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ON THE SUPERSTABILITY OF THE PEXIDER TYPE SINE FUNCTIONAL EQUATION

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Abstract. The aim of this paper is to investigate the superstability of the pexider type sine(hyperbolic sine) functional equation

$$f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 = \lambda g(x)h(y), \qquad \lambda : constant$$

which is bounded by the unknown functions $\varphi(x)$ or $\varphi(y)$.

As a consequence, we have generalized the stability results for the sine functional equation by P. M. Cholewa, R. Badora, R. Ger, and G. H. Kim.

1. Introduction

The stability problem of the functional equation was conjectured by Ulam [18] during the conference in the university of Wisconsin in 1940. In the next year, it was solved by Hyers [10] in the case of additive mapping, which is called the Hyers-Ulam stability. Thereafter, this problem was improved by D. G. Bourgin [6] , T. Aoki [1], Th. M. Rassias [17], R. Ger [9], and P. Găvruta [8], in which Rassias's result was called the Hyers-Ulam-Rassias stability.

In 1979, J. Baker et al. in [5] developed the superstability, which is if f satisfies the stability inequality $|E_1(f) - E_2(f)| \leq \varepsilon$, then either f is bounded or $E_1(f) = E_2(f)$.

Baker [4] showed the superstability of the cosine functional equation (also called the d'Alembert functional equation)

$$\text{(A)} \qquad f(x+y) + f(x-y) = 2f(x)f(y).$$

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The stability of the generalized functional equations of (A)

$$(A_{fg}) \qquad f(x+y)+f(x-y)=2f(x)g(y),$$
$$(T_{gf}) \qquad f(x+y)-f(x-y)=2g(x)f(y),$$

was researched in papers ([2], [3], [11], [13], [14], [16]), in which (A_{fg}), as the generalization of the cosine functional equation (A), is called the Wilson functional equation, and (T_{gf}) is the mixed trigonometric functional equation.

The superstability bounded by a constant for the sine functional equation

$$\text{(S)} \qquad f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 = f(x)f(y),$$

was investigated by P. W. Cholewa [7] and was improved by R. Badora and R. Ger [3]. Thereafter, these equations were studied by many mathematician such as Badora, Ger, Kalra, Parnami, Vasudeva, Kannappan, Kim, et al([3], [11], [12], [15],[16]).

In their work, we can have doubts about the boundedness of function. Because, as despite the fact that the cosine and sine functions are essentially bounded, their proof had used the assumption of the unboundedness.

However, due to the evidence in the following paragraph, we can know that no doubt about this problem disappears.

The hyperbolic cosine function, hyperbolic sine function, hyperbolic trigonometric function, and some exponential functions also satisfy the above mentioned equations; thus they can be called the *hyperbolic* cosine (sine, trigonometric) exponential, and Jensen functional equation, respectively. For example,

$$\cosh(x+y)+\cosh(x-y)=2\cosh(x)\cosh(y)$$
$$\sinh^2\left(\tfrac{x+y}{2}\right)-\sinh^2\left(\tfrac{x-y}{2}\right)=\sinh(x)\sinh(y)$$
$$=\cosh^2\left(\tfrac{x+y}{2}\right)-\cosh^2\left(\tfrac{x-y}{2}\right)$$
$$\sinh(x+y)+\sinh(x-y)=2\sinh(x)\cosh(y)$$
$$ca^{x+y}+ca^{x-y}=2ce^{x}\,\frac{a^y+a^{-y}}{2}$$
$$e^{x+y}+e^{x-y}=2e^x\cosh(y)$$
$$(n(x+y)+c)+(n(x-y)+c)=2(nx+c) : \text{Jensen equation},$$

for $f(x)=nx+c$, where a and c are constants.

In this paper, we will study the pexider type sine functional equation

$$(\widetilde{S_{gh}^{\lambda}}) \qquad f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 = \lambda g(x)h(y)$$

of the sine functional equation (S), where f, g, and h are the unknown functions to be determined, and its special cases are as follows :

$$(\widetilde{S^{\lambda}}) \qquad f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 = \lambda f(x)f(y),$$

$$(\widetilde{S_{gf}^{\lambda}}) \qquad f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 = \lambda g(x)f(y),$$

$$(\widetilde{S_{fh}^{\lambda}}) \qquad f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 = \lambda f(x)h(y),$$

$$(\widetilde{S_{gg}^{\lambda}}) \qquad f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 = \lambda g(x)g(y),$$

The other example of equation $(\widetilde{S_{gh}^{\lambda}})$ with $\sigma(x) = -x$ is following:

$$\left(\alpha a^{\beta\left(\frac{x+y}{2}\right)+\gamma}\right)^2 - \left(\alpha a^{\beta\left(\frac{x-y}{2}\right)+\gamma}\right)^2 = \alpha^2 a^{2\gamma} \cdot a^{\beta x} \cdot \left(a^{\beta y} - a^{-\beta y}\right),$$

in which $f(x) = \alpha a^{\beta x+\gamma}$, $g(x) = a^{\beta x}$, $h(y) = a^{\beta y} - a^{-\beta y}$, and $\lambda = \alpha^2 a^{2\gamma}$, where $a, \alpha, \beta, \gamma, \lambda$ are constants.

