \cdot \frac{f'(x_n)}{f'(y_n)} \cdot \frac{27f'(x_n)-7f'(y_n)}{f'(x_n)+9f'(y_n)}$	$\frac{1}{6}\left(\frac{27/v_n-7}{1+9v_n}\right)$
6	$(1, 0, \frac{1}{28})$	$\frac{f'(x_n)}{3} \cdot \frac{37f'(x_n)-9f'(y_n)}{f'(x_n)^2+27f'(y_n)^2}$	$\frac{1}{3}\left(\frac{37-9v_n}{1+27v_n^2}\right)$
7	$(1, -\frac{7}{11}, \frac{18}{11})$	$\frac{1}{24f'(x_n)} \cdot \frac{27f'(y_n)^2-115f'(x_n)^2}{7f'(x_n)-18f'(y_n)}$	$\frac{1}{24}\left(\frac{27v_n^2-115}{7-18v_n}\right)$
8	$(1, -\frac{27}{26}, \frac{53}{26})$	$-\frac{1}{6} \cdot \frac{f'(y_n)}{f'(x_n)} \cdot \frac{115f'(x_n)-63f'(y_n)}{27f'(x_n)-53f'(y_n)}$	$-\frac{1}{6}\left(\frac{115-63v_n}{27/v_n-53}\right)$
9	$(1, \frac{27}{16}, -\frac{37}{8})$	$\frac{16}{3} \cdot \frac{f'(x_n)f'(y_n)}{27f'(x_n)^2-74f'(x_n)f'(y_n)+63f'(y_n)^2}$	$\frac{16}{3(27/v_n-74+63v_n)}$
10	$(1, \frac{7}{16}, -\frac{9}{8})$	$\frac{16}{3} \cdot \frac{f'(x_n)^2}{7f'(x_n)^2-18f'(x_n)f'(y_n)+27f'(x_n)^2}$	$\frac{16}{3(7-18v_n+27v_n^2)}$
11	$(1, \frac{63}{16}, -\frac{81}{8})$	$\frac{16}{3} \cdot \frac{f'(y_n)^2}{63f'(x_n)^2-162f'(x_n)f'(y_n)+115f'(y_n)^2}$	$\frac{16}{3(63/v_n^2-162/v_n+115)}$

* $V_f(v_n) = K_f(x_n),\ v_n = f'(y_n)/f'(x_n),\ y_n = x_n - f(x_n)/f'(x_n)$.

3. Algorithm, numerical results and discussions

The analysis described in Section 2 allows us to develop a zero-finding algorithm to be implemented with *Mathematica*[10]:

Algorithm 3.1 (Zero-Finding Algorithm)
Step 1. Construct iteration scheme (1.1) with the given function f having a simple zero α for $n \in \mathbb{N} \cup \{0\}$ as mentioned in Section 1.
Step 2. Set the minimum number of precision digits. With exact zero α or most accurate zero, supply the asymptotic error constant η, order of convergence p as well as c_2, c_3, c_4, θ and ω stated in Section 2. Set the error bound ϵ, the maximum iteration number n_{max} and the initial guess x_0. Compute $|f(x_0)|$ and $|x_0 - \alpha\ |$.
Step 3. Tabulate the computed values of $n,\ x_n,\ |f(x_n)|,\ |e_n| = |x_n - \alpha|$, $|e_{n+1}/{e_n}^p|$ and η.

Throughout the numerical experiments, the minimum number of precision digits was chosen as 256, being large enough to minimize round-off errors as well as to clearly observe the computed asymptotic error constant requiring small-number divisions. The error bound $\epsilon = 0.5 \times 10^{-192}$ was used for moderately accurate computation. The values of initial guess x_0 were selected closely to α to guarantee convergence of iterative methods. The computed asymptotic error constant agrees up to 10 significant digits with the theoretical one. The computed zero is actually rounded to be accurate up to the 192 significant digits, although displayed only up to 15 significant digits.

Iteration scheme (1.1) applied to test functions $f(x) = \sin^2 x - x^2 + 1$ and $e^{-x}\sin x + \log[1 + (x - \pi)^2]$ clearly shows successful asymptotic error constants with fourth-order convergence for suitable initial values chosen near α. Tables 2 and 3 list iteration indexes n, approximate zeros x_n, residual errors $|f(x_n)|$, errors $|e_n| = |x_n - \alpha|$ and computational asymptotic error constants $|e_{n+1}/{e_n}^4|$ as well as the theoretical asymptotic error constant η.

