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ON LYAPUNOV-TYPE FUNCTIONS FOR LINEAR DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this paper we give a necessary and sufficient condition for characterizing h-stability for linear dynamic systems on time scales by using Lyapunov functions.

1. Introduction

In the study of the stability of general nonlinear systems, the Lyapunov's direct (or second) method which introduced by Lyapunov in 1892 is the main and powerful tool. This method allows one to investigate the qualitative nature of solutions without actually determining the solutions themselves. Therefore, we regard it as one of the major tools in stability theory. For the various types of Lyapunov stability theorems for differential equations and difference equations, see [9, 13, 8] and [1, 3], respectively.

In 1988, S. Hilger introduced the calculus on time scales (a closed subset of $\mathbb{R}$) which unifies continuous and discrete analysis. The theory of dynamic equations on time scales provides the framework to handle both continuous and discrete dynamical systems.

Pinto [12] introduced the notion of h-stability for differential equations with the intention of obtaining results about for weakly stable differential systems under some perturbations. Also, Medina and Pinto

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[11] applied the h-stability to obtain a uniform treatment for the various stability notions in difference systems and extended the study of exponential stability to a variety of reasonable systems called h-system.

Choi et al. [6] studied the h-stability for linear dynamic system by using the unified time scale quadratic Lyapunov functions. Also, P. E. Kloeden and A. Zmorzynska [7] established the existence of a Lyapunov function characterizing the uniform exponential asymptotic stability for linear nonautonomous dynamic systems on time scales.

In this paper we give a necessary and sufficient condition for characterizing h-stability for linear dynamic systems on time scales by using Lyapunov functions.

2. Main results

We refer the reader to Ref. [2] for all the basic definitions and results on time scales necessary to this paper (e.g. delta differentiability, rd-continuity, exponential function and its properties).

It is assumed throughout that a time scale $\mathbb{T}$ will be unbounded above. Let $\mathbb{R}^n$ be the n-dimensional real Euclidean space. The symbol $|\cdot|$ will be used to denote any convenient vector norm in $\mathbb{R}^n$. $C_{rd}(\mathbb{T}, \mathbb{R}^{n^2})$ denotes the set of all rd-continuous functions from $\mathbb{T}$ to $\mathbb{R}^{n^2}$ and $\mathbb{R}_+ = [0, \infty)$.

We consider the linear dynamic system

$$\begin{cases} x^\Delta = A(t)x, \ t \in \mathbb{T}, \\ x(t_0) = x_0, t_0 \in \mathbb{T}, \ x_0 \in \mathbb{R}^n, \end{cases} \tag{2.1}$$

where $A \in C_{rd}(\mathbb{T}, \mathbb{R}^{n^2})$.

For the Lyapunov-like function $V \in C_{\rm rd}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+)$, we recall the following definition:

Definition 2.1. [10, Definition 3.1.1] We define the *generalized derivative* $D^+V^\Delta_{(2.1)}(t, x(t))$ of $V(t, x)$ relative to (2.1) as follows: given $\varepsilon > 0$, there exists a neighborhood $N(\varepsilon)$ of $t \in \mathbb{T}$ such that

$$\begin{aligned} \frac{1}{\sigma(t) - s}[V(\sigma(t), x(\sigma(t))) \quad &- \quad V(s, x(\sigma(t)) - (\sigma(t) - s)f(t, x(t)))] \\ &< \quad D^+V^\Delta_{(2.1)}(t, x(t)) + \varepsilon, \ s \in N(\varepsilon), \ s > t, \end{aligned}$$

where $x(t)$ is any solution of (2.1) and the *upper right Dini derivative* $V_*^\Delta(t)$ of V_* is given by

$$V_*^\Delta(t) = \begin{cases} \overline{\lim}_{\eta\to 0^+, \eta+t\in\mathbb{T}} \frac{V_*(t+\eta)-V_*(t)}{\eta}, & \text{if } t=\sigma(t), \\ \frac{V_*(\sigma(t))-V_*(t)}{\mu(t)}, & \text{if } t<\sigma(t), \end{cases} \tag{2.2}$$

where $V_*(t) = V(t, x(t))$.

Then it is well-known that

$$V_{(2.1)}^\Delta(t, x(t)) = V_*^\Delta(t)$$

We recall the notions of h-stability for dynamic systems on time scales in [6].

Definition 2.2. System (2.1) is called an *h-system* if there exist a positive rd-continuous function $h : \mathbb{T} \to \mathbb{R}$, a constant $c \geq 1$, and $\delta > 0$ such that

$$|x(t, t_0, x_0)| \leq c|x_0|h(t)h(t_0)^{-1}, \ t \geq t_0$$

for $|x_0| < \delta$ (here $h(t)^{-1} = \frac{1}{h(t)}$).

