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GENERALIZED DUAL GOTTLIEB SETS AND COCATEGORIES

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Abstract. In this paper, we introduce the concepts of DC_k^p-spaces for maps which are the dual concepts of C_k^f-spaces for maps, and characterize DC_k^p-spaces for maps using the dual Gottlieb sets for maps and the LS cocategories.

1. Introduction

A space X is called a T-space [1] if the fibration $\Omega X \to X^{S^1} \to X$ is fibre homotopically trivial. It is well known that any H-space is a T-space, but the converse does not hold. We introduced [5] the concepts of C_k-spaces as intermediate stages between H-spaces and T-spaces. The loop space ΩX of any space X has a homotopy type of an associative H-space. A 0-connected space X is filtered by the projective spaces of ΩX by a result of Milnor [8] and Stasheff [10];

$$\Sigma\Omega X = P^1(\Omega X) \hookrightarrow P^2(\Omega X) \hookrightarrow \cdots \hookrightarrow P^\infty(\Omega X) \simeq X.$$

For each k, let $e_k^X : P^k(\Omega X) \to P^\infty(\Omega X) \simeq X$ be the natural inclusion. We write $e^X = e_1^X : \Sigma\Omega X = P^1(\Omega X) \to X$. It is shown [1] that X is a T-space if and only if $e = e_1 : \Sigma\Omega X \to X$ is cyclic. We see that $e_\infty^X \sim 1_X : X \to X$. For a map $f : A \to X$, a based map $g : B \to X$ is called *f-cyclic* [9] if there exist a map $G : A \times B \to X$ such that $Gj \sim \nabla(f \vee g)$, where $j : A \vee B \to A \times B$ is the inclusion and $\nabla : X \vee X \to X$ is the folding map. The *Gottlieb set for a map* $f : A \to X$, $G^f(B, X)$, is the set of all homotopy classes of f-cyclic maps from B to X. For a map $f : A \to X$, a connected space X is called [5] a *C_k^f-space* if the inclusion $e_k^X : P^k(\Omega X) \to X$ is f-cyclic. In fact, a C_k-space is C_k^1-space for the

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identity map $1 : X \to X$. We showed [5] that a space X is a C_k^f-space for a map $f : A \to X$ if and only if $G^f(Z, X) = [Z, X]$ for any space Z with *cat* $Z \leq k$. Let $f : A \to X$ and $g : B \to Y$ be maps. Then we also showed that $X \times Y$ is a $C_k^{f \times g}$-space if and only if X is a C_k^f-space and Y is a C_k^g-space.

In this paper, we introduce the concepts of DC_k^p-spaces for maps which are generalizations of DC_k-spaces [16]. We show that a space X is a DC_k^p-space for a map $p : X \to A$ if and only if $DG^p(X, Z) = [X, Z]$ for any space Z with *cocat* $Z \leq k$. Let $p : X \to A$ and $q : Y \to B$ be any maps. Then we show that the wedge space $X \vee Y$ is a $DC_k^{\nabla(p \vee q)}$-space if and only if X is a DC_k^p-space and Y is a DC_k^q-space.

2. DC_k^p-spaces for a map $p : X \to A$

We now recall the following Ganea's theorem [3].

THEOREM 2.1. ([3],[4]) *Let $k \geq 1$ be an integer or $k = \infty$ and assume that X is a 0-connected space. The category cat $X \leq k$ if and only if $e_k^X : P^k(\Omega X) \to X$ has a right homotopy inverse.*

In [3], Ganea introduced the concept of cocategory of a space as follows; Let X be a any space. Define a sequence of cofibrations

$$\mathcal{C}_k : X \xrightarrow{e'_k} F_k \xrightarrow{s'_k} B_k \ (k \geq 0)$$

as follows, let $\mathcal{C}_0 : X \overset{e'_0}{\to} cX \overset{s'_0}{\to} \Sigma X$ be the standard cofibration. Assuming $\mathcal{C}_k$ to be defined, let F'_{k+1} be the fibre of s'_k and $e_{k+1}" : X \to F'_{k+1}$ lift e'_k. Define F_{k+1} as the reduced mapping cylinder of $e_{k+1}"$, let $e'_{k+1} : X \to F_{k+1}$ is the obvious inclusion map, and let $B_{k+1} = F_{k+1}/e'_{k+1}(X)$ and $s'_{k+1} : F_{k+1} \to F_{k+1}/e_{k+1}(X)$ the quotient map.

DEFINITION 2.2. [3] *The cocategory of X, cocat X, is the least integer $k \geq 0$ for which there is a map $r : F_k \to X$ such that $r \circ e'_k \sim 1$. If there is no such integer, cocat $X = \infty$.*

The following remark can easily obtained from the above definition.