Convergence behavior was confirmed for further test functions that are listed below:

$f_1(x) = \cos(\frac{\pi x}{2}) + \log(x^2 + 2x + 2),\ \alpha = -1,\ x_0 = -0.8$
$f_2(x) = x^3 + 4x^2 - 10,$
$\alpha = -\frac{4}{3} + \frac{1}{3}(71 - 3\sqrt{105})^{1/3} + \frac{1}{3}(71 + 3\sqrt{105})^{1/3}, x_0 = 0.8$
$f_3(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5,\ \alpha = -1.207647827130918, x_0 = -2$
$f_4(x) = e^x \sin x + \log(1 + x^2),\ \alpha = 0, x_0 = 0.1$
$f_5(x) = (x^2 + 1)\sin(\frac{\pi}{x}) + \frac{1}{(x^4+1)} + \frac{84}{17},\ \alpha = -2, x_0 = -1.2$

TABLE 2. Asymptotic error constant for $f(x) = \sin^2 x - x^2 + 1$ with $\alpha \approx 1.40449164821534$

n	x_n	$\|f(x_n)\|$	$\|e_n\| = \|x_n - \alpha\|$	$\|e_{n+1}/e_n{}^4\|$	η
0	0.8	0.874600	0.604492		
1	1.72313078087797	0.992207	0.318639	2.386370609	
2	1.40723164558655	0.00681658	0.00274	0.2657992194	
3	1.40449164824959	8.50264×10^{-11}	3.42507×10^{-11}	0.6076716039	0.6121130898
4	1.40449164821534	2.09119×10^{-42}	8.42383×10^{-43}	0.6121130898	
5	1.40449164821534	7.65165×10^{-169}	3.08227×10^{-169}	0.6121130898	
6	1.40449164821534	$0.\times10^{-256}$	$0.\times10^{-255}$		

TABLE 3. Convergence for $f(x) = e^{-x}\sin x + \log[1+(x-\pi)^2]$ with $\alpha = \pi$

n	x_n	$\|f(x_n)\|$	$\|e_n\| = \|x_n - \alpha\|$	$\|e_{n+1}/e_n{}^4\|$	η
0	2.0	0.957374	1.14159		
1	2.94096965534875	0.0499855	0.200623	0.1181234316	
2	3.10197254278276	0.00334939	0.0396201	24.45641638	
3	3.13862854247199	0.000137257	0.00296411	1202.906127	20311.89896
4	3.14159149745019	4.99627×10^{-8}	1.15614×10^{-6}	14977.25958	
5	3.14159265358979	1.56806×10^{-21}	3.62859×10^{-20}	20309.40092	
6	3.14159265358979	1.52169×10^{-75}	3.52130×10^{-74}	20311.89896	
7	3.14159265358979	$0.\times10^{-256}$	$0.\times10^{-255}$		

TABLE 4. Comparison of (ν, N_f) for various iterative methods

		(ν, N_f)								
$f(x)$	x_0	NM	JM	TM	KM	KLW	CH	YK1	YK2	YK7
f_1	-0.8	(8,16)	(4,12)	(4,12)	(4,12)	(4,12)	(4,12)	(4,12)	(4,12)	(4,12)
f_2	0.8	(9,18)	(5,15)	(5,15)	(7,21)	(5,15)	(5,15)	(5,15)	((5,15)	(5,15)
f_3	-2	(12,24)	(6,18)	(6,18)	(7,21)	(6,18)	(7,21)	(6,18)	(6,18)	(5,15)
f_4	0.1	(8,16)	(4,12)	(4,12)	(5,15)	(5,15)	(5,15)	(5,15)	(5,15)	(4,12)
f_5	-1.2	(9,18)	(5,15)	(5,15)	(5,15)	(5,15)	(5,15)	(4,12)	(4,12))	(5,15)
f_6	$1.5i$	(9,18)	(5,15)	(5,15)	(6,18)	(5,15)	(5,15)	(5,15)	(5,15)	(4,12)
f_7	0.9	(8,16)	(5,15)	(5,15)	(5,15)	(5,15)	(5,15))	(5,15)	(5,15)	(5,15)

$$f_6(x) = x^2+\pi-\sin x^2+\log(x^2+\pi+1),\ \alpha = i\sqrt{\pi},\ x_0 = 1.5i,\ i = \sqrt{-1}$$
$$f_7(x) = x^4 + \sin(\tfrac{\pi}{x^2}) - 5,\ \alpha = \sqrt{2}, x_0 = 0.9$$

Table 4 lists pairs of iteration number ν and number of function evaluations N_f within the prescribed error bound for various fourth-order methods with the same efficiency index[9] plus classical Newton's method.

