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ON THE GENERALIZED HYERS-ULAM STABILITY OF A BI-JENSEN FUNCTIONAL EQUATION ON A PUNCTURED DOMAIN

Yang-Hi Lee*

Abstract. In this paper, we investigate the stability of a bi-Jensen functional equation

$$2f(\tfrac{x+y}{2}, z) - f(x, z) - f(y, z) = 0,$$
$$2f(x, \tfrac{y+z}{2}) - f(x, y) - f(x, z) = 0$$

in the spirit of P.Găvruta.

1. Introduction

In 1940, S. M. Ulam [13] raised a question concerning the stability of homomorphisms: Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta$$

for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \varepsilon$$

for all $x \in G_1$? The case of approximately additive mappings was solved by D. H. Hyers [3] under the assumption that G_1 and G_2 are Banach spaces. In 1978 Th. M. Rassias [12] gave a generalization of Hyers' Theorem by allowing the Cauchy difference to be controlled by a sum of powers like

$$\|h(x + y) - h(x) - h(y)\| \le \varepsilon(\|x\|^p + \|y\|^p).$$

P. Găvruta [2] provided a further generalization of Th.M.Rassias' Theorem in which he replaced the bound $\varepsilon(\|x\|^p+\|y\|^p)$ by a general function.

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During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians[7,9-11].

Throughout this paper, let X and Y be a normed space and a Banach space, respectively. A mapping $g : X \to Y$ is called a Jensen mapping if g satisfies the functional equation $2g(\frac{x+y}{2}) = g(x) + g(y)$. For a given mapping $f : X \times X \to Y$, we define

$$J_1f(x, y, z) := 2f(\tfrac{x+y}{2}, z) - f(x, z) - f(y, z),$$
$$J_2f(x, y, z) := 2f(x, \tfrac{y+z}{2}) - f(x, y) - f(x, z)$$

for all $x, y, z \in X$. A mapping $f : X \times X \to Y$ is called a bi-Jensen mapping if f satisfies the functional equations $J_1f = 0$ and $J_2f = 0$.

Bae and Park [1] obtained the generalized Hyers-Ulam stability of a bi-Jensen mapping. The Jun et al [4, 5] improved the results of Bae and Park.

In this paper, we investigate the generalized Hyers-Ulam stability of a bi-Jensen functional equation on the punctured domain.

2. Stability of a bi-Jensen functional equation

Jun et al [6] established the basic properties of a bi-Jensen mapping in the following lemma.

Lemma 2.1. *Let $f : X \times X \to Y$ be a bi-Jensen mapping. Then the following equalities hold:*

$$\begin{aligned} f(x, y) &= \frac{f(2^n x, 2^n y)}{2^n} + \frac{2^n - 1}{2^{2n+1}}(f(2^n x, -2^n y) + f(-2^n x, 2^n y)) \\ &\quad + (1 - \frac{1}{2^n})^2 f(0, 0) \end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$.

From Lemma 3.1 in [8], we get the following lemma.

Lemma 2.2. *Let A be a subset of X satisfying the following condition: for every $x \neq 0$, there exists a positive integer n_x such that $nx \notin A$ for all with $|n| \geq n_x$, and such that $nx \in A$ for all with $|n| < n_x$. Let $f : X \times X \to Y$ be a mapping such that*

$$J_1f(x, y, z) = 0, \quad J_2f(x, y, z) = 0$$

for all $x, y, z \in X\backslash A$. *Then there exists a unique bi-Jensen mapping* $F : X \times X \to Y$ *such that*

$$F(x, y) = f(x, y)$$

for all $x, y \in X\backslash A$. *Moreover,*

$$F(x, y) = f(x, y)$$

holds for all $(x, y) \in (X \times X)\backslash(A \times A)$.