REMARK 2.3.
(1) *cocat $X \leq k$ if and only if $e'_k : X \to F_k$ has a left homotopy inverse.*
(2) *cocat $X = 0$ if and only if X is contractible.*

For a map $p : X \to A$, a based map $g : X \to B$ is *p-cocyclic* [9] if there is a map $\theta : X \to A \vee B$ such that $j\theta \sim (p \times g)\Delta$, where $j : A \vee B \to A \times B$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. *The dual Gottlieb set for a map* $p : X \to A$, $DG^p(X, B)$, is the set of all homotopy classes of p-cocyclic maps from X to B. In fact, a 1-cocyclic map is a cocyclic map. A space X is called a *co-T-space* [12] if $e' : X \to \Omega\Sigma X$ is cocyclic. Clearly, any co-H-space is a co-T-space. Let $p : X \to A$ be a map. A space X is called [15] a *co-T^p-space* if $e' = e'_1 : X \to \Omega\Sigma X$ is p-cocyclic. Clearly, a co-T^1-space is a co-T-space.

Definition 2.4. *Let $p : X \to A$ be a map. A space X is called a DC^p_k-space if the inclusion $e'_k : X \to F_k$ is p-cocyclic.*

Clearly, a DC^1_k-space for the identity map $1 : X \to X$ is a DC_k-space [16]. We can easily show that F_1 and $\Omega\Sigma X$ have the same homotopy type. Thus we know that a DC_1-space is a co-T-space. Clearly, DC^p_1-spaces and co-T^p-spaces are equivalent. We showed [15] that X is a co-T^p-space if and only if $DG^p(X, \Omega B) = [X, \Omega B]$ for any space B. It is shown [2] that *cocat* $Z \leq 1$ if and only if Z can be dominated by a loop space. Thus we can generalize the above result as following theorem. It says also that DC^p_k-spaces are closely related by the dual Gottlieb sets for maps and cocategory of spaces.

Theorem 2.5. *Let $p : X \to A$ be a map. Then a space X is a DC^p_k-space if and only if $DG^p(X, Z) = [X, Z]$ for any space Z with cocat $Z \leq k$.*

Proof. Suppose X is a DC^p_k-space for a map $p : X \to A$. Since $e'_k : X \to F_k$ is p-cocyclic, there is a map $\theta : X \to A \vee F_k$ such that $j\theta \sim (p \times e'_k)\Delta$, where $j : A \vee F_k \to A \times F_k$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. Let Z be a space with *cocat* $Z \leq k$. Let $g : X \to Z$ be any map. Since *cocat* $Z \leq k$, there is a map $s : F_k \to Z$ such that $s \circ e'_k \sim 1_Z$. Interpreting F_k as a functor, we have the following homotopy commutative diagram;

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "e'_k"] & Z \arrow[d, "e'_k"] \arrow[dr, "1"] & \\
F_k(X) \arrow[r, "F_k(g)"] & F_k(Z) \arrow[r, "s"] & Z.
\end{tikzcd}$$

Also, we consider the following homotopy commutative diagram;

$$\begin{array}{ccccccc}
X \times X & \xrightarrow{(p\times e_k')} & A \times F_k(X) & \xrightarrow{(1\times F_k(g))} & A \times F_k(Z) & \xrightarrow{(1\times s)} & A \times Z \\
\Delta\Big\uparrow & & j\Big\uparrow & & j\Big\uparrow & & j\Big\uparrow \\
X & \xrightarrow{\theta} & A \vee F_k(X) & \xrightarrow{(1\vee F_k(g))} & A \vee F_k(Z) & \xrightarrow{(1\vee s)} & A \vee Z.
\end{array}$$

Thus we have a map $\phi = (1 \vee s)(1 \vee F_k(g))\theta : X \to A \vee Z$ such that $j\phi \sim (p \times g)\Delta$, where $j : A \vee Z \to A \times Z$ is the inclusion. Thus $g : X \to Z$ is p-cocyclic. On the other hand, we assume that for any space Z with *cocat* $Z \leq k$, $DG^p(X, Z) = [X, Z]$. It is well known [2] that if $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration, then *cocat* $F \leq$ *cocat* $E + 1$. From the fact that $F_k \simeq F_k' \to F_{k-1} \xrightarrow{s_{k-1}'} B_{k-1}$ is a fibration, we know that *cocat* $F_k \leq$ *cocat* $F_{k-1} + 1$. Then we have, by induction, *cocat* $F_k \leq k$. Thus we know, by our assumption, that $e_k' : X \to F_k$ is p-cocyclic and X is a DC_k^p-space for a map $p : X \to A$. □

If we take $p = 1 : X \to X$ in the above theorem, then we have the following corollary.

COROLLARY 2.6. [16] *X is a DC_k-space if and only if $DG(X, Z) = [X, Z]$ for any space Z with cocat $Z \leq k$.*

It is well known fact [14] that a space X is a co-H^p-spaces if and only if $1 : X \to X$ is p-cocyclic. Moreover, it is also known [9] that if $f : X \to Y$ is p-cocyclic and $g : Y \to Z$ is any map, then $gf : X \to Z$ is p-cocyclic. Any co-H-space X is a co-H^p-space and any co-H^p-space X is a co-T^p-space for any map $p : X \to A$. Thus we have the following corollary from the definition of cocategory and the above theorem.