In a particular case of it, it satisfies the hyperbolic sine functional equation , that is,

$$\left(\frac{e^{\frac{x+y}{2}} - e^{-\left(\frac{x+y}{2}\right)}}{2}\right)^2 - \left(\frac{e^{\frac{x-y}{2}} - e^{-\left(\frac{x-y}{2}\right)}}{2}\right)^2 = \frac{e^x - e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2}.$$

In this paper, let $(G, +)$ be a uniquely 2-divisible Abelian group, $\mathbb{C}$ the field of complex numbers, $\mathbb{R}$ the field of real numbers, λ a nonzero scalar, and let σ be an endomorphism of G with $\sigma(\sigma(x)) = x$ for all $x \in G$ with a notation $\sigma(x) = \sigma x$. The properties $g(x) = g(\sigma x)$ and $g(x) = -g(\sigma x)$ with respect to σ will be represented as the even and odd functions, respectively. In each equation, if "σy" is replaced by "$-y$", and as is also in the case "$\lambda = 1$", it will be remove the notation tilde and the notation λ as (S).

We assume that f and g are nonzero functions, ε is a nonnegative real constant, and $\varphi : G \to \mathbb{R}$ be a mapping.

Given the mappings $f, g, h : G \to \mathbb{C}$, we define the difference operator $\widetilde{DS^{\lambda}_{gh}} : G \times G \to \mathbb{C}$ as

$$(\widetilde{DS^{\lambda}_{gh}}) \qquad \widetilde{DS^{\lambda}_{gh}}(x,y) := f\big(\frac{x+y}{2}\big)^2 - f\big(\frac{x+\sigma y}{2}\big)^2 - \lambda g(x)h(y).$$

The aim of this paper is to investigate the superstability of the pexider type sine functional equation $(\widetilde{S^{\lambda}_{gh}})$ under the conditions $|\widetilde{DS_{gh}}(x,y)| \le \varphi(x)$, $\varphi(y)$ and $\min\{\varphi(x), \varphi(y)\}$. As a consequence, the obtained results validate justly the superstability of the equations $(\widetilde{S^{\lambda}_{gf}})$, $(\widetilde{S^{\lambda}_{fh}})$, $(\widetilde{S^{\lambda}_{gg}})$, and $(\widetilde{S^{\lambda}})$, which are bounded by $\varphi(x)$, $\varphi(y)$, $\min\{\varphi(x), \varphi(y)\}$, and a constant. Furthermore, the range of the function in all the results expands to the Banach algebra. Applying with $\sigma(x) = -x$ and $\lambda = 1$ in the obtained results, as corollaries, we can obtain the superstability of the sine type functional equations (S), (S_{fg}), (S_{gf}), (S_{gg}), and (S_{gh}), which can be found in the papers([3], [7], [15], [16]). For simplicity, we will use the notations of the equation as follows :

$$(\widetilde{A^{\lambda}}) \qquad f(x+y) + f(x+\sigma y) = \lambda f(x)f(y)$$

$$(\widetilde{S}) \qquad f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 = f(x)f(y).$$

2. Stability of the equation $(\widetilde{S^{\lambda}_{gh}})$

We will now investigate the superstability of the Pexider type functional equation $(\widetilde{S^{\lambda}_{gh}})$ of the sine functional equation (S).

THEOREM 2.1. *Suppose that $f, g, h : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 - \lambda g(x)h(y) \right| \le \varphi(y) \tag{2.1}$$

for all $x, y \in G$. Then either g is bounded or h satisfies $(\widetilde{S})$. In addition, if g satisfies $(\widetilde{A^{\lambda}})$, then h and g satisfy

$$(\widetilde{T^{\lambda}_{gh}}) \qquad h(x+y) - h(x+\sigma y) = \lambda g(x)h(y).$$

Proof. Let g be unbounded. Then we can choose a sequence $\{x_n\}$ in G such that

$$0 \ne |g(2x_n)| \to \infty, \quad \text{as} \quad n \to \infty. \tag{2.2}$$

Inequality (2.1) may equivalently be written as

$$|f(x+y)^2 - f(x+\sigma y)^2 - \lambda g(2x)h(2y)| \le \varphi(2y) \quad \forall\, x,y \in G \tag{2.3}$$

Taking $x = x_n$ in (2.3), we obtain

$$\left| \frac{f(x_n+y)^2 - f(x_n+\sigma y)^2}{\lambda g(2x_n)} - h(2y) \right| \le \frac{\varphi(2y)}{|\lambda g(2x_n)|},$$

that is, using (2.2)

$$h(2y) = \lim_{n\to\infty} \frac{f(x_n+y)^2 - f(x_n+\sigma y)^2}{\lambda g(2x_n)}. \tag{2.4}$$

