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FRACTIONAL RELLICH-KONDRACHOV COMPACTNESS THEOREM

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Abstract. It is proved that the fractional Sobolev spaces $W_p^s(\mathbb{R}^n)$, $0 < s < n$, are compactly embedded into Lebesgue spaces $L^q(\Omega)$ for some bounded set Ω.

1. The main result

It has been derived a fractional version of Rellich-Kondrachov compactness theorem. The classical theorem says that some Sobolev spaces $W_p^1(\mathbb{R}^n)$ *with regularity one* are compactly embedded to some Lebesgue spaces $L^q(\Omega)$ for some bounded open set Ω (with smooth boundary). This paper proves that one may still have the same kind of compactness result with only small amount of regularity s, $0 < s < 1$. The result is stated as follows:

Theorem 1.1. *Let $0 < s < n$, $1 < p < \frac{n}{s}$ and $1 \leq q \leq \frac{np}{n-sp}$. Also, let $\{u_m\}$ be a sequence in $L^q(\mathbb{R}^n)$ and Ω be a bounded open set with smooth boundary. Suppose that*

$$\int_{\mathbb{R}^n} |\sqrt{1-\Delta}^s u_m(x)|^p dx$$

are uniformly bounded, then $\{u_m\}$ has a convergent subsequence in $L^q(\Omega)$.

Among some equivalent definitions of the fractional Laplacian, we employ it as

$$\sqrt{-\Delta}^s \phi := \mathcal{F}^{-1}(|\cdot|^s \mathcal{F}(\phi))$$

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and also $\sqrt{1-\Delta}^{s}\phi := \mathcal{F}^{-1}\left((1+|\cdot|)^{s}\mathcal{F}(\phi)\right)$, where $\hat{u}=\mathcal{F}(u)$ represents the Fourier transform of u on $\mathbb{R}^n$ defined by

$$\hat{f}(\xi)=\mathcal{F}(f)(\xi)=\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,dx$$

for $f\in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$.

2. The proof

Let ϕ be a smooth non-negative function with support in $\{x:|x|\le 1\}$ and with $\int_{|x|\le 1}\phi(x)dx=1$, and define $\phi^{\ell}(x):=\ell^n\phi(\ell x)$. In this proof, the notation $X\lesssim Y$ means that $X\le CY$ for some fixed but unspecified constant C.

By virtue of the fractional Sobolev inequality[3, 6], it can be observed that

$$\|u_m\|_{L^r(K)}\lesssim\left\|\sqrt{-\Delta}^{s}u_m\right\|_{L^p(\mathbb{R}^n)}\lesssim\left\|\sqrt{1-\Delta}^{s}u_m\right\|_{L^p(\mathbb{R}^n)}\lesssim 1 \tag{1}$$

with $r=\frac{np}{n-ps}$ for any compact subset K of Ω. Hence in the *spirit* of Frechet-Kolmogorov theorem, it suffices to show the following (see page 50 in [7]): for any $\varepsilon>0$ and any compact subset K of Ω, there is a constant $M>0$ such that for $\ell\ge M$,

$$\|\phi^{\ell}*u-u\|_{L^q(K)}<\varepsilon,$$

for all $u\in\mathcal{S}(\mathbb{R}^n)$ with $\|\sqrt{1-\Delta}^{s}u\|_{L^p(\mathbb{R}^n)}\lesssim 1$. Then using the interpolation inequality, we have

$$\|\phi^{\ell}*u-u\|_{L^q(K)}\le 2^{1-\theta}\|u\|_{L^r(K)}^{1-\theta}\|\phi^{\ell}*u-u\|_{L^1(K)}^{\theta},$$

with $\frac{1-\theta}{r}+\theta=\frac{1}{q}$ and $r=\frac{np}{n-ps}$. Consequently, (1) implies that

$$\|\phi^{\ell}*u-u\|_{L^q(K)}\lesssim\|\phi^{\ell}*u-u\|_{L^1(K)}^{\theta}.$$

Now we define $g:=\sqrt{1-\Delta}^{s}u$ to have $u=G_s*g$ and $\|g\|_{L^p}\lesssim 1$, where G_s is the Bessel kernel of order s. Therefore we obtain

$$\|\phi^{\ell}*u-u\|_{L^1(K)}\lesssim\|(\phi^{\ell}*G_s-G_s)*g\|_{L^p(\mathbb{R}^n)}\lesssim\|\phi^{\ell}*G_s-G_s\|_{L^1(\mathbb{R}^n)}\to 0$$

as $\ell\to\infty$. This completes the proof. □

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