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ON THE STATISTICAL CONVERGENCE OF SEQUENCES OF FUZZY POINTS IN A FUZZY NORMED LINEAR SPACE

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Abstract. In this paper, we introduce the notion of the statistcally fuzzy convergent sequence and the statistical fuzzy α -Cauchy sequence of fuzzy points in a fuzzy normed linear space. And investigate some related properties.

1. Introduction

The idea of statistical convergence was introduced for the first time by Fast [4] as a generalization of ordinary convergence for real and complex sequences. Since then it has been studied by many authors. Statistical convergence has also been discussed in more general abstract spaces such as the fuzzy number space [1,12], locally convex spaces and Banach spaces [3].

The concept of fuzzy norm was first introduced by Katsaras [7] in studying fuzzy topological vector spaces. Krishna and Sarma [8] observed the convergence of sequences of fuzzy points in a fuzzy normed linear space. In [9], Rhie et al. introduced the fuzzy α-Cauchy sequence of fuzzy points and studied the fuzzy completeness in a fuzzy normed linear space.

In the present paper, as a generalization of those described above, we introduce the notion of the statistcal fuzzy convergent sequence and the statistical fuzzy α -Cauchy sequence of fuzzy points in a fuzzy normed linear space. And investigate some related properties.

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2. Preliminaries

Throughout this paper, X is a vector space over the field $\Phi(\mathbb{R}$ or $\mathbb{C})$. Fuzzy subsets of X are denoted by Greek letters in general. χ_A denotes the characteristic function of the set A. By a fuzzy point μ we mean a fuzzy subset $\mu : X \to [0,1]$ such that

$$\mu(z) = \begin{cases} \alpha & \text{if} \quad z = x \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha \in (0,1)$ and I^X denotes the set $\{\mu \mid \mu : X \to [0,1]\}$. We usually denote the fuzzy point with support x and value α by (x, α)

Definition 2.1. [5] For two fuzzy subset μ_1 and μ_2 of X, the fuzzy subset $\mu_1 + \mu_2$ is defined by

$$(\mu_1 + \mu_2)(x) = \vee\{\mu_1(x_1) \wedge \mu_2(x_2) \mid \ x = x_1 + x_2\}.$$

And for a scalar t of Φ and a fuzzy subset μ of X, the fuzzy subset $t\mu$ is defined by

$$(t\mu)(x) = \begin{cases} \mu(x/t) & \text{if} \quad t \neq 0 \\ 0 & \text{if} \quad t = 0 \ \text{ and } \ x \neq 0 \\ \vee\{\mu(y) \mid \ y \in X\} & \text{if} \quad t = 0 \ \text{ and } \ x = 0. \end{cases}$$

Definition 2.2. [7] $\mu \in I^X$ is said to be

1. *convex* if $t\mu + (1-t)\mu \subseteq \mu$ for each $t \in [0,1]$
2. *balanced* if $t\mu \subseteq \mu$ for each $t \in \Phi$ with $\mid t \mid \leq 1$
3. *absorbing* if $\vee\{t\mu(x) \mid t > 0\} = 1$ for all $x \in X$.

Definition 2.3. [7] Let (X, τ) be a topological space and $\omega(\tau) = \{f : (X,\tau) \to [0,1] \mid f$ is lower semicontinuous$\}$. Then $\omega(\tau)$ is a fuzzy topology on X. This topology is called the *fuzzy topology generated by* τ on X. The fuzzy usual topology on Φ means the fuzzy topology generated by the usual topology of Φ.

Definition 2.4. [6] A *fuzzy linear topology* on a vector space X over Φ is a fuzzy topology on X such that the two mappings

$$\begin{aligned} + \ &: \ X \times X \to X, \qquad (x,y) \to x + y \\ \cdot \ &: \ \Phi \times X \to X, \qquad (t,x) \to tx \end{aligned}$$

are continuous when Φ has the fuzzy usual topology and $\Phi \times X$ and $X \times X$ have the corresponding product fuzzy topologies. A linear space with a fuzzy linear topology is called a *fuzzy topological linear space* or a *fuzzy topological vector space.*

Definition 2.5. [10] Let (X, τ) be a fuzzy topological space. A fuzzy subset μ in X is called a *neighbourhood* of (x, α) if there exists $\psi \in \tau$ with $\psi(x) \geq \alpha$ and $\psi \leq \mu$.