Using (2.1), we have

$$\begin{aligned}
2\varphi(y) \ge & \left| f\left(x_n + \frac{x+y}{2}\right)^2 - f\left(x_n + \frac{x+\sigma y}{2}\right)^2 - \lambda g(2x_n+x)h(y) \right| \\
& + \left| f\left(x_n + \frac{\sigma x+y}{2}\right)^2 - f\left(x_n + \frac{\sigma(x+y)}{2}\right)^2 - \lambda g(2x_n+\sigma x)h(y) \right| \\
& = \left| \left(f\left(x_n + \frac{x+y}{2}\right)^2 - f\left(x_n + \frac{\sigma(x+y)}{2}\right)^2 \right) \right. \\
& \quad - \left(f\left(x_n + \frac{x+\sigma y}{2}\right)^2 - f\left(x_n + \frac{\sigma(x+\sigma y)}{2}\right)^2 \right) \\
& \quad \left. -\lambda \left(g(2x_n+x) + g(2x_n+\sigma x)\right) h(y) \right|.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\frac{2\varphi(y)}{|\lambda g(2x_n)|} \ge & \left| \frac{f\left(x_n + \frac{x+y}{2}\right)^2 - f\left(x_n + \frac{\sigma(x+y)}{2}\right)^2}{\lambda g(2x_n)} \right. \\
& \quad - \frac{f\left(x_n + \frac{x+\sigma y}{2}\right)^2 - f\left(x_n + \frac{\sigma(x+\sigma y)}{2}\right)^2}{\lambda g(2x_n)} \\
& \quad \left. - \frac{\lambda g(2x_n+x) + g(2x_n+\sigma x)}{\lambda g(2x_n)} h(y) \right|
\end{aligned}$$

for all $x, y, x_n \in G$ and every $n \in \mathbb{N}$.

Taking the limit as $n \longrightarrow \infty$ with the use of (2.2) and (2.4), we conclude that, for every $x \in G$, there exists the limit

$$k(x) := \lim_{n\to\infty} \frac{g(2x_n+x) + g(2x_n+\sigma x)}{\lambda g(2x_n)},$$

where the function $k : G \to \mathbb{C}$ satisfies the equation

$$h(x+y) - h(x+\sigma y) = \lambda k(x)h(y) \quad \forall x, y \in G. \tag{2.5}$$

From the definition of k, we get the equality $k(0) = 2/\lambda$, which, joined with (2.5), implies that h has to be odd w.r.t. σ, that is, $h(y) = -h(\sigma y)$. Keeping this in mind, by means of (2.5), we can infer the equality

$$\begin{aligned} h(x+y)^2 - h(x+\sigma y)^2 &= \lambda[h(x+y) + h(x+\sigma y)]k(x)h(y) \\ &= \big[h(2x+y) + h(2x+\sigma y)\big]h(y) \\ &= \big[h(y+2x) - h(y+\sigma(2x))\big]h(y) \\ &= \lambda k(y)h(2x)h(y). \end{aligned}$$

Since the oddness of h forces $h(x+\sigma x) = 0$ for all $x \in G$, by putting $x = y$ in (2.5), we obtain a duplication equation

$$h(2y) = \lambda h(y)k(y) \quad \forall y \in G.$$

This, in return, leads to the equation

$$h(x+y)^2 - h(x+\sigma y)^2 = h(2x)h(2y), \tag{2.6}$$

valid for all $x, y \in G$ which, in light of the unique 2-divisibility of G, states nothing else but $(\widetilde{S})$.

In addition, if g satisfies $(\widetilde{A^\lambda})$, since the limit function k validates g, then (2.5) becomes $(\widetilde{T^\lambda_{gh}})$. $\square$

Theorem 2.2. *Suppose that $f, g, h : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 - \lambda g(x)h(y) \right| \le \varphi(x), \tag{2.7}$$

which meets one of the two cases $g(0) = 0$, $f(x)^2 = f(\sigma x)^2$ for all $x, y \in G$. Then either h is bounded or g satisfies $(\widetilde{S})$. In addition, if h satisfies $(\widetilde{A^\lambda})$, then g and h satisfy the Wilson equation

$$g(x+y) + g(x+\sigma y) = \lambda g(x)h(y). \tag{$\widetilde{A^\lambda_{gh}}$}$$

Proof. We obtain

$$|f(x+y)^2 - f(x+\sigma y)^2 - \lambda g(2x)h(2y)| \le \varphi(2x) \qquad \forall\, x, y \in G \tag{2.8}$$

as an equivalent of (2.7).

Let h be unbounded. Then we can choose a sequence $\{y_n\}$ in G such that $0 \neq |h(2y_n)| \to \infty$ as $n \to \infty$. An obvious slight change in the proof steps applied in Theorem 2.1 gives us

$$g(2x) = \lim_{n\to\infty} \frac{f(x+y_n)^2 - f(x+\sigma y_n)^2}{\lambda h(2y_n)} \quad \text{for all} \quad x \in G. \tag{2.9}$$

Which, by applying (2.9), allows one to state the existence of a functional equation

$$g(x+y) + g(x+\sigma y) = \lambda g(x)p(y) \qquad \forall x, y \in G, \tag{2.10}$$

where $p : G \to \mathbb{C}$ satisfies the limit equation

$$p(y) := \lim_{n\to\infty} \frac{h(y+2y_n) + h(\sigma y + 2y_n)}{\lambda h(2y_n)}.$$

From the definition of p, we get the equality $p(y) = p(\sigma y)$.

