

q -EXTENSION OF A GENERALIZATION OF GOTTLIEB POLYNOMIALS IN TWO VARIABLES

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ABSTRACT. Gottlieb polynomials were introduced and investigated in 1938, and then have been cited in several articles. Very recently Khan and Akhlaq introduced and investigated Gottlieb polynomials in two and three variables to give their generating functions. Subsequently, Khan and Asif investigated the generating functions for the q -analogue of Gottlieb polynomials. Also, by modifying Khan and Akhlaq's method, Choi presented a generalization of the Gottlieb polynomials in m variables to give two generating functions of the generalized Gottlieb polynomials $\varphi_n^m(\cdot)$. Here, we aim at defining a q -extension of the generalized two variable Gottlieb polynomials $\varphi_n^2(\cdot)$ and presenting their several generating functions.

1. Introduction and preliminaries

Generating functions play an important role in the investigation of various useful properties of the sequences which they generate. They are used in finding certain properties and formulas for numbers and polynomials in a wide variety of research subjects, indeed, in modern combinatorics. For a systematic introduction to, and several interesting (and useful) applications of the various methods of obtaining linear, bilinear, bilateral or mixed multilateral generating functions for a fairly wide variety of sequences of special functions (and polynomials) in one, two and more variables, among much abundant literature, we refer to the extensive work by Srivastava and Manocha [10]. While concerning some orthogonal polynomials on a finite or enumerable set of points, Gottlieb

Received January 12, 2012; Accepted April 17, 2012.

2010 Mathematics Subject Classification: Primary 33C65, 33C99; Secondary 33C05, 33C20.

Key words and phrases: Pochhammer symbol, generating functions, generalized hypergeometric function ${}_pF_q$, (generalized) Gottlieb polynomials, Lauricella series, a q -analogue of Gottlieb polynomials, q -shifted factorial, q -binomial theorem, Basic (q -) hypergeometric series.

[4] developed the following interesting polynomials (see also [2]; [5]; [6]; [8, p. 303]; [10, pp. 185–186]):

$$(1.1) \quad \begin{aligned} \varphi_n(x; \lambda) &:= e^{-n\lambda} \sum_{k=0}^n \binom{n}{k} \binom{x}{k} (1 - e^\lambda)^k \\ &= e^{-n\lambda} {}_2F_1 \left(-n, -x; 1; 1 - e^\lambda \right), \end{aligned}$$

where ${}_2F_1$ denotes Gauss's hypergeometric series whose natural generalization of an arbitrary number of p numerator and q denominator parameters ($p, q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and \mathbb{N} the set of positive integers) is called and denoted by the generalized hypergeometric series ${}_pF_q$ defined by

$$(1.2) \quad \begin{aligned} {}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n z^n}{(\beta_1)_n \cdots (\beta_q)_n n!} \\ &= {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z). \end{aligned}$$

Here $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by

$$(1.3) \quad \begin{aligned} (\lambda)_n &:= \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (n \in \mathbb{N}) \end{cases} \\ &= \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-) \end{aligned}$$

and \mathbb{C} and \mathbb{Z}_0^- denotes the set of nonpositive integers and the set of complex numbers, respectively, and $\Gamma(\lambda)$ is the familiar Gamma function.

Gottlieb [4] presented many interesting identities for his polynomials $\varphi_n(x; \lambda)$, which is denoted by $l_n(x)$ in [4], including the following two generating functions (see also [5]; [6]; [8, p. 303]; [10, pp. 185–186]):

$$(1.4) \quad \sum_{n=0}^{\infty} \varphi_n(x; \lambda) t^n = (1 - t)^x (1 - t e^{-\lambda})^{-x-1} \quad (|t| < 1);$$

$$(1.5) \quad \begin{aligned} \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \varphi_n(x; \lambda) t^n \\ = (1 - t e^{-\lambda})^{-\mu} {}_2F_1 \left[\begin{matrix} \mu, -x; \\ 1; \end{matrix} \frac{(1 - e^{-\lambda}) t}{1 - t e^{-\lambda}} \right]. \end{aligned}$$

Recently Khan and Akhlaq [5] introduced and investigated Gottlieb polynomials in two and three variables to give their generating functions. Subsequently, Khan and Asif [6] investigated the generating functions

for a q -analogue of Gottlieb polynomials. Also, by modifying Khan and Akhlaq's method [5], Choi [2] presented a generalization of the Gottlieb polynomials in m variables to give two generating functions of the generalized Gottlieb polynomials $\varphi_n^m(\cdot)$.