COROLLARY 2.7. *Let $p : X \to A$ be any map.*
(1) *X is a DC_1^p-space $\Leftrightarrow$ X is a co-T_1^p-space $\Leftrightarrow$ X is a co-T^p-space.*
(2) *If X is a co-T_m^p-space, then X is a co-T_n^p-space for any $n < m$.*
(3) *If X is a DC_m^p-space, then X is a DC_n^p-space for any $n < m$.*
(4) *If X is a DC_k^p-space and cocat $X \leq k$, then X is a co-H^p-space.*

PROPOSITION 2.8. *Let $p : X \to A$ and $q : Y \to A$ be any maps. Then the relation*

$$DG^{\nabla(p\vee q)}(X \vee Y, B) \equiv DG^p(X, B) \times DG^q(Y, B)$$

holds for any space B.

Proof. Let $g : X \vee Y \to A \vee B$ be a $\nabla(p \vee q)$-cocyclic map. Then there is a map $G : X \vee Y \to A \vee B$ such that $jG \sim (\nabla(p \vee q) \times g)\Delta$, where $j : A \vee B \to A \times B$ is the inclusion and $\Delta : X \vee Y \to (X \vee Y) \times (X \vee Y)$

is the diagonal. Consider the maps $G_1 = Gi_1 : X \to A \vee B$ and $G_2 = Gi_2 : Y \to A \vee B$, where $i_1 : X \to X \vee Y$, $i_2 : Y \to X \vee Y$ are natural inclusions. Then $jG_1 \sim (p \times gi_1)\Delta$, $jG_2 \sim (q \times gi_2)\Delta$, where $i_1 : X \to X \vee Y$, $i_2 : Y \to X \vee Y$ are the inclusions. Thus we have $(gi_1, gi_2) \in DG^p(X,B) \times DG^q(Y,B)$. On the other hand, let $(g_1, g_2) \in DG^p(X,B) \times DG^q(Y,B)$. Since $g_1 : X \to B$ is p-cocyclic and $g_2 : Y \to B$ is q-cocyclic, there are maps $G_1 : X \to A \vee B$ and $G_2 : Y \to A \vee B$ such that $jG_1 \sim (p \times g_1)\Delta$ and $jG_2 \sim (q \times g_2)\Delta$ respectively. Let $T : A \vee (B \vee A) \vee B \to A \vee (A \vee B) \vee B$ be the switching map. Then consider the map $G = (\nabla \vee \nabla)T(G_1 \vee G_2) : X \vee Y \to A \vee B$. Then $jG \sim ((\nabla(p \vee q) \times \nabla(g_1 \vee g_2))\Delta$, where $\Delta : X \vee Y \to (X \vee Y) \times (X \vee Y)$ is the diagonal map. Thus we know $\nabla(g_1 \vee g_2) \in DG^{\nabla(p\vee q)}(X \vee Y, B)$. $\square$

THEOREM 2.9. *Let $p : X \to A$ and $q : Y \to A$ be any maps. Then the wedge space $X \vee Y$ is a $DC_k^{\nabla(p\vee q)}$-space if and only if X is a DC_k^p-space and Y is a DC_k^q-space.*

Proof. If $X \vee Y$ is a $DC^{\nabla(p\vee q)}$-space for a map $\nabla(p \vee q) : X \vee Y \to A$, then we know, from Theorem 2.5 and Proposition 2.8, that $DG^p(X,Z) \times DG^q(Y,Z) \equiv DG^{\nabla(p\vee q)}(X \vee Y, Z) = [X \vee Y, Z] \equiv [X,Z] \times [Y,Z]$ for any space Z with *cocat* $Z \leq k$. Then we have $DG^p(X,Z) = [X,Z]$ and $DG^q(Y,Z) = [Y,Z]$. Thus we know that X is a DC_k^p-space for a map $p : X \to A$ and Y is a DC_k^q-space for a map $q : Y \to A$. On the other hand, suppose that X is a DC_k^p-space for a map $p : X \to A$ and Y is a DC_k^q-space for a map $q : Y \to A$. Then $DG^p(X,Z) = [X,Z]$, $DG^q(Y,Z) = [Y,Z]$ for any space Z with *cocat* $Z \leq k$. Thus we know $DG^{\nabla(p\vee q)}(X \vee Y, Z) \equiv DG^p(X,Z) \times DG^q(Y,Z) = [X,Z] \times [Y,Z] \equiv [X \vee Y, Z]$ for any space Z with *cocat* $Z \leq k$. Thus $X \vee Y$ is a $DC_k^{\nabla(p\vee q)}$-space for a map $\nabla(p \vee q) : X \vee Y \to A$. $\square$

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