Definition 2.6. [7] A *fuzzy seminorm* on X is a fuzzy set ρ in X which is convex ,balanced and absorbing. If in addition $\wedge\{(t\rho)(x) \mid t > 0\} = 0$ for $x \neq 0$, then ρ is called a *fuzzy norm.*

Theorem 2.7. [7] *If ρ is a fuzzy seminorm on X, then the family $B_\rho = \{\theta \wedge (t\rho) \mid 0 < \theta \leq 1,\ t > 0\}$ is a base for a fuzzy linear topology τ_ρ, where $\theta \wedge (t\rho)$ is the function $X \to [0,1]$ such that $(\theta \wedge (t\rho))(x) = \min\{\theta, (t\rho)(x)\}$.*

Definition 2.8. [7] Let ρ be a fuzzy seminorm on X. The fuzzy topology τ_ρ in Theorem 2.7 is called the *fuzzy topology induced by the fuzzy seminorm ρ* . And a linear space equipped with a fuzzy seminorm (resp. fuzzy norm) is called a *fuzzy seminormed (resp. fuzzy normed) linear space.*

Theorem 2.9. [10] *Let ρ is a fuzzy seminorm on X, then the function $P_\epsilon : X \to \mathbb{R}^+$ defined by*

$$P_\epsilon(x) = \wedge\{t > 0 \mid t\rho(x) \geq \epsilon\}$$

for each $\epsilon \in (0,1)$, is a seminorm on X . Further P_ϵ is a norm on X for each $\epsilon \in (0,1)$ if and only if ρ is a fuzzy norm on X.

3. Statistical fuzzy convergence of sequences of fuzzy points

In this section, we introduce the statistical fuzzy convergence of sequences and the statistical fuzzy α-Cauchy sequence of fuzzy points in a fuzzy normed lilear space, and investigate some related properties.

Definition 3.1. [1] The natural density of a set K of positive integers is defined by $\delta(K) = \lim_{n\to\infty} \frac{1}{n} \mid \{k \in K : k \leq n\} \mid$, where $\mid \{k \in K : k \leq n\} \mid$ denotes the number of elements of K not exceeding n.

Remark 3.2. It is clear that for a finite set K, we have $\delta(K) = 0$. The natural density may not exist for each set K and is different from zero which means $\delta(K) > 0$. Besides that, $\delta(K^c) = 1 - \delta(K)$.

Definition 3.3. Let (X, τ) be a fuzzy topological space, $\{\mu_n = (x_n, \alpha_n)\}$ be a sequence of fuzzy points in X and μ be a fuzzy point in X. We say that $\{\mu_n\}$ *statistically fuzzy converges to* μ, written as $\mu_n \to \mu$ if and only if for every neighbourhood N of μ, there exists a positive integer set K having natural density one such that $\mu_k \leq N$ for all $k \in K$. Equivalently, $\{\mu_n\}$ statistically fuzzy converges to μ if and only if for every neighbourhood N of μ, there exists a positive integer set K having naural densety zero such that $\mu_k \leq N$ for all $k \in K^c$.

THEOREM 3.4. *Let (X, ρ) be a fuzzy normed linear space and $\{\mu_n = (x_n, \alpha_n)\}$ be a sequence of fuzzy points in X and $\mu = (x, \alpha)$ a fuzzy point in X. Then $\mu_n \to \mu$ if and only if for every $t > 0$, there exists a positive integer set K having natural density one such that for all $k \in K$, $\alpha_k \leq \alpha$ and $P_{\alpha_k}(x_k - x) < t$.*

Proof. Suppose $\mu_n \to \mu$. Let $t > 0$. Write $N = x + \alpha \wedge \frac{1}{2}t\rho$. Then N is a neighbourhood of μ. So by Definition 3.3, there exists a positive integer set K having natural density one such that for all $k \in K$, $\mu_k \leq x + \alpha \wedge \frac{1}{2}t\rho$; that is for all $k \in K$, $\alpha_k \leq \alpha \wedge \frac{1}{2}t\rho(x_k - x)$, which means for all $k \in K$, $\alpha_k \leq \alpha$ and $\alpha_k \leq \frac{1}{2}t\rho(x_k - x)$, that is for all $k \in K$, $\alpha_k \leq \alpha$ and $P_{\alpha_k}(x_k - x) \leq \frac{t}{2} < t$. Hence for every $t > 0$, there exists a positive integer set K having natural density one such that for all $k \in K$, $\alpha_k \leq \alpha$ and $P_{\alpha_k}(x_k - x) < t$.