First, consider the case $g(0) = 0$. We know from (2.10) that

$$g(0) = 0 \Leftrightarrow g(x) = -g(\sigma x) \Leftrightarrow g(x+\sigma x) = 0. \tag{2.11}$$

Putting $y = x$ in (2.10), by (2.11) we obtain a duplication equation

$$g(2x) = \lambda g(x)p(x). \tag{2.12}$$

Using (2.11) and (2.12) for g, we obtain, by means of (2.10), the equation

$$\begin{aligned} g(x+y)^2 - g(x+\sigma y)^2 &= \lambda g(x)p(y)[g(x+y) - g(x+\sigma y)] \\ &= g(x)[g(x+2y) - g(x+2\sigma y)] \\ &= g(x)[g(x+2y) + g(\sigma x+2y)] \\ &= \lambda g(x)g(2y)p(x) = g(2x)g(2y), \end{aligned}$$

which holds true for all $x, y \in G$, and, in light of the unique 2-divisibility of G, states nothing else but $(\widetilde{S})$.

In the next case $f(x)^2 = f(\sigma x)^2$, it is enough to show that $g(0) = 0$.

Suppose that this does not hold, i.e. $g(0) \neq 0$. Putting $x = 0$ in (2.8) with $f(y)^2 = f(\sigma y)^2$ and the 2-divisibility of group G, we obtain the inequality

$$|h(y)| \leq \frac{\varphi(0)}{|\lambda g(0)|} \qquad \forall\, y \in G.$$

This inequality means that h is globally bounded, which is a contradiction. Thus, the claimed $g(0) = 0$ holds, hence g satisfies $(\widetilde{S})$.

For the additive case, assume that h satisfies $(\widetilde{A^\lambda})$. Then, since the limited function p becomes h, the equation (2.10) implies $(\widetilde{A^\lambda_{gh}})$. The proof of the theorem is completed. □

From Theorems 2.1 and 2.2, we can obtain the following result immediately :

THEOREM 2.3. *Suppose that $f, g, h : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 - \lambda g(x)h(y) \right| \leq \min\{\varphi(x), \varphi(y)\} \qquad \forall\, x, y \in G. \tag{2.13}$$

Then,

(i) either g is bounded or h satisfies $(\widetilde{S})$. In addition, if g satisfies $(\widetilde{A^\lambda})$, g and h satisfy $(\widetilde{T^\lambda_{gh}})$.

(ii) either h is bounded or g satisfies $(\widetilde{S})$ under one of the two cases $g(0) = 0$, $f(x)^2 = f(\sigma x)^2$. If, additionally, h satisfies $(\widetilde{A^\lambda})$, then g and h satisfy $(\widetilde{A^\lambda_{gh}})$.

COROLLARY 2.4. *Suppose that $f, g, h : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 - \lambda g(x)h(y) \right| \leq \varepsilon \qquad \forall\, x, y \in G.$$

Then,

(i) either g is bounded or h satisfies $(\widetilde{S})$. In addition, if g satisfies $(\widetilde{A^\lambda})$, g and h satisfy $(\widetilde{T^\lambda_{gh}})$.

(ii) either h is bounded or g satisfies $(\widetilde{S})$ under one of the two cases $g(0) = 0$, $f(x)^2 = f(\sigma x)^2$. If, additionally, h satisfies $(\widetilde{A^\lambda})$, then g and h satisfy $(\widetilde{A^\lambda_{gh}})$.

COROLLARY 2.5 ([15]). *Suppose that $f, g, h : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 - g(x)h(y) \right| \leq \varepsilon \qquad \forall\, x, y \in G.$$

Then,

(i) either g is bounded or h satisfies (S). In addition, if g satisfies (A), g and h satisfy (T_{gh}).

(ii) either h is bounded or g satisfies (S) under one of the two cases $g(0) = 0$, $f(x)^2 = f(-x)^2$. In addition, if h satisfies (A), then g and h satisfy (A_{gh}).

Remark 2.6. (i) In all results of the section, by putting $\sigma(x) = -x$, by putting $\lambda = 1$ in sine type equations ($\widetilde{S}$, $\widetilde{S^{\lambda}_{gh}}$) and $\lambda = 2$ in cosine and trigonometric type equations ($\widetilde{A^{\lambda}}$,$\widetilde{A^{\lambda}_{gh}}$, $\widetilde{T^{\lambda}_{gh}}$), we obtain the same numbers of corollaries.

(ii) Replacing h and g by f in Corollary 2.5, it is found in Cholewa [7].

3. Application to equations ($\widetilde{S^{\lambda}_{gf}}$), ($\widetilde{S^{\lambda}_{fh}}$), ($\widetilde{S^{\lambda}_{gg}}$), and ($\widetilde{S^{\lambda}}$).