Basic (or q -) hypergeometric series are useful in a wide variety of fields including, for example, theory of partitions, number theory, combinatorial analysis, finite vector space, Lie theory, particle physics, non-linear electric circuit theory, mechanical engineering, theory of heat conduction, quantum mechanics, cosmology, and statistics (see [9, 346–351] and also see the cited references therein).

Here, we aim at defining a q -extension of the generalized two variable Gottlieb polynomials $\varphi_n^2(\cdot)$ and presenting their three generating functions.

For our purpose we recall here the following definitions and notations in the q -theory (see, for example, [3]). The q -shifted factorial $(a; q)_n$ is defined by

$$(1.6) \quad (a; q)_n := \begin{cases} 1 & (n = 0) \\ \prod_{k=0}^{n-1} (1 - a q^k) & (n \in \mathbb{N}), \end{cases}$$

where $a, q \in \mathbb{C}$ and it is assumed that $a \neq q^{-m}$ ($m \in \mathbb{N}_0$). It is noted that some other notations that have been used in the literature for the product $(a; q)_n$ in (1.6) are $(a)_{q,n}$, $[a]_n$, and even $(a)_n$ when the Pochhammer symbol (1.3) is not used and the base q is understood.

The q -shifted factorial for negative subscript is defined by

$$(1.7) \quad (a; q)_{-n} := \frac{1}{(1 - a q^{-1})(1 - a q^{-2}) \cdots (1 - a q^{-n})} \quad (n \in \mathbb{N}_0),$$

which yields

$$(1.8) \quad (a; q)_{-n} = \frac{1}{(a q^{-n}; q)_n} = \frac{(-q/a)^n q^{\binom{n}{2}}}{(q/a; q)_n} \quad (n \in \mathbb{N}_0).$$

We also write

$$(1.9) \quad (a; q)_\infty := \prod_{k=0}^{\infty} (1 - a q^k) \quad (a, q \in \mathbb{C}; |q| < 1).$$

It is noted that, when $a \neq 0$ and $|q| \geq 1$, the infinite product in (1.9) diverges. So, whenever $(a; q)_\infty$ is involved in a given formula, the constraint $|q| < 1$ will be tacitly assumed.

It follows from (1.6), (1.7) and (1.9) that

$$(1.10) \quad (a; q)_n = \frac{(a; q)_\infty}{(a q^n; q)_\infty} \quad (n \in \mathbb{Z}),$$

\mathbb{Z} being the set of integers, which can be extended to $n = \alpha \in \mathbb{C}$ as follows:

$$(1.11) \quad (a; q)_\alpha = \frac{(a; q)_\infty}{(a q^\alpha; q)_\infty} \quad (\alpha \in \mathbb{C}; |q| < 1),$$

where the principal value of q^α is taken.

Two easily-verified required identities are given:

$$(1.12) \quad (q^{-n}; q)_k = \frac{(q; q)_n}{(q; q)_{n-k}} (-1)^k q^{\binom{k}{2} - nk} \quad (n, k \in \mathbb{Z})$$

and

$$(1.13) \quad (a; q)_{n+k} = (a; q)_n (a q^n; q)_k \quad (n, k \in \mathbb{Z}).$$

The notation $[z]_q$ is defined by

$$(1.14) \quad [z]_q := \frac{1 - q^z}{1 - q} = \frac{q^z - 1}{q - 1} \quad (z \in \mathbb{C}; q \in \mathbb{C} \setminus \{1\}; q^z \neq 1).$$

A special case of (1.14) when $z \in \mathbb{N}$ is

$$(1.15) \quad [n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1} \quad (n \in \mathbb{N}),$$

which is called the q -analogue (or q -extension) of $n \in \mathbb{N}$, since

$$\lim_{q \rightarrow 1} [n]_q = \lim_{q \rightarrow 1} (1 + q + \cdots + q^{n-1}) = n.$$