Conversely, suppose that for every $t > 0$, there exists a positive integer set K having natural density one such that for all $k \in K$, $\alpha_k \leq \alpha$ and $P_{\alpha_k}(x_k - x) < t$. Let N be a neighbourhood of μ. Then N contains a neighbourhood of the form $x + \alpha \wedge t\rho$ for some $t > 0$. For this t, by hypothesis, there exists a positive integer set K having natural density one such that for all $k \in K$, $\alpha_k \leq \alpha$ and $P_{\alpha_k}(x_k - x) < t$. Now, for all $k \in K$, we have

$$\begin{aligned} P_{\alpha_k}(x_k - x) < t \quad &\Rightarrow \quad t\rho(x_k - x) \geq \alpha_k \\ &\Rightarrow \quad x + \alpha \wedge t\rho(x_k) \geq \alpha_k \\ &\Rightarrow \quad \mu_k \leq x + \alpha \wedge t\rho. \end{aligned}$$

This implies that $\mu_k \leq N$ for all $k \in K$. This completes the proof. □

THEOREM 3.5. *Let (X, ρ) be a fuzzy normed linear space over the filed Φ.*

(a) If $(x_n, \alpha_n) = \mu_n \to \mu = (x, \alpha)$ and $(y_n, \beta_n) = \nu_n \to \nu = (y, \beta)$, then $\mu_n + \nu_n = (x_n + y_n, \alpha_n \wedge \beta_n) \to ((x + y, \alpha \wedge \beta)) = \mu + \nu$.

(b) If $\{t_n\} \subseteq \Phi$, $t \in \Phi$ and $t_n \to t$, $\mu_n = (x_n, \alpha_n) \to \mu = (x, \alpha)$, then $t_n\mu_n = (t_n x_n,\ \alpha_n) \to t\mu = (tx, \alpha)$.

Proof. (a) Let $t > 0$ be given. So there exists a positive integer set K having natural density one such that for all $k \in K$, $\alpha_k \leq \alpha$, $P_{\alpha_k}(x_k - x) < \frac{t}{2}$, $\beta_k \leq \beta$, $P_{\beta_k}(y_k - y) < \frac{t}{2}$. Now, $P_{\alpha_k \wedge \beta_k}(x_k + y_k -$

$(x+y)) \leq P_{\alpha_k}(x_k - x) + P_{\beta_k}(y_k - y)$. Hence, for all $k \in K$. we have $\alpha_k \wedge \beta_k \leq \alpha \wedge \beta$ and $P_{\alpha_k \wedge \beta_k}(x_k + y_k - (x+y)) < t$ which proves (a).

(b) Let $s > 0$ be given. Then there exists a positive integer set K_1 having natural density one such that for all $k \in K_1$, $\alpha_k \leq \alpha$ and $P_{\alpha_k}(x_k - x) < \frac{s}{2M}$ where M is a positive number such that $\mid t_k \mid \leq M$ for all k. Also, since $t_k \to t$ there exists a positive integer set K_2 having natural density one such that for all $k \in K_2$, $\mid t_k - t \mid \leq \frac{s}{[2P_\alpha(x)+1]}$. Let$K = K_1 \cup K_2$. Now,

$$\begin{aligned} P_{\alpha_k}(t_k x_k - tx) &= P_{\alpha_k}(t_k(x_k - x) + (t_k - t)x) \\ &< \ \mid t_k \mid P_{\alpha_k}(x_k - x) + \mid t_k - t \mid P_{\alpha_k}(x) \\ &< \ M \cdot \tfrac{s}{2M} + \tfrac{s}{[2P_\alpha(x)+1]} \cdot P_{\alpha_k}(x) \quad \text{for all } \mathrm{k} \in \mathrm{K} \\ &< \ \tfrac{s}{2} + \tfrac{s}{2} \ = \ s, \end{aligned}$$

which proves (b). □

Definition 3.6. Let $\alpha \in (0,1)$. A sequence of fuzzy points $\{\mu_n = (x_n, \alpha_n)\}$ is said to be a *statistical fuzzy α-Cauchy sequence* in a fuzzy normed linear space (X, ρ) if for each zero neighbourhood N with $N(0) > \alpha$, there exists a positive integer set K having natural density one such that for all $m, n \in K$ implies $\mu_n - \mu_m = (x_n - x_m, \alpha_n \wedge \alpha_m) \leq N$.

