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ANTI-CYCLOTOMIC EXTENSION AND HILBERT CLASS FIELD

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Abstract. In this paper, we show how to construct the first layer k_1^a of anti-cyclotomic $\mathbb{Z}_3$-extension of imaginary quadratic fields $k(=\mathbb{Q}(\sqrt{-d}))$ when the Sylow subgroup of class group of k is 3-elementary, and give an example. This example is different from the one we obtained before in the sense that when we write $k_1^a = k(\eta)$, η is obtained from non-units of $\mathbb{Q}(\sqrt{3d})$.

1. Introduction

Let k be an imaginary quadratic field, and L an abelian extension of k. L is called an anti-cyclotomic extension of k if it is Galois over $\mathbb{Q}$, and $Gal(k/\mathbb{Q})$ acts on $Gal(L/k)$ by -1. For each prime number p, the compositum K of all $\mathbb{Z}_p$-extensions over k becomes a ${\mathbb{Z}_p}^2$-extension, and K is the compositum of the cyclotomic $\mathbb{Z}_p$-extension and the anti-cyclotomic $\mathbb{Z}_p$-extension of k. The layers k_n^c of cyclotomic $\mathbb{Z}_p$-extension are well understood by definition. However, little is known in the anti-cyclotomic case. In the paper[4],using Kummer theory and class field theory, we constructed the first layer k_1^a of the anti-cyclotomic $\mathbb{Z}_3$-extension of an imaginary quadratic field whose class number is not divisible by 3. In the paper [6], we applied the same method as in [4] to construct 3-Hilbert class fields of certain imaginary quadratic fields k which also become the first layers k_1^a of anti-cyclotomic $\mathbb{Z}_3$-extension of k. In this paper, we give a method to construct k_1^a for imaginary quadratic number fields k whose 3-part of ideal class group is 3-elementary. In the examples of papers [4] and [6], the class number of k is 1 or 3. We briefly explain our method to

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compute η. Note that $k(\zeta_3)k_1^a = k(\zeta_3)(\sqrt[3]{\beta})$ for some $\beta \in k(\zeta_3)$ by Kummer theory . By Lemma 1 of this paper, we see that β is a combination of the fundamental unit and generators of ideals of $\mathbb{Q}(\sqrt{3d})$. Then, by Lemma 2 and Theorem 3 of this paper, β can be determined. Hence, by Kummer theory again, we can determine η such that $k_1^a = k(\eta)$. To illustrate the method, we give an example at the end of this paper.

2. Proof of theorems

Throughout this section, we denote by H_k, h_k, A_k, M_k the p-part of Hilbert class field, the p-class number, and p-part of ideal class group of k, the maximal abelian p-extension of k unramified outside above p, respectively. Let k be an imaginary quadratic field and ζ_p a primitive p-th root of unity. We denote $F = k(\zeta_p)$. The first layer of anti-cyclotomic $\mathbb{Z}_p$-extension of an imaginary quadratic field k may be or may not be contained in the p-Hilbert class field of k. The following Theorem gives an answer for this question when $p = 3$.

THEOREM 2.1. (See [5, Theorem 2]) *Let $d \not\equiv 3$ mod 9 be a squarefree positive integer, $k = \mathbb{Q}(\sqrt{-d})$ an imaginary quadratic field and K the compositum of all $\mathbb{Z}_3$-extensions over k. Assume that $A_{\mathbb{Q}(\sqrt{-d})}$ is 3-elementary.Then*

$$H_k \cap K = k \Longleftrightarrow$$

$$rank_{\mathbb{Z}/3} A_{\mathbb{Q}(\sqrt{3d})} = rank_{\mathbb{Z}/3} A_{\mathbb{Q}(\sqrt{-d})}.$$

REMARK 2.2. It is well-known that

$$rank_{\mathbb{Z}/3} A_{\mathbb{Q}(\sqrt{3d})} \leq rank_{\mathbb{Z}/3} A_{\mathbb{Q}(\sqrt{-d})}$$

$$\leq rank_{\mathbb{Z}/3} A_{\mathbb{Q}(\sqrt{3d})} + 1.$$

Next we describe the $k(\zeta_3)k_1^a$ by Kummer Theory.