The q -analogue of $n!$ is then defined by

$$(1.16) \quad [n]_q! := \begin{cases} 1 & \text{if } n = 0, \\ [n]_q [n-1]_q \cdots [2]_q [1]_q & \text{if } n \in \mathbb{N}, \end{cases}$$

from which the q -binomial coefficient (or the *Gaussian polynomial*) analogous to $\binom{n}{k}$ is defined by

$$(1.17) \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[n-k]_q! [k]_q!} \quad (n, k \in \mathbb{N}_0; 0 \leq k \leq n).$$

It is easily seen from (1.6) and (1.17) that

$$(1.18) \quad (q; q)_n = (1 - q)^n [n]_q! \quad (n \in \mathbb{N}_0).$$

The q -binomial coefficient in (1.17) can be generalized as follows:

$$(1.19) \quad \begin{bmatrix} \alpha \\ k \end{bmatrix}_q := \frac{[\alpha]_{q;k}}{[k]_q!} \quad (\alpha \in \mathbb{C}; k \in \mathbb{N}_0),$$

where $[\alpha]_{q;k}$ is defined by

$$(1.20) \quad [\alpha]_{q;k} := [\alpha]_q [\alpha - 1]_q \cdots [\alpha - k + 1]_q \quad (\alpha \in \mathbb{C}; k \in \mathbb{N}_0).$$

The generalized q -binomial coefficient in (1.19) can be expressed in the following form:

$$(1.21) \quad \begin{bmatrix} \alpha \\ k \end{bmatrix}_q = \frac{(q^{-\alpha}; q)_k}{(q; q)_k} (-q^\alpha)^k q^{-\binom{k}{2}} \quad (\alpha \in \mathbb{C}; k \in \mathbb{N}_0).$$

The following notations are also frequently used:

$$(1.22) \quad (a_1, a_2, \dots, a_m; q)_n := (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n$$

and

$$(1.23) \quad (a_1, a_2, \dots, a_m; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty.$$

In order to introduce q -binomial theorem, we begin by recalling the well-known Ramanujan's ${}_1\Psi_1$ -sum:

$$(1.24) \quad {}_1\Psi_1(a; b; q, z) := \sum_{k=-\infty}^{\infty} \frac{(a; q)_k}{(b; q)_k} z^k = \frac{(az; q)_\infty \left(\frac{a}{az}; q\right)_\infty (q; q)_\infty \left(\frac{b}{a}; q\right)_\infty}{(z; q)_\infty \left(\frac{b}{az}; q\right)_\infty (b; q)_\infty \left(\frac{a}{a}; q\right)_\infty}$$

$$(|q| < 1; |a| > |q|; |b| < 1; |b/a| < |z| < 1).$$

A special case of (1.24) when $b = q$ yields the q -binomial theorem:

$$(1.25) \quad {}_1\Phi_0(a; -; q, z) := \sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} z^k = \frac{(az; q)_\infty}{(z; q)_\infty} \quad (|q| < 1; |z| < 1).$$

Two special cases of (1.25) when $a = 0$ and when z is replaced by $z a^{-1}$ and $a \rightarrow \infty$ yield Euler's formulas:

$$(1.26) \quad \sum_{k=0}^{\infty} \frac{z^k}{(q; q)_k} = \frac{1}{(z; q)_\infty} \quad (|q| < 1; |z| < 1)$$

and

$$(1.27) \quad \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q; q)_k} z^k = (z; q)_\infty \quad (|q| < 1; |z| < 1),$$

respectively.

It is observed that

$$(1.28) \quad \lim_{q \downarrow 1} \frac{(q^a z; q)_\infty}{(z; q)_\infty} = \lim_{q \downarrow 1} {}_1\Phi_0(q^a; -; q, z) = {}_1F_0(a; -; z) = (1 - z)^{-a}$$

$$(|z| < 1; a \in \mathbb{C}),$$

which, by the principle of analytic continuation, holds true for $z \in \mathbb{C}$ cut along the positive real axis from 1 to ∞ , with $(1 - z)^{-a}$ positive when z is real and less than 1.