Theorem 3.7. *Let (X, ρ) be a fuzzy normed linear space and $\alpha \in (0,1)$. Then $\{(x_n, \alpha_n)\}$ is a statistical fuzzy α-Cauchy sequence if and only if for each $t > 0$, there exists a positive integer set K having natural density one such that for all $m, n \in K$, $\alpha_n \wedge \alpha_m \leq \alpha$ and $P_{(\alpha_n \wedge \alpha_m)}(x_n - x_m) < t$.*

Proof. Assume that $\{(x_n, \alpha_n)\}$ is a statistical fuzzy α -Cauchy sequence and $t > 0$. Let $\beta > \alpha$ and $N = \beta \wedge \frac{1}{2}t\rho$, where $(\beta \wedge \frac{1}{2}t\rho)(x) = \min\{\beta, \rho(\frac{2x}{t})\}$. Since $N(0) > \alpha$, there exists a positive integer set K having natural density one such that for all $m, n \in K$,

$$\begin{aligned} &\beta \wedge \tfrac{1}{2}t\rho(x_n - x_m) \geq \alpha_n \wedge \alpha_m \\ &\quad \Rightarrow \quad \alpha_n \wedge \alpha_m \leq \beta \quad \text{and} \quad \tfrac{1}{2}t\rho(x_n - x_m) \geq \alpha_n \wedge \alpha_m \\ &\quad \Rightarrow \quad \alpha_n \wedge \alpha_m \leq \beta \quad \text{and} \quad P_{(\alpha_n \wedge \alpha_m)}(x_n - x_m) \leq \tfrac{t}{2} < t. \end{aligned}$$

Therefore, $\alpha_n \wedge \alpha_m \leq \alpha$ and $P_{(\alpha_n \wedge \alpha_m)}(x_n - x_m) < t$. For the converse, let N be a neighbourhood of zero with $N(0) > \alpha$. Then there exist $t > 0$ and $\alpha' > \alpha$ such that $\alpha' \wedge t\rho \leq N$. For this t , there exists a positive integer set K having natural density one such that for all $m, n \in K$,

$\alpha_n \wedge \alpha_m \leq \alpha'$ and

$$\begin{array}{rcl} P_{(\alpha_n \wedge \alpha_m)}(x_n - x_m) < t & \Rightarrow & t\rho(x_n - x_m) \geq \alpha_n \wedge \alpha_m \\ & \Rightarrow & \alpha' \wedge t\rho(x_n - x_m) \geq \alpha_n \wedge \alpha_m \\ & \Rightarrow & \alpha' \wedge t\rho \geq (x_n - x_m, \alpha_n \wedge \alpha_m). \end{array}$$

Therefore, $N \geq \alpha' \wedge t\rho \geq \alpha \wedge t\rho \geq (x_n - x_m, \alpha_n \wedge \alpha_m)$. The proof is completed. □

COROLLARY 3.8. *Any subsequence of a statistical fuzzy α-Cauchy sequence is also a statistical fuzzy α-Cauchy sequence.*

Note. Let $\alpha < \alpha'$. Every statistical fuzzy α-Cauchy sequence is a statistical fuzzy α'-Cauchy sequence and statistical $P_{\alpha'}$-Cauchy sequence is a statistical P_α-Cauchy sequence.

THEOREM 3.9. *If $\{(x_n, \alpha_n)\}$ statistically fuzzy converges to (x, α) then it is a statistical fuzzy α -Cauchy sequence.*

Proof. Since $\{(x_n, \alpha_n)\}$ statistically fuzzy converges to (x, α), for every $t > 0$, there exists a positive integer set K having natural density one such that for all $k \in K$, $\alpha_k \leq \alpha$ and $P_{\alpha_k}(x_k - x) < \frac{t}{2}$. If $m, n \in K$, then $\alpha_n \wedge \alpha_m \leq \alpha$ and

$$\begin{array}{rcl} P_{(\alpha_n \wedge \alpha_m)}(x_n - x_m) & \leq & P_{(\alpha_n \wedge \alpha_m)}(x_n - x) + P_{(\alpha_n \wedge \alpha_m)}(x_m - x) \\ & \leq & P_{\alpha_n}(x_n - x) + P_{\alpha_m}(x_m - x) \\ & < & \frac{t}{2} + \frac{t}{2} = t. \end{array}$$

Therefore, $\{(x_n, \alpha_n)\}$ is a statistical fuzzy α-Cauchy sequence. The proof is completed. □

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