The listed method names are abbreviated by the following:

NM: classical Newton's method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

JM: Jarratt's method:

$$x_{n+1} = x_n - \big(1 - \frac{3}{2} \cdot \frac{f'(x_n) - f'(y_n)}{f'(x_n) - 3f'(y_n)}\big) \cdot \frac{f(x_n)}{f'(x_n)},\ with\ y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}.$$

TM: Traub-Ostrowski's method:

$$x_{n+1} = x_n - \frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)} \cdot \frac{f(x_n)}{f'(x_n)},\ with\ y_n = x_n - \frac{f(x_n)}{f'(x_n)}.$$

KM: King's method with $\beta = 3$:

$$x_{n+1} = y_n - \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2)f(y_n)} \cdot \frac{f(x_n)}{f'(x_n)},\ with\ y_n = x_n - \frac{f(x_n)}{f'(x_n)}.$$

KLW: Kou-Li-Wang's method[7]:

$$x_{n+1} = x_n - \frac{f(x_n)^2 + f(y_n)^2}{(f(x_n) - f(y_n))} \cdot \frac{1}{f'(x_n)},\ with\ y_n = x_n - \frac{f(x_n)}{f'(x_n)}.$$

CH: Chun-Ham's method[3]:

$$x_{n+1} = y_n - \big(\frac{f(x_n) + f(y_n)}{f(x_n)}\big)^2 \cdot \frac{f(y_n)}{f'(x_n)},\ with\ y_n = x_n - \frac{f(x_n)}{f'(x_n)}.$$

YKi: proposed method (1.1) that is identified by case number $i(1, 2, 7)$ in Table 1.

The comparison in Table 4 suggests that proposed method (1.1) can compete with classical Jarratt's method. In particular, YK7 shows slightly better performance than other listed methods. The efficiency index[9] defined by $^*EFF = p^{1/d}$, with p as the order of convergence and d the number of new evaluations of $f(x)$ or its derivatives per iteration, is found to be $4^{1/3} \approx 1.5874$ which is better than $\sqrt{2}$, the efficiency index of Newton's method.

References

[1] I.K. Argyros, D. Chen, Q. Qian, *The Jarratt method in Banach space setting*, J. Comput. Appl. Math. **51** (1994), 103-106.

[2] C. Chun, Y. Ham, *A one-parameter fourth-order family of iterative methods for nonlinear equations*, Applied Mathematics and Computation **189** (2007), Issue 1, 610-614.

[3] C. Chun, Y. Ham, *Some fourth-order modifications of Newton's method*, Applied Mathematics and Computation, **197** (2008), 654-658.

[4] P. Jarratt, *Some fourth-order multipoint iterative methods for solving equations*, Math. Comput. **20** (1966), no. 95, 434-437.

[5] R. King, *A family of fourth-order methods for nonlinear equations*, SIAM J. Numer. Anal. **10** (1973), no. 5, 876-879.

[6] J. Kou, Y. Li and X. Wang, *Fourth-order iterative methods free from second derivative*, Applied Mathematics and Computation **184** (2007), Issue 2, 880-885.

[7] J. Kou, Y. Li and X. Wang, *A composite fourth-order iterative method for solving non-linear equations*, Int. J. Comput. Math. **184** (2007), 471-475.

[8] M. Noor, F. Ahmad, *Fourth-order convergent iterative method for non-linear equation*, Applied Mathematics and Computation **182** (2006), Issue 2, 1149-1153.

[9] J. F. Traub, *Iterative Methods for the Solution of Equations*, Chelsea Publishing Company, 1982.

[10] S. Wolfram, *The Mathematica Book, 4th ed.*, Cambridge University Press, 1999.

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