A q -analogue of the classical exponential function e^z is defined by

$$(1.29) \quad e_q(z) := \sum_{k=0}^{\infty} \frac{z^k}{[k]_q!}$$

and another q -analogue of the classical exponential function e^z is defined by

$$(1.30) \quad E_q(z) := \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{z^k}{[k]_q!}.$$

It is easily seen by applying (1.29) and (1.30) that

$$(1.31) \quad e_q(z) E_q(-z) = 1.$$

We conclude this section by giving a widely-investigated generalization ${}_r\Phi_s$ of the function ${}_1\Phi_0(a; -; q, z)$ in (1.25), which is defined by

$$(1.32) \quad {}_r\Phi_s \left[\begin{matrix} a_1, \dots, a_r; \\ b_1, \dots, b_s; \end{matrix} \middle| q, z \right] = {}_r\Phi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, z)$$

$$:= \sum_{k=0}^{\infty} (-1)^{(1-r+s)k} q^{(1-r+s)\binom{k}{2}} \frac{(a_1; q)_k \cdots (a_r; q)_k}{(b_1; q)_k \cdots (b_s; q)_k} \frac{z^k}{(q; q)_k},$$

provided that the *generalized basic* (or q -) *hypergeometric series* in (1.32) converges.

2. Definitions of q -Extensions of Gottlieb and two variable Gottlieb polynomials

We begin by recalling the definition of a several variable analogue of the Gottlieb polynomials $\varphi_n(x; \lambda)$ and one of their generating functions in [2].

DEFINITION 1.1. An extension of the Gottlieb polynomials $\varphi_n(x; \lambda)$ in m variables is defined by

$$\begin{aligned}
 & \varphi_n^m(x_1, x_2, \dots, x_m; \lambda_1, \lambda_2, \dots, \lambda_m) \\
 (2.1) \quad & = \exp(-n \sigma_m) \sum_{r_1=0}^n \sum_{r_2=0}^{n-r_1} \sum_{r_3=0}^{n-r_1-r_2} \cdots \sum_{r_m=0}^{n-r_1-r_2-\dots-r_{m-1}} \\
 & \cdot \frac{(-n)_{\delta_m} \cdot \prod_{j=1}^m (-x_j)_{r_j} \cdot \prod_{j=1}^m (1 - e^{\lambda_j})^{r_j}}{\prod_{j=1}^m r_j! \cdot \delta_m!} \quad (n, m \in \mathbb{N}),
 \end{aligned}$$

where, for convenience,

$$(2.2) \quad \sigma_m := \sum_{j=1}^m \lambda_j \quad \text{and} \quad \delta_m := \sum_{j=1}^m r_j.$$

It is noted that the special case $m = 1$ of (2.1) reduces immediately to the second one of the Gottlieb polynomials $\varphi_n(x; \lambda)$ in (1.1) and the cases of (2.1) when $m = 2$ and $m = 3$ correspond with those in [5, 6].

The following generating function for $\varphi_n^m(x_1, x_2, \dots, x_m; \lambda_1, \lambda_2, \dots, \lambda_m)$ holds true:

$$\begin{aligned}
 (2.3) \quad & \sum_{n=0}^{\infty} (\mu)_n \varphi_n^m(x_1, x_2, \dots, x_m; \lambda_1, \lambda_2, \dots, \lambda_m) \frac{t^n}{n!} = (1 - t e^{-\sigma_m})^{-\mu} \\
 & \cdot F_D^{(m)} \left[\mu, -x_1, \dots, -x_m; 1; \frac{t(e^{\lambda_1} - 1) e^{-\sigma_m}}{1 - t e^{-\sigma_m}}, \dots, \frac{t(e^{\lambda_m} - 1) e^{-\sigma_m}}{1 - t e^{-\sigma_m}} \right],
 \end{aligned}$$

where $F_D^{(m)}[\cdot]$ denotes one of the Lauricella series in m variables (see [9, p. 33, Eq. (4)]; see also [7]) defined by

$$\begin{aligned}
 (2.4) \quad & F_D^{(m)}[a, b_1, \dots, b_m; c; x_1, \dots, x_m] \\
 & = \sum_{r_1=0, \dots, r_m=0}^{\infty} \frac{(a)_{\delta_m} (b_1)_{r_1} \cdots (b_m)_{r_m}}{(c)_{\delta_m}} \frac{x_1^{r_1}}{r_1!} \cdots \frac{x_m^{r_m}}{r_m!} \\
 & \quad (\max\{|x_1|, \dots, |x_m|\} < 1),
 \end{aligned}$$

and σ_m, δ_m are given in (2.2).