THEOREM 2.3. *Let* $\varphi, \psi : X \times X \times X \to [0, \infty)$ *be two functions satisfying*

$$\sum_{j=0}^{\infty}(\varphi(2^j x, 2^j y, 2^j z) + \psi(2^j x, 2^j y, 2^j z)) < \infty \tag{2.1}$$

for all $x, y, z \in X\backslash A$. *Let* $f : X \times X \to Y$ *be a mapping such that*

$$\|J_1 f(x, y, z)\| \le \varphi(x, y, z), \tag{2.2}$$

$$\|J_2 f(x, y, z)\| \le \psi(x, y, z) \tag{2.3}$$

for all $x, y, z \in X\backslash A$. *Then there exists a (unique for the bi-Jensen mapping* F' *with* $F(0, 0) = F'(0, 0)$*) bi-Jensen mapping* $F : X \times X \to Y$ *such that*

$$\|f(x, y) - F(x, y)\| \quad \le M(x, y) \tag{2.4}$$

for all $x, y \in X\backslash A$, *where*

$$\begin{aligned}
M(x, y) &= \sum_{j=0}^{\infty}[(\frac{1}{4^{j+2}} + \frac{1}{2^{j+4}})\Phi_1(2^j x, 2^j y) + \frac{1 + 2^j}{2^{j+2}}\Phi_2(2^j x, 2^j y) \\
&\quad + \frac{1 + 2^{j+1}}{2^{j+3}}\Phi_2(2^j x, 2^{j+1} y) + \frac{1 + 2^{j+1}}{2^{j+4}}\Psi_1(2^j x, 2^j y) \\
&\quad + \frac{\Psi_1(2^{j+1} x, 2^j y)}{2^{2j+5}} + \frac{\Psi_2(2^j x, 2^j y)}{2^{j+2}} + \frac{\Psi_2(2^{j+1} x, 2^j y)}{2^{j+3}}] \\
&\quad + \frac{\varphi(x, -x, y)}{2} + \frac{\Psi_2(x, y)}{4}, \\
\Phi_1(x, y) &= \sum_{l=0}^{1}\sum_{m=0}^{1}\sum_{n=0}^{1}\varphi((-1)^l \cdot 3x, (-1)^m x, (-1)^n y), \\
\Phi_2(x, y) &= \varphi(x, -x, y) + \varphi(x, -x, -y), \\
\Psi_1(x, y) &= \sum_{l=0}^{1}\sum_{m=0}^{1}\sum_{n=0}^{1}\psi((-1)^l x, (-1)^m \cdot 3y, (-1)^n y), \\
\Psi_2(x, y) &= \psi(x, y, -y) + \psi(-x, y, -y).
\end{aligned}$$

The mapping $F : X \times X \to Y$ is given by

$$F(x,y) \quad := \lim_{j\to\infty}[\tfrac{\tilde{f}(2^jx,2^jy)}{4^j} + \tfrac{f(0,2^jy)}{2^j} + \tfrac{f(2^jx,0)}{2^j}] + y_0$$

for all $x, y \in X$, where $\tilde{f} : X \times X \to Y$ and $y_0 \in Y$ are given by

$$\tilde{f}(x,y) := \frac{f(x,y) - f(-x,y) - f(x,-y) + f(-x,-y)}{4}$$

for all $x, y \in X$ and

$$y_0 = \lim_{j\to\infty} \frac{f(0,2^jw) + f(0,-2^jw)}{2}$$

for any $w \in X\backslash\{0\}$.

Proof. By (2.1), (2.2) and (2.3), we get the following relations

$$\begin{aligned}
&\|\frac{\tilde{f}(2^jx,2^jy)}{4^j} - \frac{\tilde{f}(2^{j+1}x,2^{j+1}y)}{4^{j+1}}\| \\
&\quad = \|\sum_{l=0}^{1}\sum_{m=0}^{1}\sum_{n=0}^{1}\frac{(-1)^{m+n+1}}{2^{2j+4}}J_1f(2^j(-1)^lx, 2^j(-1)^mx, 2^j(-1)^ny) \\
&\qquad + \sum_{l=0}^{1}\sum_{m=0}^{1}\sum_{n=0}^{1}\frac{(-1)^{l+n+1}}{2^{2j+5}}J_2f(2^{j+1}(-1)^lx, 3(-1)^m2^jy, 2^j(-1)^ny)\| \\
&\quad \le \frac{1}{4^{j+2}}\Phi_1(2^jx,2^jy) + \frac{1}{2^{2j+5}}\Psi_1(2^{j+1}x,2^jy),
\end{aligned}$$