From Section 2's results, by replacing h (or g) by f, replacing h by g, and replacing h and g by f, we obtain the same numbers of corollaries in each case. These imply the superstability of the functional equations ($\widetilde{S^{\lambda}_{gf}}$), ($\widetilde{S^{\lambda}_{fh}}$), ($\widetilde{S^{\lambda}_{gg}}$), and ($\widetilde{S^{\lambda}}$). Each proof also runs along the same processes as those mentioned in section 2. Hence, almost of the proofs will be skipped.

3.1. Stability of equation ($\widetilde{S^{\lambda}_{gf}}$)

Replacing h by f in Theorems 2.1, 2.2, 2.3, and Corollary 2.4. We obtain the following results, respectively.

Corollary 3.1. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 - \lambda g(x)f(y) \right| \leq \varphi(y)$$

for all $x, y \in G$. Then either g is bounded or f satisfies ($\widetilde{S}$). In addition, if g satisfies ($\widetilde{A^{\lambda}}$), then f and g satisfy the equation

$$(\widetilde{T^{\lambda}_{ffgf}}) \qquad f(x+y) - f(x+\sigma y) = \lambda g(x)f(y).$$

Corollary 3.2. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 - \lambda g(x)f(y) \right| \leq \varphi(x),$$

which satisfies one of the two cases $g(0) = 0$, $f(x)^2 = f(\sigma x)^2$ for all $x, y \in G$. Then either f is bounded or g satisfies ($\widetilde{S}$).

In addition, if f satisfies $(\widetilde{A^\lambda})$, then f and g satisfy the Wilson equation

$$(\widetilde{A^\lambda_{gggf}}) \qquad g(x+y)+g(x+\sigma y)=\lambda g(x)f(y).$$

Corollary 3.3. *Suppose that $f,g:G\to\mathbb{C}$ satisfy the inequality*

$$\left|f\left(\frac{x+y}{2}\right)^2-f\left(\frac{x+\sigma y}{2}\right)^2-\lambda g(x)f(y)\right|\le\min\{\varphi(x),\varphi(y)\} \tag{3.1}$$

for all $x,y\in G$.

Then,

(i) either g is bounded or f (and g) satisfy $(\widetilde{S})$. In addition, if g satisfies $(\widetilde{A^\lambda})$, f and g satisfy $(\widetilde{T^\lambda_{ffgf}})$.

(ii) either f is bounded or g satisfies $(\widetilde{S})$ under one of the two cases $g(0)=0$, $\widetilde{f(x)^2}=f(\sigma x)^2$. If, additionally, f satisfies $(\widetilde{A^\lambda})$, then g and f satisfy $(\widetilde{A^\lambda_{gggf}})$.

Proof. It is enough to show that either g is bounded or g satisfies $(\widetilde{S})$. The other cases follow from Corollary 3.1 and Corollary 3.2, immediately.

Inequality (3.1) can equivalently written as

$$\begin{aligned}&|f(x+y)^2-f(x+\sigma y)^2-\lambda g(2x)f(2y)|\\ &\qquad\le\min\{\varphi(2x),\varphi(2y)\}\quad\forall\, x,y\in G.\end{aligned} \tag{3.2}$$

First, if f is bounded, choose $y_0\in G$ such that $f(2y_0)\neq 0$, and then, by (3.2), we obtain

$$\begin{aligned}&\left|\frac{f(x+y_0)^2-f(x-y_0)^2}{f(2y_0)}\right|-|\lambda g(2x)|\\ &\qquad\le\left|\frac{f(x+y_0)^2-f(x-y_0)^2}{f(2y_0)}-\lambda g(2x)\right|\\ &\qquad\le\frac{\min\{\varphi(2x),\varphi(2y_0)\}}{|f(2y_0)|}\le\frac{\varphi(2y_0)}{|f(2y_0)|}\end{aligned}$$

and it follows that g is also bounded on G. Since the unboundedness of g exacts it of f, we can run along the step of Theorem 2.2.

Under the situation that h is replaced by f, the process applied after (2.8) of Theorem 2.2 gives us the equation

$$g(2x)=\lim_{n\to\infty}\frac{f(x+y_n)^2-f(x+\sigma y_n)^2}{\lambda f(2y_n)}\quad\text{for all}\quad x\in G, \tag{3.3}$$

which, since f satisfies $(\widetilde{S})$ by Corollary 1 whenever g is unbounded, validates

$$g(2x) = f(2x) \qquad \forall x \in G.$$

By the 2-divisibility of group G, we obtain $g = f$. Hence, we conclude that g also satisfies $(\widetilde{S})$. $\square$

COROLLARY 3.4. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 - \lambda g(x) f(y) \right| \le \varepsilon$$

for all $x, y \in G$. Then,

(i) *either g is bounded or f (and g) satisfy $(\widetilde{S})$. In addition, if g satisfies $(\widetilde{A^\lambda})$, then f and g satisfy $(\widetilde{T^\lambda_{ffgf}})$.*

(ii) *either f is bounded or g satisfies $(\widetilde{S})$ under one of the two cases $g(0) = 0$, $f(x)^2 = f(\sigma x)^2$. In addition, if f satisfies $(\widetilde{A^\lambda})$, then g and f satisfy $(\widetilde{A^\lambda_{gggf}})$.*

3.2. Stability of equation $(\widetilde{S^\lambda_{fh}})$

Replacing g by f in Theorems 2.1, 2.2, 2.3, and Corollary 2.4. We obtain the following results, respectively.