Khan and Asif [6] defined a q -analogue of Gottlieb polynomials in (1.1) given below.

DEFINITION 1.2. A q -extension of the Gottlieb polynomials $\varphi_n(x; \lambda)$ is given as follows:

$$(2.5) \quad \varphi_{n;q}(x; \lambda) = \{E_q(-\lambda)\}^n \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} x \\ k \end{bmatrix}_q \cdot q^{k(k-1)-xk} \cdot (1 - e_q(\lambda))^k.$$

By using (1.12), (1.17) and (1.19), Khan and Asif [6] gave another expression of the q -extension of the Gottlieb polynomials $\varphi_n(x; \lambda)$ in (2.5):

$$(2.6) \quad \varphi_{n;q}(x; \lambda) = \{E_q(-\lambda)\}^n {}_2\Phi_1 \left[\begin{matrix} q^{-n}, & q^{-x}; \\ & q; \end{matrix} \begin{matrix} q^n (1 - e_q(\lambda)) \end{matrix} \right].$$

Khan and Asif [6] used some of the identities given in Section 1 to present the following three generating functions for q -Gottlieb polynomials in (2.5):

$$(2.7) \quad \begin{aligned} & \sum_{n=0}^{\infty} \varphi_{n;q}(x; \lambda) t^n \\ &= (1 - t E_q(-\lambda))^{-1} {}_1\Phi_1 \left[\begin{matrix} q^{-x}; \\ q t E_q(-\lambda); \end{matrix} q, -(1 - E_q(-\lambda)) t \right]; \end{aligned}$$

$$(2.8) \quad \begin{aligned} & \sum_{n=0}^{\infty} \varphi_{n;q}(x; \lambda) \frac{t^n}{(q; q)_n} \\ &= e_q(t E_q(-\lambda)) {}_1\Phi_1 \left[\begin{matrix} q^{-x}; \\ q; \end{matrix} q, -(1 - E_q(-\lambda)) t \right]; \end{aligned}$$

$$(2.9) \quad \begin{aligned} & \sum_{n=0}^{\infty} \frac{(q^c; q)_n}{(q; q)_n} \varphi_{n;q}(x; \lambda) t^n \\ &= \frac{(t q^c E_q(-\lambda); q)_{\infty}}{(t E_q(-\lambda); q)_{\infty}} {}_2\Phi_2 \left[\begin{matrix} q^c, & q^{-x}; \\ q, & t q^c E_q(-\lambda); \end{matrix} q, -(1 - E_q(-\lambda)) t \right]. \end{aligned}$$

We find from Definition 1 that the two variable Gottlieb polynomials $\varphi_n^2(x_1, x_2; \lambda_1, \lambda_2)$ is given by

$$(2.10) \quad \begin{aligned} & \varphi_n^2(x_1, x_2; \lambda_1, \lambda_2) = \exp(-n(\lambda_1 + \lambda_2)) \\ & \cdot \sum_{k=0}^n \sum_{j=0}^{n-k} \frac{(-n)_{k+j} (-x_1)_k (-x_2)_j (1 - e^{\lambda_1})^k (1 - e^{\lambda_2})^j}{k! j! (k + j)!} \quad (n \in \mathbb{N}), \end{aligned}$$

which is easily rewritten as follows:

$$(2.11) \quad \begin{aligned} \varphi_n^2(x_1, x_2; \lambda_1, \lambda_2) &= \exp(-n(\lambda_1 + \lambda_2)) \\ &\cdot \sum_{k=0}^n \sum_{j=0}^{n-k} \binom{n}{k+j} \binom{x_1}{k} \binom{x_2}{j} (1 - e^{\lambda_1})^k (1 - e^{\lambda_2})^j \quad (n \in \mathbb{N}). \end{aligned}$$

We define a q -extension of $\varphi_n^2(x_1, x_2; \lambda_1, \lambda_2)$.