$$\begin{aligned}
&\|\frac{1}{2^{j+1}}(f(0,2^jy) - f(0,-2^jy)) - \frac{1}{2^{j+2}}(f(0,2^{j+1}y) - f(0,-2^{j+1}y))\| \\
&\quad = \frac{1}{2^{j+4}}\|4J_1f(2^jx,-2^jx,2^jy) - 4J_1f(2^jx,-2^jx,-2^jy) \\
&\qquad -2J_1f(2^jx,-2^jx,2^{j+1}y) + 2J_1f(2^jx,-2^jx,-2^{j+1}y) \\
&\qquad -\sum_{l=0}^{1}\sum_{m=0}^{1}\sum_{n=0}^{1}(-1)^nJ_2f((-1)^l\cdot 2^jx, (-1)^m\cdot 3\cdot 2^jy, (-1)^n\cdot 2^jy) \\
&\quad \le \frac{1}{2^{j+4}}[4\Phi_2(2^jx,2^jy) + 2\Phi_2(2^jx,2^{j+1}y) + \Psi_1(2^jx,2^jy)],
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\|f(0,2^jy)+f(0,-2^jy)-(f(0,2^{j+1}y)+f(0,-2^{j+1}y))\| \\
&= \frac{1}{8}\|2J_1f(2^jx,-2^jx,2^jy)+2J_1f(2^jx,-2^jx,-2^jy) \\
&\quad -2J_1f(2^jx,-2^jx,2^{j+1}y)-2J_1f(2^jx,-2^jx,-2^{j+1}y) \\
&\quad -\sum_{l=0}^{1}\sum_{m=0}^{1}\sum_{n=0}^{1}(-1)^{m+n}J_2f((-1)^l\cdot 2^jx,(-1)^m\cdot 3\cdot 2^jy,(-1)^n\cdot 2^jy)\| \\
&\le \frac{1}{8}[2\Phi_2(2^jx,2^jy)+2\Phi_2(2^jx,2^{j+1}y)+\Psi_1(2^jx,2^jy)],
\end{aligned}$$

$$\begin{aligned}
&\|\frac{1}{2^{j+1}}\quad(f(2^jx,0)-f(-2^jx,0))-\tfrac{1}{2^{j+2}}(f(2^{j+1}x,0)-f(-2^{j+1}x,0))\| \\
&\qquad\le \tfrac{1}{2^{j+4}}[4\Psi_2(2^jx,2^jy)+2\Psi_2(2^{j+1}x,2^jy)+\Phi_1(2^jx,2^jy)]
\end{aligned}$$

for all $x,y\in X\backslash A$ and $j\in\mathbb{N}$. From the above relations, we can easily show that, for given integers l,m $(0\le l<m)$, the inequalities

$$\begin{aligned}
&\|\frac{\tilde f(2^lx,2^ly)}{4^l}-\frac{\tilde f(2^mx,2^my)}{4^m}\| \\
&\le \sum_{j=l}^{m-1}[\frac{1}{4^{j+2}}\Phi_1(2^jx,2^jy)+\frac{1}{2^{2j+5}}\Psi_1(2^{j+1}x,2^jy)],
\end{aligned}\tag{2.5}$$

$$\begin{aligned}
&\|\frac{f(0,2^ly)-f(0,-2^ly)}{2^{l+1}}-\frac{f(0,2^my)-f(0,-2^my)}{2^{m+1}}\| \\
&\le \sum_{j=l}^{m-1}\frac{1}{2^{j+4}}[4\Phi_2(2^jx,2^jy)+2\Phi_2(2^jx,2^{j+1}y)+\Psi_1(2^jx,2^jy)],
\end{aligned}\tag{2.6}$$