LEMMA 2.3. *Let $k = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field, and χ be a nontrivial character of $Gal(k/\mathbb{Q})$. Assume that both $A_{\mathbb{Q}(\sqrt{3d})}$ and $A_{\mathbb{Q}(\sqrt{-d})}$ are 3-elementary. Denote $F = k(\zeta_3)$. Then the compositum Fk_1^a of F and k_1^a is contained in $F(\sqrt[3]{\varepsilon}, \sqrt[3]{\alpha_1}, \cdots, \sqrt[3]{\alpha_t})$ where ε is the fundamental unit of $\mathbb{Q}(\sqrt{3d})$ and α_i such that $\mathfrak{p}^3 = (\alpha_i)$ for ideals $\mathfrak{p}$ of $\mathbb{Q}(\sqrt{3d})$.*

Proof. Let $X_F := Gal(M_F/F)/3Gal(M_F/F)$ and $X_{F,\chi}$ be the χ-component of X_F for the nontrivial character χ of $Gal(k/\mathbb{Q})$. Let S be a subset of $F^\times/(F^\times)^3$ corresponding to the X_F. Then, by Kummer theory, we have a perfect pairing $S_{\chi\omega} \times X_{F,\chi} \longrightarrow \mu_3$, where ω is the nontrivial character of $Gal(\mathbb{Q}(\sqrt{-3})/\mathbb{Q})$ and $S_{\chi\omega}$ is the $\chi\omega$-component of S. Note that $S \simeq E_F/{E_F}^3 \times A_F/{A_F}^3 \times < 3 > / < 3 >^3$, where E_F is the group of units of F and A_F is the 3-part of the ideal class group of F.(See [3] for example). Therefore Lemma 1 follows since the $\chi\omega$-component $E_{F,\chi\omega}$ of the group of units E_F is the group of the units of the real quadratic subfield $F^+(=\mathbb{Q}(\sqrt{3d}))$ of F, and the $\chi\omega$-component $E_{F,\chi\omega}$ of the group of units E_F and the $\chi\omega$-component $A_{F,\chi\omega}$ of A_F is the idea class group of the real quadratic field $\mathbb{Q}(\sqrt{3d})$. Note that $X_{F,\chi} \simeq X_{k,\chi}$. □

In this paper, we are assuming that the first layer of anti-cyclotomic $\mathbb{Z}_3$-extension of k is a part of Hilbert class field. We wiil describe a criterion for telling k_1^a from the rest of Hilber class field of k. The following statement is used in [2] to give an example with the Iwasawa invariants $\mu = \lambda = 0$ without proof. Here we give a proof of it.

LEMMA 2.4. *Let p be an odd prime, $k = \mathbb{Q}(\sqrt{-d})$ an imaginary quadratic field such that $A_{\mathbb{Q}(\sqrt{-d})}$ is p-elementary, p is unramified in $k/\mathbb{Q}$, and $\zeta_3 \notin k$. Assume that $k_\infty^a \cap H_k = k_1^a$. Then the image of $Gal(X_{k,\chi}/k_\infty^a)$ in $Gal(H_k/k)$ corresponds to a subgroup B_k of the ideal class group A_k of k consisting of classes c with the following property: If $\mathfrak{a} \in c$, then $\mathfrak{a}^p = (\alpha)$, where α is an $\mathfrak{L}$-adic p-th power for every prime $\mathfrak{L}$ of k lying above p.*

Proof. Denote the ray class field of k of conductor p^2 by $\mathfrak{R}(p^2)$. Then, by class field theory, we see that $\mathfrak{R}(p^2) = k_1^c k_2^a H_k R$ for some abelian extension R of k of degree m prime to p. By assumption, $\mathfrak{p}^p = (\alpha)$ for some $\alpha \in k$, and if $\mathfrak{p}$ corresponds to a class in B_k, then $\mathfrak{p}$ splits in k_1^a/k. Hence $(\mathfrak{p}, \mathfrak{R}(p^2)/k)^{pm}$ is the identity map. Hence $\alpha^m \equiv 1 \bmod p^2$, which implies that α^m is a $\mathfrak{L}$-adic p-th power for every prime $\mathfrak{L}$ above p. Since m is prime to p, α is also a $\mathfrak{L}$-adic p-th power. Now assume that $\mathfrak{p}$ corresponds to a class in A_k but not in B_k. Then $(\mathfrak{p}, \mathfrak{R}(p^2)/k)^p$ is not the identity map. Hence $\alpha \not\equiv 1 \bmod p^2$., which implies that α is not a $\mathfrak{L}$-adic cubic for some $\mathfrak{L}$ lying above p. □

Now we state the main theorem of this paper, and give an example. Choose prime ideals $\mathfrak{p}_i$'s which represents the classes of B_k and does not lie above 3.

