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AN EXPLICIT FORM OF POWERS OF A 2×2 MATRIX USING A RECURSIVE SEQUENCE

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Abstract. The purpose of this paper is to derive powers A^n using a system of recursive sequences for a given 2×2 matrix A. Introducing a recursive sequence we have a quadratic equation. Solutions to this quadratic equation are related with eigenvalues of A. By solving this quadratic equation we can easily obtain an explicit form of A^n. Our method holds when A is defined not only on the real field but also on the complex field.

1. Introduction

An $m \times m$ matrix A is said to be diagonalisable if it is similar to a diagonal matrix; i.e., if there exists an invertible matrix U and a diagonal matrix D such that $A = UDU^{-1}$. A matrix A is diagonallisable if and only if A has m linearly independent eigenvectors, Searle (2006). When A is diagonalisable, we can obtain the powers A^n as $A^n = UD^nU^{-1}$ where n is a positive integer. But if A is not diagonalisable, it is not easy to derive the powers A^n explicitly.

In this paper we are interested in deriving an explicit form of the powers A^n of a 2×2 matrix $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. As mentioned above if A has two eigenvalues, then A is diagonalisable, hence the powers A^n can be obtained easily. But if A has only one eigenvalue with multiplicity 2, then A is not diagonalisable. In this case it is not easy to derive the powers A^n explicitly.

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An explicit form of A^n may be derived in various ways. For example, an explicit form of A^n can be derived using a diagonalisation of A, also using Cayley-Hamilton theorem, Bronson (1991).

In this paper we derive an explicit form of A^n by solving a system of recursive sequences. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real or complex numbers satisfying

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}, \quad \text{for} \quad n = 1, 2, 3, \cdots . \tag{1.1}$$

This equation can be rewritten as a system of recursive equations

$$a_{n+1} = pa_n + qb_n \tag{1.2}$$

$$b_{n+1} = ra_n + sb_n. \tag{1.3}$$

Computing $(1.2) + \lambda \cdot (1.3)$ gives us

$$a_{n+1} + \lambda b_{n+1} = (p + \lambda r) \left(a_n + \frac{q + \lambda s}{p + \lambda r} b_n \right). \tag{1.4}$$

Let $\lambda = \frac{q+\lambda s}{p+\lambda r}$, then equation (1.4) becomes

$$a_{n+1} + \lambda b_{n+1} = (p + \lambda r)(a_n + \lambda b_n), \tag{1.5}$$

it follows that, for $n = 1, 2, 3, \cdots$,

$$a_{n+1} + \lambda b_{n+1} = (p + \lambda r)^n (a_1 + \lambda b_1). \tag{1.6}$$

From the equation $\lambda = \frac{q+\lambda s}{p+\lambda r}$, we have a quadratic equation

$$\lambda^2 r + (p - s)\lambda - q = 0. \tag{1.7}$$

By solving the equation (1.7) an explicit form of the powers A^n can be derived in terms of the components of A and λ. The derivation of an explicit form of A^n depends on whether (1.7) has two distinct solutions or not.

Before we proceed to describe our main results, we introduce some results of diagonalisation theorem.

If an $n \times n$ matrix A has eigenvalues λ_k with multiplicity m_k for $k = 1, 2, \cdots, s$ and $\sum_{k=1}^{s} m_k = n$, diagonalisation theorem is given by:

THEOREM 1.1. (Searle, 2006) *$A_{n\times n}$, having egivenvalues λ_k with multiplity m_k has n eigenvectors that are linearly independent if and only if $rank(A - \lambda_k I) = n - m_k$ for all $k = 1, 2, \cdots, s$; whereupon there exists an invertible matrix U and A is diagonalisable as $U^{-1}AU = D$, where I is an identity matrix and D is a diagonal matrix.*

Throughout this paper we consider only a 2×2 matrix A that is not a diagonal matrix.

It is easy to see that if A has two eigenvalues λ_1 and λ_2 with eigenvectors x_1 and x_2, respectively, then x_1 and x_2 are linearly independent. Hence A is diagonalisable.

Assume that a 2×2 matrix A has only one eigenvalue λ with multiplicity 2. Then unless A is of the form kI, $rank(A - \lambda I)$ can not be zero, whereas $n - m_k = 0$. Hence A can not be diagonalisable.

A derivation of A^n when $r \neq 0$ is described in section 2. In section 3 we derive A^n when $r = 0$.

2. When r is not equal to 0.

Firstly we consider the case when $r \neq 0$, so that the equation (1.7) has two different solutions or one multiple solution.