$$\begin{aligned}
&\frac{1}{2}\|f(0,2^ly)+f(0,-2^ly)-(f(0,2^my)+f(0,-2^my))\| \\
&\le \sum_{j=l}^{m-1}\frac{1}{8}[2\Phi_2(2^jx,2^jy)+2\Phi_2(2^jx,2^{j+1}y)+\Psi_1(2^jx,2^jy)],
\end{aligned}\tag{2.7}$$

$$\begin{aligned}
&\|\frac{f(2^lx,0)-f(-2^lx,0)}{2^{l+1}}-\frac{f(2^mx,0)-f(-2^mx,0)}{2^{m+1}}\| \\
&\le \sum_{j=l}^{m-1}\frac{1}{2^{j+4}}[4\Psi_2(2^jx,2^jy)+2\Psi_2(2^{j+1}x,2^jy)+\Phi_1(2^jx,2^jy)]
\end{aligned}\tag{2.8}$$

hold for all $x, y \in X\backslash A$. By (2.1), (2.5), (2.6), (2.7), and (2.8), the sequences $\{\frac{\tilde{f}(2^jx,2^jy)}{4^j}\}$, $\{\frac{f(0,2^jy)-f(0,-2^jy)}{2^{j+1}}\}$, $\{\frac{f(0,2^jy)+f(0,-2^jy)}{2}\}$ and

$$\{\frac{f(2^jx,0)-f(-2^jx,0)}{2^{j+1}}\}$$

are Cauchy sequences for all $x, y \in X\backslash A$. Since Y is complete, the sequences $\{\frac{\tilde{f}(2^jx,2^jy)}{4^j}\}$, $\{\frac{f(0,2^jy)-f(0,-2^jy)}{2^{j+1}}\}$, $\{\frac{f(0,2^jy)+f(0,-2^jy)}{2}\}$, and

$$\{\frac{f(2^jx,0)-f(-2^jx,0)}{2^{j+1}}\}$$

converge for all $x, y \in X\backslash A$. Using the inequality

$$\begin{aligned}
&\|\frac{f(0,2^jy)+f(0,-2^jy)}{2}-\frac{f(2^jx,0)+f(-2^jx,0)}{2}\| \\
&= \frac{1}{4}\|J_1f(2^jx,-2^jx,2^jy)+J_1f(2^jx,-2^jx,-2^jy) \\
&\quad -J_2f(2^jx,2^jy,-2^jy)-J_2f(-2^jx,2^jy,-2^jy)\| \\
&\leq \frac{1}{4}[\Phi_2(2^jx,2^jy)+\Psi_2(2^jx,2^jy)]
\end{aligned}$$

for all $x, y \in X\backslash A$ and $j \in \mathbb{N}$, we get

$$\lim_{j\to\infty}[\frac{f(2^jx,0)+f(-2^jx,0)}{2}-\frac{f(0,2^jy)+f(0,-2^jy)}{2}]=0$$

for all $x, y \in X\backslash\{0\}$. From this and the Cauchy sequence $\{\frac{f(0,2^jy)+f(0,-2^jy)}{2}\}$, there exists a unique $y_0 \in Y$ such that

$$y_0=\lim_{j\to\infty}\frac{f(2^jx,0)+f(-2^jx,0)}{2}=\lim_{j\to\infty}\frac{f(0,2^jy)+f(0,-2^jy)}{2}$$

for all $x, y \in X\backslash\{0\}$. Therefore, we get

$$\lim_{j\to\infty}\frac{f(2^jx,0)+f(-2^jx,0)}{2^{j+1}}=\lim_{j\to\infty}\frac{f(0,2^jy)+f(0,-2^jy)}{2^{j+1}}=0$$

for all $x, y \in X$ and we can define $F_1, F_2, F_3 : X \times X \to Y$ by

$$\begin{aligned}
F_1(x,y) &:= \lim_{j\to\infty}\frac{\tilde{f}(2^jx,2^jy)}{4^j}, \\
F_2(x,y) &:= \lim_{j\to\infty}\frac{f(0,2^jy)-f(0,-2^jy)}{2^{j+1}}=\lim_{j\to\infty}\frac{f(0,2^jy)}{2^j}, \\
F_3(x,y) &:= \lim_{j\to\infty}\frac{f(2^jx,0)-f(-2^jx,0)}{2^{j+1}}=\lim_{j\to\infty}\frac{f(2^jx,0)}{2^j}.
\end{aligned}$$