COROLLARY 3.5. *Suppose that $f, h : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 - \lambda f(x) h(y) \right| \le \varphi(y) \tag{3.4}$$

for all $x, y \in G$. Then either f is bounded or h satisfies $(\widetilde{S})$. In addition, if f satisfies $(\widetilde{A^\lambda})$, then f and h satisfy

$$(\widetilde{T^\lambda_{hhfh}}) \qquad h(x+y) - h(x+\sigma y) = \lambda f(x) h(y).$$

COROLLARY 3.6. *Suppose that $f, h : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 - \lambda f(x) h(y) \right| \le \varphi(x), \tag{3.5}$$

which satisfies one of the two cases $f(0) = 0$, $f(x)^2 = f(\sigma x)^2$ for all $x, y \in G$. Then either h is bounded or f satisfies $(\widetilde{S})$. In addition, if h satisfies $(\widetilde{A^\lambda})$, then f and h satisfy

$$(\widetilde{A^\lambda_{fh}}) \qquad f(x+y) + f(x+\sigma y) = \lambda f(x) h(y).$$

Corollary 3.7. *Suppose that $f, h : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 - \lambda f(x)h(y) \right| \leq \min\{\varphi(x), \varphi(y)\}, \tag{3.6}$$

for all $x, y \in G$. Then,

(i) *either f (or h) is bounded or h satisfy $(\widetilde{S})$. In addition, if f satisfies $(\widetilde{A^\lambda})$, then h and f satisfy $(\widetilde{T^\lambda_{hhfh}})$.*

(ii) *either h is bounded or f satisfies $(\widetilde{S})$ under one of the two cases $f(0) = 0$, $f(x)^2 = f(\sigma x)^2$. In addition, if h satisfies $(\widetilde{A^\lambda})$, then f and h satisfy $(\widetilde{A^\lambda_{fh}})$.*

Proof. Except for that h in the case (i) is bounded, it is enough from Corollary 3.5 and Corollary 3.6.

Then, through the same method as Corollary 3.3, we can easily check that h is also bounded when f is bounded. Hence, we can apply Corollary 3.5. □

Corollary 3.8. *Suppose that $f, h : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 - \lambda f(x)h(y) \right| \leq \varepsilon$$

for all $x, y \in G$. Then,

(i) *either f (or h) is bounded or h satisfy $(\widetilde{S})$. In addition, if f satisfies $(\widetilde{A^\lambda})$, then h and f satisfy $(\widetilde{T^\lambda_{hhfh}})$.*

(ii) *either h is bounded or f satisfies $(\widetilde{S})$ under one of the two cases $f(0) = 0$, $f(x)^2 = f(\sigma x)^2$. In addition, if h satisfies $(\widetilde{A^\lambda})$, then f and h satisfy $(\widetilde{A^\lambda_{fh}})$.*

3.3. Stability of equation $(\widetilde{S^\lambda_{gg}})$

Replacing h by g in Theorems 2.1, 2.2, 2.3, and Corollary 2.4. We obtain the following results, respectively.

Corollary 3.9. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 - \lambda g(x)g(y) \right| \leq \varphi(y)$$

for all $x, y \in G$. Then either g is bounded or g satisfies $(\widetilde{S})$.

Corollary 3.10. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 - \lambda g(x)g(y) \right| \le \varphi(x),$$

which satisfies one of the two cases $g(0) = 0$, $f(x)^2 = f(\sigma x)^2$ for all $x, y \in G$. Then either g is bounded or g satisfies $(\widetilde{S})$.

Corollary 3.11. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 - \lambda g(x)g(y) \right| \le \min\{\varphi(x), \varphi(y)\}$$

for all $x, y \in G$.

Then either g is bounded or g satisfy $(\widetilde{S})$.

Corollary 3.12. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 - \lambda g(x)g(y) \right| \le \varepsilon$$

for all $x, y \in G$. Then either g is bounded or g satisfies $(\widetilde{S})$.

3.4. Stability of equation $(\widetilde{S^{\lambda}})$

Replacing g and h by f in Theorems 1, 2, 3, and Corollary 2.4. We also obtain the following results as corollaries.

Corollary 3.13. *Suppose that $f : G \to \mathbb{C}$ satisfies the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 - \lambda f(x)f(y) \right| \le \varphi(y) \tag{3.7}$$

for all $x, y \in G$. Then either f is bounded or f satisfies $(\widetilde{S})$.

Corollary 3.14. *Suppose that $f : G \to \mathbb{C}$ satisfies the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 - \lambda f(x)f(y) \right| \le \varphi(x). \tag{3.8}$$

Then either f is bounded or f satisfies $(\widetilde{S})$.