DEFINITION 1.3. A q -extension of the two variable Gottlieb polynomials $\varphi_n^2(x_1, x_2; \lambda_1, \lambda_2)$ is defined as follows:

$$(2.12) \quad \begin{aligned} \varphi_{n;q}^2(x_1, x_2; \lambda_1, \lambda_2) &:= \{E_q(-\lambda_1)\}^n \{E_q(-\lambda_2)\}^n \sum_{k=0}^n \sum_{j=0}^{n-k} \begin{bmatrix} n \\ k+j \end{bmatrix}_q \begin{bmatrix} x_1 \\ k \end{bmatrix}_q \begin{bmatrix} x_2 \\ j \end{bmatrix}_q \\ &\cdot q^{\binom{k+j}{2} + \binom{k}{2} + \binom{j}{2} - x_1 k - x_2 j} \cdot (1 - e_q(\lambda_1))^k (1 - e_q(\lambda_2))^j, \end{aligned}$$

which, upon making use of Equations (1.12), (1.17) and (1.21), can be expressed in the following form:

$$(2.13) \quad \begin{aligned} \varphi_{n;q}^2(x_1, x_2; \lambda_1, \lambda_2) &= \{E_q(-\lambda_1)\}^n \{E_q(-\lambda_2)\}^n \\ &\cdot \sum_{k=0}^n \sum_{j=0}^{n-k} \frac{(q^{-n}; q)_{k+j} (q^{-x_1}; q)_k (q^{-x_2}; q)_j}{(q; q)_{k+j} (q; q)_k (q; q)_j} \\ &\cdot \{q^n (1 - e_q(\lambda_1))\}^k \{q^n (1 - e_q(\lambda_2))\}^j. \end{aligned}$$

REMARK 1.4. In the definitions (1.32) and (3.4) of the q -hypergeometric series and a generalized basic double series, the complementary inserting factors

$$(-1)^{(1-r+s)k} q^{(1-r+s)\binom{k}{2}} \quad \text{and} \quad q^{\frac{1}{2}ir(r-1) + \frac{1}{2}js(s-1) + krs}$$

play an important role to deduce certain identities corresponding to those in the generalized hypergeometric series. Likewise, in Definitions 2 and 3, the inserting factors

$$q^{k(k-1)-xk} \quad \text{and} \quad q^{\binom{k+j}{2} + \binom{k}{2} + \binom{j}{2} - x_1 k - x_2 j}$$

also play a smooth role to present their respective generating functions.

Proof. We will prove Equation (3.3) only. The other two Equations (3.1) and (3.2) will be verified in a similar way. For convenience, let the left hand side of (3.3) be denoted by $\mathcal{L}_{c;q}(x_1, x_2; \lambda_1, \lambda_2)$. By using Equation (2.13) and then Equation (1.12), we obtain

$$\begin{aligned} \mathcal{L}_{c;q}(x_1, x_2; \lambda_1, \lambda_2) &= \sum_{n=0}^{\infty} (q^c; q)_n \{t E_q(-\lambda_1) E_q(-\lambda_2)\}^n \\ &\cdot \sum_{k=0}^n \sum_{j=0}^{n-k} \frac{(-1)^{k+j} q^{\binom{k+j}{2}} (q^{-x_1}; q)_k (q^{-x_2}; q)_j}{(q; q)_{n-k-j} (q; q)_{k+j} (q; q)_k (q; q)_j} \\ &\cdot \{1 - e_q(\lambda_1)\}^k \{1 - e_q(\lambda_2)\}^j. \end{aligned}$$