By (2.2), (2.3), and the definition of F_1, we obtain

$$\begin{aligned} J_1F_1(x,y,z) &= \lim_{j\to\infty}[\frac{J_1f(2^jx,2^jy,2^jz)-J_1f(-2^jx,-2^jy,2^jz)}{4^{j+1}} \\ &\quad -\frac{J_1f(2^jx,2^jy,-2^jz)-J_1f(-2^jx,-2^jy,-2^jz)}{4^{j+1}}]=0, \\ J_2F_1(x,y,z) &= \lim_{j\to\infty}[\frac{J_2f(2^jx,2^jy,2^jz)-J_2f(-2^jx,2^jy,2^jz)}{4^{j+1}} \\ &\quad -\frac{J_2f(2^jx,-2^jy,-2^jz)-J_2f(-2^jx,-2^jy,-2^jz)}{4^{j+1}}]=0 \end{aligned}$$

for all $x, y, z \neq 0$. Since

$$\begin{aligned} &J_2F_2(x,y,-y)=0, \\ &J_2F_2(x,y,z)=\lim_{j\to\infty}[\frac{J_1f(2^jx,-2^jx,2^{j-1}(y+z))}{2^j}-\frac{J_1f(2^jx,-2^jx,2^jy)}{2^{j+1}} \\ &\quad -\frac{J_1f(2^jx,-2^jx,2^jz)}{2^{j+1}}+\frac{J_2f(2^jx,2^jy,2^jz)}{2^{j+1}}+\frac{J_2f(-2^jx,2^jy,2^jz)}{2^{j+1}}]=0 \end{aligned}$$

for all $x, y, z \neq 0$ with $y+z \neq 0$, we have

$$J_1F_2(x,y,z)=0,\ \ J_2F_2(x,y,z)=0$$

for all $x, y, z \neq 0$. Similarly the equalities

$$J_1F_3(x,y,z)=0,\ \ J_2F_3(x,y,z)=0$$

hold for all $x, y, z \neq 0$. By Lemma 2, there exist bi-Jensen mappings $F_1', F_2', F_3' : X \times X \to Y$ such that

$$F_1'(x,y)=F_1(x,y),\ \ F_2'(x,y)=F_2(x,y),\ \ F_3'(x,y)=F_3(x,y) \tag{2.9}$$

for all $(x, y) \neq (0, 0)$. From the definitions of F_1, F_2, F_3, we have $J_1F_1(x, -x, 0) = 0$, $J_1F_2(x, -x, 0) = 0$, and $J_1F_3(x, -x, 0) = 0$ for any $x \neq 0$. From (2.9) and the equalities

$$\begin{aligned} F_1'(0,0)-F_1(0,0)&=\tfrac{1}{2}J_1F_1'(x,-x,0)-\tfrac{1}{2}J_1F_1(x,-x,0)=0, \\ F_2'(0,0)-F_2(0,0)&=\tfrac{1}{2}J_1F_2'(x,-x,0)-\tfrac{1}{2}J_1F_2(x,-x,0)=0, \\ F_3'(0,0)-F_3(0,0)&=\tfrac{1}{2}J_1F_3'(x,-x,0)-\tfrac{1}{2}J_1F_3(x,-x,0)=0, \end{aligned}$$

we conclude that F_1, F_2, F_3 are bi-Jensen mappings.