Proof. When f is unbounded, we can find $f(0) = 0$ in the proof of Theorem 5 [3]. Hence, we eliminate the assumption $f(0) = 0$. □

Corollary 3.15. *Suppose that $f : G \to \mathbb{C}$ satisfies the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 - \lambda f(x)f(y) \right| \leq \varepsilon$$

for all $x, y \in G$. Then either f is bounded or f satisfies $(\widetilde{S})$.

Remark 3.16. In all results of the Section 3, by putting $\sigma(x) = -x$, and $\lambda = 1$ in the sine type equations ($\widetilde{S}$, $\widetilde{S^\lambda}$, $\widetilde{S^\lambda_{gf}}$, $\widetilde{S^\lambda_{fh}}$, $\widetilde{S^\lambda_{gg}}$) and $\lambda = 2$ in the cosine and the trigonometric type equations ($\widetilde{A^\lambda}$, $\widetilde{A^\lambda_{fh}}$, $\widetilde{A^\lambda_{gh}}$, $\widetilde{A^\lambda_{gggf}}$, $\widetilde{T^\lambda_{ffgf}}$, $\widetilde{T^\lambda_{hhfh}}$, $\widetilde{T^\lambda_{hhfh}}$), we obtain the same numbers of corollaries for each subsection, which is found in the papers ([3], [15], [16]).

4. Extension of the results to Banach space

In this section, let $(E, \|\cdot\|)$ be a semisimple commutative Banach space. All results in the Section 2 and the Section 3 can be extended to the superstability on the Banach space. For simplicity, we will combine the three theorems of the same functional equation in Section 2 and Section 3 into the one theorem, respectively, the proofs of the results also will skip except for the case (i) of (a).

Theorem 4.1. *Assume that $f, g, h : G \to E$ satisfy one of the inequalities*

$$\begin{aligned} &\left\| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 - \lambda g(x)h(y) \right\| \\ &\leq \begin{cases} (i) & \varphi(y) \\ (ii) & \varphi(x) \\ (iii) & \min\{\varphi(x), \varphi(y)\}. \end{cases} \end{aligned} \tag{4.1}$$

For an arbitrary linear multiplicative functional $x^* \in E^*$,

(a) *if the superposition $x^* \circ g$ fails to be bounded,*
then, in the cases (i) and (iii), h satisfies $(\widetilde{S})$.
In addition, if $x^* \circ g$ satisfies $(\widetilde{A^\lambda})$, g and h satisfy $(\widetilde{T^\lambda_{gh}})$.

(b) *in the cases (ii) and (iii), let the inequality (4.1) satisfies one of the two cases $g(0) = 0$, $f(x)^2 = f(\sigma x)^2$.*
if the superposition $x^* \circ h$ fails to be bounded,
then, g satisfies $(\widetilde{S})$.
In addition, if $x^* \circ h$ satisfies $(\widetilde{A^\lambda})$, g and h satisfy $(\widetilde{A^\lambda_{gh}})$.

Proof. The proofs of each case are very similar, thus it suffices to show only the proof of case (i) in (a). Assume that (i) holds and fix arbitrarily a linear multiplicative functional $x^* \in E$. As is well known, we have $\|x^*\| = 1$, thus, for every $x, y \in G$, we have

$$\begin{aligned}\varphi(y) &\geq \left\| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 - \lambda g(x)h(y) \right\| \\ &= \sup_{\|y^*\|=1} \left| y^*\left(f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 \right) - \lambda g(x)h(y) \right| \\ &\geq \left| x^*\left(f\left(\frac{x+y}{2}\right)^2 \right) - x^*\left(f\left(\frac{x+\sigma y}{2}\right)^2 \right) - \lambda \cdot x^*(g(x)) \cdot x^*(h(y)) \right|,\end{aligned}$$

which states that the superpositions $x^* \circ f$, $x^* \circ g$ and $x^* \circ h$ yield a solution of stability inequality (2.1) of Theorem 2.1. By assumption, since the superposition $x^* \circ g$ is unbounded, an appeal to Theorem 2.1 shows that the function $x^* \circ h$ solves $(\widetilde{S})$. In other words, bearing the linear multiplicativity of x^* in mind, for all $x, y \in G$, the difference of $(\widetilde{S^\lambda_{gh}})$

$$\widetilde{DS^\lambda_h}(x,y) := h\left(\frac{x+y}{2}\right)^2 - h\left(\frac{x+\sigma y}{2}\right)^2 - \lambda h(x)h(y)$$

falls into the kernel of x^*. Therefore, in view of the unrestricted choice of x^*, we infer that

$$\widetilde{DS^\lambda_h}(x,y) \in \bigcap \{\ker x^* : x^* \text{ is a multiplicative member of } E^*\}$$

for all $x, y \in G$. Since the algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, i.e.

$$h\left(\frac{x+y}{2}\right)^2 - h\left(\frac{x+\sigma y}{2}\right)^2 - \lambda h(x)h(y) = 0 \quad \text{for all} \quad x, y \in G,$$

as claimed.