Employing a formal manipulation of double series (see [1]):

$$(3.5) \quad \sum_{n=0}^{\infty} \sum_{l=0}^n A(l, n) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} A(l, n+l),$$

we have

$$\begin{aligned} &\mathcal{L}_{c;q}(x_1, x_2; \lambda_1, \lambda_2) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^n (q^c; q)_{n+k} \{t E_q(-\lambda_1) E_q(-\lambda_2)\}^{n+k} \\ &\cdot \frac{(-1)^{k+j} q^{\binom{k+j}{2}} (q^{-x_1}; q)_k (q^{-x_2}; q)_j}{(q; q)_{n-j} (q; q)_{k+j} (q; q)_k (q; q)_j} \{1 - e_q(\lambda_1)\}^k \{1 - e_q(\lambda_2)\}^j. \end{aligned}$$

Using (3.5) again, we find

$$\begin{aligned} &\mathcal{L}_{c;q}(x_1, x_2; \lambda_1, \lambda_2) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (q^c; q)_{n+k+j} \{t E_q(-\lambda_1) E_q(-\lambda_2)\}^{n+k+j} \\ &\cdot \frac{(-1)^{k+j} q^{\binom{k+j}{2}} (q^{-x_1}; q)_k (q^{-x_2}; q)_j}{(q; q)_n (q; q)_{k+j} (q; q)_k (q; q)_j} \{1 - e_q(\lambda_1)\}^k \{1 - e_q(\lambda_2)\}^j. \end{aligned}$$

Applying (1.13) and (1.31), and rearranging the resulting triple series, we get

$$\begin{aligned} \mathcal{L}_{c;q}(x_1, x_2; \lambda_1, \lambda_2) &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(q^c; q)_{k+j} q^{\binom{k+j}{2}} (q^{-x_1}; q)_k (q^{-x_2}; q)_j}{(q; q)_{k+j} (q; q)_k (q; q)_j} \\ &\cdot \{t E_q(-\lambda_2) (1 - E_q(-\lambda_1))\}^k \{t E_q(-\lambda_1) (1 - E_q(-\lambda_2))\}^j \\ &\cdot \sum_{n=0}^{\infty} \frac{(q^{c+k+j}; q)_n}{(q; q)_n} \{t E_q(-\lambda_1) E_q(-\lambda_2)\}^n. \end{aligned}$$

By using the q -binomial theorem (1.25) for the most inner infinite series, we obtain

$$\begin{aligned} \mathcal{L}_{c;q}(x_1, x_2; \lambda_1, \lambda_2) &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(q^c; q)_{k+j} q^{\binom{k+j}{2}} (q^{-x_1}; q)_k (q^{-x_2}; q)_j}{(q; q)_{k+j} (q; q)_k (q; q)_j} \\ &\cdot \{t E_q(-\lambda_2) (1 - E_q(-\lambda_1))\}^k \{t E_q(-\lambda_1) (1 - E_q(-\lambda_2))\}^j \\ &\cdot \frac{(q^{c+k+j} t E_q(-\lambda_1) E_q(-\lambda_2); q)_{\infty}}{(t E_q(-\lambda_1) E_q(-\lambda_2); q)_{\infty}}. \end{aligned}$$

If we apply (1.10) to the last line, we find

$$\begin{aligned} \mathcal{L}_{c;q}(x_1, x_2; \lambda_1, \lambda_2) &= \frac{(t q^c E_q(-\lambda_1) E_q(-\lambda_2); q)_{\infty}}{(t E_q(-\lambda_1) E_q(-\lambda_2); q)_{\infty}} \\ &\cdot \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(q^c; q)_{k+j} q^{\binom{k+j}{2}} (q^{-x_1}; q)_k (q^{-x_2}; q)_j}{(q; q)_{k+j} (t q^c E_q(-\lambda_1) E_q(-\lambda_2); q)_{k+j}} \\ &\cdot \frac{\{t E_q(-\lambda_2) (1 - E_q(-\lambda_1))\}^k}{(q; q)_k} \frac{\{t E_q(-\lambda_1) (1 - E_q(-\lambda_2))\}^j}{(q; q)_j}. \end{aligned}$$

Finally, by considering the generalized basic double series in (3.4), the last resulting double series corresponds with Equation (3.3). The other two Equations (3.1) and (3.3) will be proved by a similar argument. This completes the proof of Theorem. \square

Acknowledgements

This research was supported by the Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (2012-0002957).

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