Putting $l = 0$ and taking $m \to \infty$ in (2.5), (2.6), and (2.8), one can obtain the inequalities

$$\begin{aligned}
&\|\tilde{f}(x,y) - F_1(x,y)\| \\
(2.10)\qquad &\le \sum_{j=0}^{\infty} \frac{1}{4^{j+2}}\Phi_1(2^j x, 2^j y) + \frac{1}{2^{2j+5}}\Psi_1(2^{j+1}x, 2^j y), \\
&\left\|\frac{f(0,y) - f(0,-y)}{2} - F_2(x,y)\right\| \\
(2.11)\qquad &\le \sum_{j=0}^{\infty} \frac{1}{2^{j+4}}[4\Phi_2(2^j x, 2^j y) + 2\Phi_2(2^j x, 2^{j+1}y) + \Psi_1(2^j x, 2^j y)], \\
&\left\|\frac{f(x,0) - f(-x,0)}{2} - F_3(x,y)\right\| \\
(2.12)\qquad &\le \sum_{j=0}^{\infty} \frac{1}{2^{j+4}}[4\Psi_2(2^j x, 2^j y) + 2\Psi_2(2^{j+1} x, 2^j y) + \Phi_1(2^j x, 2^j y)], \\
&\left\|\frac{f(0,y) + f(0,-y)}{2} - y_0\right\| \\
&\le \sum_{j=0}^{\infty} \frac{1}{8}[2\Phi_2(2^j x, 2^j y) + 2\Phi_2(2^j x, 2^{j+1}y) + \Psi_1(2^j x, 2^j y)]
\end{aligned}$$

for all $x, y \in X \backslash A$. By the above inequalities and the following two inequalities

$$\begin{aligned}
&\|f(x,y) - F(x,y)\| \\
&\le \left\|f(x,y) - \tilde{f}(x,y) - f(0,y) - \frac{f(x,0) - f(-x,0)}{2}\right\| \\
&+\|\tilde{f}(x,y) - \tilde{f}(x,y)\| + \left\|\frac{f(0,y) - f(0,-y)}{2} - F_2(x,y)\right\| \\
(2.13)\qquad &+\left\|\frac{f(x,0) - f(-x,0)}{2} - F_3(x,y)\right\| + \left\|\frac{f(0,y) + f(0,-y)}{2} - y_0\right\|, \\
&\left\|f(x,y) - \tilde{f}(x,y) - f(0,y) - \frac{f(x,0) - f(-x,0)}{2}\right\| \\
&\le \left\| - \frac{1}{2}J_1 f(x,-x,y) - \frac{1}{4}J_2 f(x,y,-y) + \frac{1}{4}J_2 f(-x,y,-y)\right\| \\
(2.14)\qquad &\le \frac{1}{2}\varphi(x,-x,y) + \frac{1}{4}\Psi_2(x,y),
\end{aligned}$$

we obtain that F is a bi-Jensen mapping satisfying (2.4) for all $x, y \in X \backslash A$, where F is given by

$$F(x,y) = F_1(x,y) + F_2(x,y) + F_3(x,y) + y_0$$

for all $x, y \in X$. Now, let $F' : X \times X \to Y$ be another bi-Jensen mapping satisfying (2.4) with $F'(0,0) = F(0,0)$. By Lemma 1 and (2.4), we have the inequality

$$\begin{aligned}
&\|F(x,y) - F'(x,y)\| \\
&\le \|\frac{1}{2^n}(F-f)(2^n x, 2^n y)\| + \frac{1}{2^{n+1}}\|(F-f)(-2^n x, 2^n y)\| \\
&\quad + \frac{1}{2^{n+1}}\|(F-f)(2^n x, -2^n y)\| + \|\frac{1}{2^n}(f-F')(2^n x, 2^n y)\| \\
&\quad + \frac{1}{2^{n+1}}\|(f-F')(-2^n x, 2^n y)\| + \frac{1}{2^{n+1}}\|(f-F')(2^n x, -2^n y))\| \\
&\le \frac{2M(2^n x, 2^n y) + M(-2^n x, 2^n y) + M(2^n x, -2^n y)}{2^n}
\end{aligned}$$

for all $x, y \in X \backslash A$ and $n \in \mathbb{N}$. Taking $n \to \infty$ in the above inequality, we get

$$F(x,y) = F'(x,y)$$

for all $x, y \in X \backslash A$. By Lemma 2, such a bi-Jensen mapping $F : X \times X \to Y$ is unique. □