THEOREM 2.5. *Let $k = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field such that $Gal(H_k/k)$ is 3-elementary and $\zeta_3 \notin k$. Assume that $k_\infty^a \cap H_k = k_1^a$. Then there exists a unique extension M_3 of F in $M_{F,\chi}$ such that every prime ideal of F above $\mathfrak{p}_i (1 \le i \le t)$ split completely in M_3/F. Moreover, $M_3 = F(\sqrt[3]{\beta})$, and hence $k_1^a = k(\eta)$, where $\beta \in S_{\chi\omega}$ and $\eta = Tr_{M_3/k_1^a}(\sqrt[3]{\beta})$.*

Proof. Since the rank of $X_{F,\chi}$ is the same as that of A_k by Theorem 1 and Lemma 1, the extension field N_3 of degree 3 of F in $M_{F,\chi}$ is always equal to the compositum FL of F and L, where L is an extension of degree 3 of k in H_k. Moreover L is uniquely determined when N_3 is given because $Gal(F/k)$ is cyclic group of order 6. Let M_3 be the extension of F satisfying properties in Theorem 3. Then the primes of F above B_k splits completely in M_3/F and $M_3 = FL$. Hence the prime of k in B_k splits completely in L/k, which shows that $L = k_1^a$. The last statement of Theorem 3 comes from Theorem 5.3.5 in [1]. □

Now we give an example.

Let $k = \mathbb{Q}(\sqrt{-4027})$ be an imaginary quadratic field and $\mathfrak{p}_2$ is a prime ideal of $\mathbb{Q}(\sqrt{12081})$ above 2. Then

$$k_1^a = k(\sqrt[3]{\varepsilon^2\alpha} - 2\sqrt[3]{\varepsilon^{-2}\alpha^{-1}})$$

where ε is the fundamental unit of $\mathbb{Q}(\sqrt{12081})$ and $\mathfrak{p}_2^3 = (\alpha)$.

We can take $\alpha = 81((-1+\sqrt{12081})/2)+4492$ and $\varepsilon = (17288113122+157288204\sqrt{12081})^2/12$. By Lemma 1 and Theorem 2, β is one of the followings; $\varepsilon, \varepsilon\alpha, \varepsilon^2\alpha, \alpha$. We choose a prime ideal $\mathfrak{p}_{19,k}$ of k lying above 19. We see that $\mathfrak{p}_{19,k}$ is actually in B_k since ${\mathfrak{p}_{19,k}}^3 = (\gamma)$ and γ is a $\mathfrak{L}$-adic 3-rd power.(See [2]) Hence, by Lemma 2, β shoud be a cubuc modulo $\mathfrak{p}_{19,\mathbb{Q}(\sqrt{12081})}$, where $\mathfrak{p}_{19,\mathbb{Q}(\sqrt{12081})}$ is a prime ideal of $\mathbb{Q}(\sqrt{12081})$ lying above 19. We can easily check, by Maple, only $\varepsilon^2\alpha$ is a cubic modulo $\mathfrak{p}_{19,\mathbb{Q}(\sqrt{12081})}$. Since $\sigma \in Gal(M_3/k_1^a)$ satisfies $\sigma^2 = 1$, we have $\sqrt[3]{\varepsilon^2\alpha}^{\sigma} = -2\sqrt[3]{\varepsilon^{-2}\alpha^{-1}}$ and therefore $\eta = Tr_{M_3/k_1^a}(\sqrt[3]{\beta}) = \sqrt[3]{\varepsilon^2\alpha} - 2\sqrt[3]{\varepsilon^{-2}\alpha^{-1}}$.

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