Moreover, system (2.1) is said to be

(hS) *h-stable* if h is a bounded function in the definition of h-system,

(GhS) *globally h-stable* if system (2.1) is hS for every $x_0 \in \mathbb{R}^n$.

Remark 2.3. For the linear dynamic systems on time scales, we see that

$$\text{G}h\text{S} \iff h\text{S}.$$

Lemma 2.4. [6] *If a differentiable function $h : \mathbb{T} \to \mathbb{R}$ is positively regressive, then $\frac{h^\Delta(t)}{h(t)}$ is positively regressive, and $e_p(t, t_0)$ satisfies*

$$e_p(t, t_0) = \frac{h(t)}{h(t_0)},$$

where $p(t) = \frac{h^\Delta(t)}{h(t)}$.

We obtain the following result by using Lyapunov function.

Theorem 2.5. *Suppose that there exist a function $V \in \mathcal{C}_{rd}^1(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^+)$ and a positive function $h \in \mathcal{C}_{rd}^1(\mathbb{T}, \mathbb{R})$ satisfying the following properties:*

(i) $a|x(t)|^2 \leq V(t, x(t)) \leq b|x(t)|^2$;

(ii) $V^\Delta(t, x(t)) \leq \gamma \frac{h^\Delta(t)}{h(t)} |x(t)|^2, \ \gamma = \begin{cases} a, & \text{if } h^\Delta(t) \geq 0, \\ b, & \text{if } h^\Delta(t) < 0, \end{cases}$

where a and b are positive constants. Then system (2.1) is an h-system.

Proof. Let $x(t,t_0,x_0)$ be any solution of (2.1). Then it follows from the conditions (i) and (ii) of $V(t,x)$ that

$$V(t,x(t)) \le V(t_0,x_0) + \int_{t_0}^{t} \frac{h^\Delta(s)}{h(s)} V(s,x(s))\Delta s,\ t \ge t_0.$$

In view of Gronwall's inequality on time scale and Lemma 2.4, we obtain

$$\begin{aligned} V(t,x(t)) &\le V(t_0,x_0) e_{\frac{h^\Delta(t)}{h(t)}}(t,t_0) \\ &\le V(t_0,x_0)h(t)h(t_0)^{-1},\ t \ge t_0. \end{aligned} \tag{2.3}$$

From (2.3) and condition (i) of $V(t,x)$, we have

$$|x(t)| \le d|x_0|H(t)H(t_0)^{-1},\ t \ge t_0,$$

where $d = \sqrt{\frac{b}{a}}$ and $H(t) = \sqrt{h(t)}$. Hence the proof is complete. □

We obtain the following Massera-type converse theorem for h-stability of linear dynamic systems on time scales.

THEOREM 2.6. *If system (2.1) is an h-system, then there exists a function $V \in \mathcal{C}^1_{rd}(\mathbb{T}\times\mathbb{R}^n, \mathbb{R}^+)$ satisfying the following properties:*

(i) $|x| \le V(t,x) \le c|x|$, $(t,x) \in \mathbb{T}\times\mathbb{R}^n$;

(ii) $|V(t,x) - V(t,x')| \le c|x-x'|$ *for any fixed* $t\in\mathbb{T}$ *and all* $x, x' \in \mathbb{R}^n$;

(iii)

$$V_*^\Delta(t) \le \frac{h^\Delta(t)}{h(t)} V_*(t),\ (t,x) \in \mathbb{T}\times\mathbb{R}^n,$$

where $V_*(t) = V(t,x(t,t_0,x))$ for any solution $x(t,t_0,x_0)$ of (2.1).

Proof. Since system (2.1) is an h-system, there exist a constant $c \ge 1$ and a positive function $h \in \mathcal{C}^1_{rd}(\mathbb{T},\mathbb{R})$ such that

$$|x(t)| \le c|x_0|h(t)h(t_0)^{-1},\ t \ge t_0,$$

where $x(t,t_0,x_0)$ is any solution of (2.1).