Corollary 2.4. *Let $p < 0$ and $\varepsilon > 0$. Let $f : X \times X \to Y$ be a mapping such that*

$$\begin{aligned}
\|J_1 f(x,y,z)\| &\le \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \\
\|J_2 f(x,y,z)\| &\le \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p)
\end{aligned}$$

for all $x, y, z \in X \backslash A$. Then there exists a (unique for the bi-Jensen mapping F' with $F(0,0) = F'(0,0)$) bi-Jensen mapping $F : X \times X \to Y$ such that

$$\begin{aligned}
\|f(x,y) - F(x,y)\| \le &\left(\frac{2 + 2^p + 2\cdot 3^p}{4 - 2^p} + \frac{10 + 2^p + 2\cdot 3^p}{2(2-2^p)} + \frac{3}{2}\right)\varepsilon\|x\|^p \\
&+ \left(\frac{3 + 3^p}{4 - 2^p} + \frac{12 + 2^p + 2\cdot 3^p}{2(2-2^p)} + 3\right)\varepsilon\|y\|^p
\end{aligned}$$

for all $x, y \in X \backslash A$.

Proof. Put $\varphi(x,y,z) := \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p)$ and $\psi(x,y,z) := \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y \in X\backslash A$ in Theorem 3. Since

$$\begin{aligned}
&\|\frac{f(0,y)+f(0,-y)}{2} - y_0\| \\
&= \|\frac{f(0,y)+f(0,-y)}{2} - \lim_{j\to\infty}\frac{f(2^j x,0)+f(-2^j x,0)}{2}\| \\
&= \frac{1}{4}\lim_{j\to\infty}\|J_1 f(2^j x,-2^j x,y) + J_1 f(2^j x,-2^j x,-y) \\
&\quad -J_2 f(2^j x,y,-y) - J_2 f(-2^j x,y,-y)\| \\
&\le \lim_{j\to\infty}\frac{3}{2}(2^{jp}\|x\|^p + \|y\|^p) = \frac{3}{2}\|y\|^p
\end{aligned}$$

for all $x, y \in X\backslash A$, we can use the inequalities (2.10), (2.11), (2.12), (2.13), and (2.14) in Theorem 3 with the above inequality to obtain the desired result. □

THEOREM 2.5. *Let $\varphi, \psi : X \times X \times X \to [0,\infty)$ be two functions satisfying*

$$\sum_{j=0}^{\infty}\frac{1}{2^j}(\varphi(2^j x,2^j y,2^j z) + \psi(2^j x,2^j y,2^j z)) < \infty$$

for all $x, y, z \in X\backslash A$. Let $\tilde{f}, \Phi_1, \Phi_2, \Psi_1, \Psi_2$ be as in Theorem 3 and let x_0 be a fixed element of $X\backslash A$. Let $f : X \times X \to Y$ be a mapping (2.2) and (2.3) for all $x, y, z \in X\backslash A$. Then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that

$$\begin{aligned}
&\|f(x,y) - F(x,y)\| \\
&\le \sum_{j=0}^{\infty}[(\frac{1}{4^{j+2}} + \frac{1}{2^{j+4}})\Phi_1(2^j x,2^j y) + \frac{\Phi_2(2^j x,2^j y)}{2^{j+2}} \\
&\quad + \frac{\Phi_2(2^j x,2^{j+1} y)}{2^{j+3}} + \frac{\Psi_1(2^j x,2^j y))}{2^{j+4}} + \frac{\Psi_1(2^{j+1} x,2^j y)}{2^{2j+5}} \\
&\quad + \frac{\Psi_2(2^j x,2^j y)}{2^{j+2}} + \frac{\Psi_2(2^{j+1} x,2^j y)}{2^{j+3}}] + \frac{\varphi(x,-x,y)}{2} \\
&\quad + \frac{\Psi_2(x,y)}{4} + \frac{\Phi_2(x_0,y)+\Psi_2(x_0,y)}{4} \qquad (2.15)
\end{aligned}$$

for all $x, y \in X\backslash A$ with $F(0,0) = y_0$, where $y_0 \in Y$ is given by $y_0 = \frac{f(x_0,0)+f(-x_0,0)}{2}$. The mapping $F : X \times X \to Y$ is given by

$$F(x,y) := \lim_{j\to\infty}[\frac{\tilde{f}(2^j x,2^j y)}{4^j} + \frac{f(0,2^j y)}{2^j} + \frac{f(2^j x,0)}{2^j}] + y_0$$

for all $x, y \in X$.