2.1. In case of two distinct solutions.

Let λ_1 and λ_2 be two different solutions (real or complex) to equation (1.7). Then we have

$$\begin{cases} a_{n+1} + \lambda_1 b_{n+1} = (p + \lambda r)^n (a_1 + \lambda b_1) \\ a_{n+1} + \lambda_2 b_{n+1} = (p + \lambda r)^n (a_1 + \lambda b_1). \end{cases} \tag{2.1}$$

This equation can be represented in matrix-vector form as follows:

$$\begin{pmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{pmatrix} \begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} (p+\lambda_1 r)^n & \lambda_1 (p+\lambda_1 r)^n \\ (p+\lambda_2 r)^n & \lambda_2 (p+\lambda_2 r)^n \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}. \tag{2.2}$$

Since $\lambda_1 \neq \lambda_2$, we have

$$\begin{aligned} &\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{pmatrix}^{-1} \begin{pmatrix} (p+\lambda_1 r)^n & \lambda_1 (p+\lambda_1 r)^n \\ (p+\lambda_2 r)^n & \lambda_2 (p+\lambda_2 r)^n \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}. \end{aligned} \tag{2.3}$$

But from the equation (1.1)

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}^n \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}. \tag{2.4}$$

Hence we have

$$\begin{aligned}
&\begin{pmatrix} p & q \\ r & s \end{pmatrix}^n \\
&= \frac{1}{\lambda_2-\lambda_1}\begin{pmatrix} \lambda_2 & -\lambda_1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} (p+\lambda_1 r)^n & \lambda_1(p+\lambda_1 r)^n \\ (p+\lambda_2 r)^n & \lambda_2(p+\lambda_2 r)^n \end{pmatrix}.
\end{aligned} \tag{2.5}$$

Now we have the result in the following theorem.

THEOREM 2.1. *Let* $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ *and* λ_1, λ_2 *are distinct solutions to the equation* $\lambda^2 r + (p-s)\lambda - q = 0$*. If* $A^n = \begin{pmatrix} P_n & Q_n \\ R_n & S_n \end{pmatrix}$*, then we have*

$$\begin{aligned}
P_n &= \frac{\lambda_2(p+\lambda_1 r)^n - \lambda_1(p+\lambda_2 r)^n}{\lambda_2-\lambda_1} \\
Q_n &= \frac{\lambda_1\lambda_2\left[(p+\lambda_1 r)^n - (p+\lambda_2 r)^n\right]}{\lambda_2-\lambda_1} \\
R_n &= \frac{-(p+\lambda_1 r)^n + (p+\lambda_2 r)^n}{\lambda_2-\lambda_1} \\
S_n &= \frac{-\lambda_1(p+\lambda_1 r)^n + \lambda_2(p+\lambda_2 r)^n}{\lambda_2-\lambda_1}
\end{aligned}$$

Note that $p + \lambda_1 r$ and $p + \lambda_2 r$ play important roles in Theorem 3.1. Actually they are eigenvalues of A as seen in the next theorem.

THEOREM 2.2. *Let a* 2×2 *matrix* A *and* λ_1, λ_2 *are defined as in Theorem 2.1. Then* $p + \lambda_1 r$ *and* $p + \lambda_2 r$ *are eigenvalues of* A*.*

Proof. Let $\alpha_1 = p+\lambda_1 r$ and $\alpha_2 = p+\lambda_2 r$. Then since $\lambda_1+\lambda_2 = -\frac{p-s}{r}$,

$$\begin{aligned}
\alpha_1 + \alpha_2 &= (p+\lambda_1 r) + (p+\lambda_2 r) \\
&= 2p + (\lambda_1+\lambda_2)r = p + s
\end{aligned}$$

and since $\lambda_1\lambda_2 = -\frac{q}{r}$,

$$\begin{aligned}
\alpha_1\alpha_2 &= (p+\lambda_1 r)(p+\lambda_2 r) \\
&= p^2 + (\lambda_1+\lambda_2)pr + \lambda_1\lambda_2 r^2 = ps - qr.
\end{aligned}$$

Hence α_1 and α_2 are solutions to the characteristic equation

$$x^2 - (p+s)x + (ps - qr) = 0, \tag{2.6}$$

therefore α_1 and α_2 are eigenvalues of A. □

Using the result of Theorem 2.2 we have the powers A^n in terms of λ_i's and eigenvalues of A in the following theorem.