Fix $t \in \mathbb{T}$, then we define

$$A_t := \{\tau \in [0,\infty) : t+\tau \in \mathbb{T}\}. \tag{2.4}$$

We define $V : \mathbb{T}\times\mathbb{R}^n \to \mathbb{R}$ by

$$V(t,x) := \sup_{\tau\in A_t} |x(t+\tau,t,x)|h(t+\tau)^{-1}h(t),$$

where $x(t) : \mathbb{T} \to \mathbb{R}^n$ is the unique solution of (2.1) with initial value $x(t,t,x) = x$. We obtain

$$\begin{aligned}|x| \le V(t,x) &= \sup_{\tau\in A_t} |x(t+\tau,t,x)|h(t+\tau)^{-1}h(t)\\ &\le \sup_{\tau\in A_t} c|x|h(t+\tau)h(t)^{-1}h(t+\tau)^{-1}h(t) = c|x|.\end{aligned}$$

Thus we have $|x| \le V(t,x(t)) \le c|x|$.

Next, we show that $V(t,x)$ is Lipchitzian in x for each $t\in\mathbb{T}$. Let $t\in\mathbb{T}$ be fixed and let $x,x'\in\mathbb{R}^n$. Then we obtain

$$\begin{aligned}|V(t,x)-V(t,x')| &= |\sup_{\tau\in A_t} |x(t+\tau,t,x)|h(t+\tau)^{-1}h(t)\\ &\qquad - \sup_{\tau\in A_t} |x(t+\tau,t,x')|h(t+\tau)^{-1}h(t)|\\ &\le \sup_{\tau\in A_t} |x(t+\tau,t,x)-x(t+\tau,t,x')|h(t+\tau)^{-1}h(t)\\ &= \sup_{\tau\in A_t} |x(t+\tau,t,x-x')|h(t+\tau)^{-1}h(t)\\ &= V(t,x-x') \le c|x-x'|,\end{aligned}$$

for each $x,x'\in\mathbb{R}^n$.

To prove the condition (iii), we consider two cases: $t<\sigma(t)$ and $t=\sigma(t)$.

Suppose that $\sigma(t)>t$. Then it follows that

$$V_*^{\Delta}(t) = \frac{V(\sigma(t))-V(t)}{\mu(t)}.$$

Since the solution of (2.1) is unique, we have the following derivative:

$$\begin{aligned}V_*^{\Delta}(t) &= \frac{V(\sigma(t))-V(t)}{\mu(t)}\\ &= \frac{1}{\mu(t)}[\sup_{\tau\in\mathbb{A}_{\sigma(t)}} |x(\sigma(t)+\tau,\sigma(t),x(\sigma(t),t,x)|h(\sigma(t)+\tau)^{-1}h(\sigma(t))\\ &\quad - \sup_{\tau\in\mathbb{A}_t} |x(t+\tau,t,x(t))|h(t+\tau)^{-1}h(t)]\\ &= \frac{1}{\mu(t)}[\sup_{\tau\in\mathbb{A}_{t+\mu(t)}} |x(t+\mu(t)+\tau,t+\mu(t),x(t+\mu(t),t,x))|\\ &\quad \times h(t+\mu(t)+\tau)^{-1}h(t+\mu(t))\\ &\quad - \sup_{\tau\in\mathbb{A}_t} |x(t+\tau,t,x(t))|h(t+\tau)^{-1}h(t)]\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\mu(t)}[\sup_{\tau\in\{\tau\in[\mu(t),\infty):t+\tau\in\mathbb{T}\}} |x(t+\tau,t,x(t))|h(t+\tau)^{-1}h(t+\mu(t)) \\
&\quad - \sup_{\tau\in\mathbb{A}_t} |x(t+\tau,t,x(t))|h(t+\tau)^{-1}h(t)] \\
&\leq \frac{1}{\mu(t)}[h(t+\mu(t))h(t)^{-1}-1]V(t,x(t)) \\
&= \frac{h^{\Delta}(t)}{h(t)}V(t,x(t)).
\end{aligned}
$$

The condition (iii) in case $t=\sigma(t)$ and the continuity of $V(t,x)$ can be proved as in similar manner of Theorem 3.6.1 in [8]. This completes the proof. □

Remark 2.7. If a positive function h in Theorem 2.5 is bounded, then Theorem 2.5 also holds.

Corollary 2.8. *If we set $\mathbb{T}=\mathbb{R}$, then Theorem 2.5 implies the Lyapunov converse theorem for h-stability for linear differential equations as a special result of Theorem 2.4 in* [4].

Corollary 2.9. *If we set $\mathbb{T}=\mathbb{Z}$, then Theorem 2.5 implies the Lyapunov converse theorem for h-stability for linear difference equations as a special result of Theorem 5 in* [5].

Corollary 2.10. *If we set $h(t)=e^{-ct}$ in Theorem 2.5 for a positive constant c, then we can obtain the Lyapunov converse theorem (Theorem 3.1 in* [7]*) as a corollary of Theorem 2.5.*

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