In addition, let $x^* \circ g$ satisfies $(\widetilde{A^\lambda})$, since $x^* \circ k$ for the limit function k defined in the Theorem 2.1 validates $x^* \circ g$, then (2.5) implies

$$(x^* \circ h)(x+y) - (x^* \circ h)(x+\sigma y) - \lambda(x^* \circ g)(x)(x^* \circ h)(y), \tag{4.2}$$

which also falls into the kernel of x^*. Hence g and h hold the claimed $(\widetilde{T^\lambda_{gh}})$. The other cases are similar. □

Corollary 4.2. *Assume that $f, g : G \to E$ satisfy one of the inequalities*

$$\left\| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 - \lambda g(x)f(y) \right\| \leq \begin{cases} (i) & \varphi(y) \\ (ii) & \varphi(x) \\ (iii) & \min\{\varphi(x), \varphi(y)\}. \end{cases} \tag{4.3}$$

For an arbitrary linear multiplicative functional $x^* \in E^*$,

(a) *if the superposition $x^* \circ g$ fails to be bounded, then,*
- *in case (i), f satisfies $(\widetilde{S})$*
- *in case (iii), f (and g) satisfy $(\widetilde{S})$, respectively.*

In each case, if $x^* \circ g$ satisfies $(\widetilde{A^\lambda})$, f and g satisfy $(\widetilde{T^\lambda_{ffgf}})$.

(b) *in the cases (ii) and (iii), let the inequality (4.3) satisfy one of the two cases $g(0) = 0$, $f(x)^2 = f(\sigma x)^2$.*

if the superposition $x^* \circ f$ fails to be bounded, then, g satisfies $(\widetilde{S})$.

In addition, if $x^* \circ f$ satisfies $(\widetilde{A^\lambda})$, g and f satisfy $(\widetilde{A^\lambda_{gggf}})$.

Corollary 4.3. *Assume that $f, h : G \to E$ satisfy one of the inequalities*

$$\left\| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 - \lambda f(x)h(y) \right\| \leq \begin{cases} (i) & \varphi(y) \\ (ii) & \varphi(x) \\ (iii) & \min\{\varphi(x), \varphi(y)\} \end{cases} \tag{4.4}$$

For an arbitrary linear multiplicative functional $x^* \in E^*$,

(a) *if the superposition $x^* \circ f$ fails to be bounded, then, in the cases (i) and (iii), h satisfies $(\widetilde{S})$.*

In addition, if $x^* \circ f$ satisfies $(\widetilde{A^\lambda})$, f and h satisfy $(\widetilde{T^\lambda_{hhfh}})$.

(b) *if the superposition $x^* \circ h$ fails to be bounded, then,*
- *in case (ii) and (iii), let the inequality (4.4) satisfy one of the two cases $g(0) = 0$, $f(x)^2 = f(\sigma x)^2$. then f satisfies $(\widetilde{S})$*
- *in case (iii), f and h satisfy $(\widetilde{S})$, respectively.*

In each case, if $x^* \circ h$ satisfies $(\widetilde{A^\lambda})$, f and h satisfy $(\widetilde{A^\lambda_{fh}})$.

Corollary 4.4. *Assume that $f, g : G \to E$ satisfy one of the inequalities*

$$\left\| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 - \lambda g(x)g(y) \right\| \le \begin{cases} (i) & \varphi(y) \\ (ii) & \varphi(x) \\ (iii) & \min\{\varphi(x), \varphi(y)\} \end{cases}$$

which satisfies one of the two cases $g(0) = 0$, $f(x)^2 = f(\sigma x)^2$ in the case (ii).

For an arbitrary linear multiplicative functional $x^* \in E^*$,
if the superposition $x^* \circ g$ fails to be bounded, then g satisfies $(\widetilde{S})$.

Corollary 4.5. *Assume that $f : G \to E$ satisfy one of the inequalities*

$$\left\| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2 - \lambda f(x)f(y) \right\| \le \begin{cases} (i) & \varphi(y) \\ (ii) & \varphi(x). \end{cases}$$

For an arbitrary linear multiplicative functional $x^* \in E^*$, either the superposition $x^* \circ f$ is bounded or f satisfies $(\widetilde{S})$.

Remark 4.6. (i) In all results of the Section 4, putting $\sigma(x) = -x$, $\lambda = 1$ in sine type equations $(\widetilde{S^\lambda_{gf}}, \widetilde{S^\lambda_{fh}}, \widetilde{S^\lambda_{gg}}, \widetilde{S^\lambda})$, and $\lambda = 2$ in cosine and trigonometric type equations $(\widetilde{A^\lambda}, \widetilde{A^\lambda_{fh}}, \widetilde{A^\lambda_{gh}}, \widetilde{T^\lambda_{hhfh}}, \widetilde{T^\lambda_{gf}})$, then we obtain the same numbers of corollaries for each equation, some of which are found in the papers ([3], [15], [16]).

(ii) Applying $\varphi(x) = \varphi(y) = \varepsilon$ in all results of this section and the above (i), then we can obtain the many numbers of corollaries.

(iii) The results of this paper also can be extended to the stability of the same type's functional equations in a group $(G, \cdot)$ with multiplication.

(iv) By adding of the Kannappan condition $f(x+y+z) = f(x+z+y)$ in [12], the Abelian group as a domain of functions in this paper can be come weak in the semigroup $(G, +)$.

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