COROLLARY 2.3. *Let α_1 and α_2 be two distinct eigenvalues of A and let $\lambda_i = \frac{\alpha_i - p}{r}$, for $i = 1, 2$. Then we have*

$$\begin{aligned} P_n &= \frac{\lambda_2 \alpha_1^n - \lambda_1 \alpha_2^n}{\lambda_2 - \lambda_1} \\ Q_n &= \frac{\lambda_1 \lambda_2 \left(\alpha_1^n - \alpha_2^n\right)}{\lambda_2 - \lambda_1} \\ R_n &= \frac{-\alpha_1^n + \alpha_2^n}{\lambda_2 - \lambda_1} \\ S_n &= \frac{-\lambda_1 \alpha_1^n + \lambda_2 \alpha_2^n}{\lambda_2 - \lambda_1} \end{aligned}$$

From this result we can see that the relationship between λ_i's and the eigenvalues of A.

2.2. In case of one multiple solution

When the equation (1.7) has only one solution λ with multiplicity 2, then the matrix A is not diagonalisable as seen in section 2. In this section we use the result of the previous section.

Since Theorem 2.1 holds for all $\lambda_1 \neq \lambda_2$, we can obtain the powers A^n by taking limits as $\lambda_2 \to \lambda_1$, that is,

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}^n = \begin{pmatrix} \lim_{\lambda_2 \to \lambda_1} P_n & \lim_{\lambda_2 \to \lambda_1} Q_n \\ \lim_{\lambda_2 \to \lambda_1} R_n & \lim_{\lambda_2 \to \lambda_1} S_n \end{pmatrix} \tag{2.7}$$

and then we have the powers by replacing λ_1 with λ.

THEOREM 2.4. *If equation (1.7) has only one solution λ with multiplicity 2 and let $A^n = \begin{pmatrix} P'_n & Q'_n \\ R'_n & S'_n \end{pmatrix}^n$, then we have*

(a) $P'_n = (p + \lambda r)^{n-1}(p + \lambda r - \lambda n r)$
(b) $Q'_n = -n r \lambda^2 (p + \lambda r)^{n-1}$
(c) $R'_n = n r (p + \lambda r)^{n-1}$
(d) $S'_n = (p + \lambda r)^{n-1}(p + \lambda r + \lambda n r)$

Proof. Since all of P_n, Q_n, R_n, and S_n are of the form $\frac{0}{0}$ in Theorem 2.1, we can use L'Hôpital's rule. Here we give a proof of part (a)

$$\begin{aligned}\lim_{\lambda_2\to\lambda_1} P_n &= \lim_{\lambda_2\to\lambda_1} \frac{\lambda_2(p+\lambda_1 r)^n - \lambda_1(p+\lambda_2 r)^n}{\lambda_2-\lambda_1}\\ &= \lim_{\lambda_2\to\lambda_1} \left\{(p+\lambda_1 r)^n - \lambda_1 nr(p+\lambda_2 r)^{n-1}\right\}\\ &= (p+\lambda_1 r)^{n-1}(p+\lambda_1 r-\lambda_1 nr)\end{aligned}$$

Replacing λ_1 with λ in the last equation we have the result. Similarly, we can show the results (b), (c), and (d). So they can be omitted. □

Since denominators and numerators of P_n, Q_n, R_n, and S_n in Theorem 2.1 are polynomials in λ_2, all of them are differentiable even if they are complex numbers. Hence we can use L'Hôpital's rule to find limits in equation (3.7), too, see Ko (2007). Therefore Theorem 2.4 holds when λ is a complex number.

3. When r is equal to 0

When $r=0$, let $A^n = \begin{pmatrix} P_n & Q_n \\ R_n & S_n \end{pmatrix}$. Then since

$$\begin{pmatrix} P_{n+1} & Q_{n+1} \\ R_{n+1} & S_{n+1} \end{pmatrix} = \begin{pmatrix} P_n & Q_n \\ R_n & S_n \end{pmatrix}\begin{pmatrix} p & q \\ 0 & s \end{pmatrix} \tag{3.1}$$

we have, for $n = 1, 2, 3, \cdots$,

$$\begin{cases} P_{n+1} = pP_n \\ Q_{n+1} = qP_n + sQ_n \\ R_{n+1} = pR_n \\ S_{n+1} = sS_n. \end{cases} \tag{3.2}$$

Solving these equations we have

$$P_n = p^n,\ Q_n = \begin{cases} q\frac{p^n-s^n}{p-s}, & \text{if } p\neq s \\ nqp^{n-1}, & \text{if } p = s \end{cases},\quad R_n = 0,\ \text{and } S_n = s^n. \tag{3.3}$$

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