Proof. As in Theorem 3, we can define F_1, F_2, F_3 by

$$F_1(x,y) := \lim_{j\to\infty} \tfrac{\tilde{f}(2^jx,2^jy)}{4^j},$$
$$F_2(x,y) := \lim_{j\to\infty} \tfrac{f(0,2^jy)-f(0,-2^jy)}{2^{j+1}},$$
$$F_3(x,y) := \lim_{j\to\infty} \tfrac{f(2^jx,0)-f(-2^jx,0)}{2^{j+1}}$$

and obtain the inequalities (2.10), (2.11), and (2.12). For a fixed $x_0 \in X\backslash A$, we get the inequality

$$\begin{aligned}
&\|\frac{f(0,y)+f(0,-y)}{2} - \frac{f(x_0,0)+f(-x_0,0)}{2}\| \\
&= \frac{1}{4}\|J_1f(x_0,-x_0,y) + J_1f(x_0,-x_0,-y) \\
&-J_2f(x_0,y,-y) - J_2f(-x_0,y,-y)\| \\
&\le \frac{1}{4}[\Phi_2(x_0,y) + \Psi_2(x_0,y)]
\end{aligned}$$

for all $y \in X\backslash A$. By (2.10), (2.11), (2.12), (2.13), (2.14) and the above inequality, we get the inequality (2.15) for all $x, y \in X\backslash A$. Since

$$\begin{aligned}
&J_2F_2(x,y,-y) = 0, \\
&J_2F_2(x,y,z) \\
&= \lim_{j\to\infty} \frac{1}{2^{j+2}}[-2J_1f(2^jx,-2^jx,-2^{j-1}(y+z)) + J_1f(2^jx,-2^jx,-2^jy) \\
&\quad +J_1f(2^jx,-2^jx,-2^jz) - J_2f(2^jx,-2^jy,-2^jz) \\
&\quad -J_2f(-2^jx,-2^jy,-2^jz) + 2J_1f(2^jx,-2^jx,2^{j-1}(y+z)) \\
&\quad -J_1f(2^jx,-2^jx,2^jy) - J_1f(2^jx,-2^jx,2^jz) + J_2f(2^jx,2^jy,2^jz) \\
&\quad +J_2f(-2^jx,2^jy,2^jz)] = 0
\end{aligned}$$

for all $x, y, z \neq 0$ with $y + z \neq 0$, we have

$$J_1F_2(x,y,z) = 0,\ J_2F_2(x,y,z) = 0$$

for all $x, y, z \neq 0$. The remainder of the proof of this theorem is same to the proof of Theorem 3. □

Corollary 2.6. *Let* $p < 1$ *and* $\varepsilon > 0$. *Let* $f : X \times X \to Y$ *be a mapping such that*

$$\|J_1f(x,y,z)\| \le \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p)$$
$$\|J_2f(x,y,z)\| \le \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X\backslash A$ and let x_0 be a fixed element of $X\backslash A$. Then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x,y) - F(x,y)\| \le \left(\frac{2+2^p+2\cdot 3^p}{4-2^p} + \frac{10+2^p+2\cdot 3^p}{2(2-2^p)} + \frac{3}{2}\right)\varepsilon\|x\|^p$$
$$+\frac{3}{2}\varepsilon\|x_0\|^p + \left(\frac{3+3^p}{4-2^p} + \frac{12+2^p+2\cdot 3^p}{2(2-2^p)} + 3\right)\varepsilon\|y\|^p$$

for all $x, y \in X\backslash A$ with $F(0,0) = \frac{f(x_0,0)+f(-x_0,0)}{2}$.

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Department of Mathematics Education
Gongju National University of Education
Gongju 314-060, Republic of Korea
E-mail: lyhmzi@pro.gjue.ac.kr