JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **24**, No. 2, June 2011

h-STABILITY OF LINEAR IMPULSIVE DIFFERENTIAL EQUATIONS VIA SIMILARITY

Sung Kyu Choi*, Namjip Koo**, and Chunmi Ryu***

Abstract. In this paper we study h-stability for the linear impulsive equations using the notion of kinematical similarity and impulsive integral inequality.

1. Introduction

The impulsive differential equations are adequate mathematical models for description of evolution processes characterized by the combination of a continuous and jump change of their states. It is now being recognized that the theory of impulsive differential equations is not only richer than the corresponding theory of differential equations but also represents a more natural framework for mathematical modelling of many real world phenomena. For a detail discussion of impulsive integral inequalities and some basic concepts about the impulsive differential equations, we refer the reader to [1, 5].

The notion of h-stability was introduced by Pinto [6, 7] with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbation. The various notions about h-stability given in [8] include several types of known stability properties as uniform stability, uniform Lipschitz stability and exponential asymptotic stability.

Received May 11, 2011; Accepted May 24, 2011.

2010 Mathematics Subject Classification: Primary 34A37.

Key words and phrases: impulsive differential equations, impulsive integral inequalities, h-stability, kinematical similarity.

Correspondence should be addressed to Namjip Koo, njkoo@cnu.ac.kr.

This work was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(NRF-2010-0008835).

Medina and Pinto investigated the important properties about h-stability for the various differential systems and difference systems [6, 7, 8, 9].

We studied h-stability for the nonlinear Volterra integro-differential system [3] and for the nonlinear perturbed systems [2]. Moreover, the notions of t_∞-similarity and Liapunov functions were used to study h-stability for nonlinear differential systems [4].

In this paper we study h-stability for the linear impulsive equations using the notion of kinematical similarity and impulsive integral inequality.

2. Main results

Let the sequence (τ_k) be fixed and satisfy the condition

$$\tau_k < \tau_{k+1},\ k \in \mathbb{Z} \text{ and } \lim_{k\to\pm\infty} \tau_k = \pm\infty. \tag{2.1}$$

Denote by $PC(\mathbb{R}, \mathbb{R}^{n\times n})$ the set of functions $\psi : \mathbb{R} \to \mathbb{R}^{n\times n}$ which are continuous for $t \in \mathbb{R},\ t \neq \tau_k$, are continuous from the left for $t \in \mathbb{R}$, and have discontinuities of the first kind at the points $\tau_k \in \mathbb{R}$ for each $k \in \mathbb{Z}$.

Consider the linear homogeneous impulsive equation

$$\begin{cases} x' = A(t)x,\ t \neq \tau_k, \\ \Delta x = A_k x,\ t = \tau_k, k \in \mathbb{Z}, \end{cases} \tag{2.2}$$

under the assumption that the following condition holds:

$$A(t) \in PC(\mathbb{R}, \mathbb{R}^{n\times n}),\ A_k \in \mathbb{R}^{n\times n},\ k \in \mathbb{Z}. \tag{2.3}$$

Consider the linear homogeneous impulsive equation

$$\begin{cases} y' = B(t)y,\ t \neq \tau_k, \\ \Delta y = B_k y,\ t = \tau_k, k \in \mathbb{Z}, \end{cases} \tag{2.4}$$

where $B(t) \in PC(\mathbb{R}, \mathbb{R}^{n\times n})$, $B_k \in \mathbb{R}^{n\times n}$, and $\det(E + B_k) \neq 0$ for each $k \in \mathbb{Z}$ and the perturbed linear homogeneous impulsive equation of (2.2)

$$\begin{cases} y' = A(t)x + D(t)x,\ t \neq \tau_k, \\ \Delta y = A_k y + D_k y,\ t = \tau_k, k \in \mathbb{Z}, \end{cases} \tag{2.5}$$

where $D(t) \in PC(\mathbb{R}, \mathbb{R}^{n\times n})$, and $D_k \in \mathbb{R}^{n\times n},\ \det(E + A_k + D_k) \neq 0$ for each $k \in \mathbb{Z}$.

Lemma 2.1. [1, Theorem 1.5] *Let conditions (2.1) and (2.3) hold. Then the following statements hold:*

(1) *For any* $(t_0, x_0) \in \mathbb{R}\times\mathbb{R}^n$, *there exists a unique solution of equation (2.2) with* $x(t_0^+) = x_0$ *(or* $x(t_0) = x_0$*)) and this solution is defined for* $t > t_0$ *(or* $t \geq t_0$*).*
(2) *If* $\det(E + A_k) \neq 0$ *for each* $k \in \mathbb{Z}$*, then this solution is defined for all* $t \in \mathbb{R}$*.*

Next result follows from a simple calculation.

LEMMA 2.2. [1] *Each solution* $y(t)$ *of (2.5) satisfies the integro-summary equation*

$$\begin{aligned} y(t) &= X(t,s)y(s) + \int_s^t X(t,\tau)D(\tau)y(\tau)d\tau \\ &\quad + \sum_{s\leq\tau_k<t} X(t,\tau_k^+)D_k y(\tau_k), \;\; t \geq s, \end{aligned}$$

where $X(t,s)$ *is the Cauchy matrix for equation (2.2).*

LEMMA 2.3. [1, Lemma 1.4] *Suppose that for* $t \geq t_0$ *the inequality*

$$u(t) \leq c + \int_{t_0}^t b(s)u(s)ds + \sum_{t_0\leq\tau_k<t} \beta_k u(\tau_k) \tag{2.6}$$

holds, where $u \in PC(\mathbb{R},\mathbb{R}), b \in PC(\mathbb{R},\mathbb{R}^+)$ *and* $\beta_k \geq 0,\; k \in \mathbb{Z}$ *and* c *are constants. Then we have*

$$\begin{aligned} u(t) &\leq c \prod_{t_0\leq\tau_k<t} (1+\beta_k) \exp(\int_{t_0}^t b(s)ds) &(2.7)\\ &\leq c\exp(\int_{t_0}^t b(s)ds + \sum_{t_0\leq\tau_k<t} \beta_k),\; t \geq t_0. &(2.8) \end{aligned}$$

REMARK 2.4. If $A(t)$ and $B(t)$ are similar(i.e., there exists an invertible bounded matrix $S(t)$ with bounded $S^{-1}(t)$ such that $SAS^{-1} = B$), then $\exp(At)$ and $\exp(Bt)$ are similar.

DEFINITION 2.5. The zero solution $x = 0$ of (2.2) is called *h-stable* if there exist a positive bounded left continuous function $h : \mathbb{R}^+ \to \mathbb{R}$ and a constant $c \geq 1$ such that

$$|x(t, t_0, x_0)| \leq c|x_0|h(t)h^{-1}(t_0),\; t \geq t_0,$$

for $|x_0|$ small enough (here $h^{-1}(t) = \frac{1}{h(t)}$).

We need the following lemma for h-stability of solutions of linear impulsive differential systems.

LEMMA 2.6. [7, Lemma 1] *The linear impulsive system (2.2) is h-stable if and only if there exist a constant $c \geq 1$ and a positive bounded left continuous functions h defined on $\mathbb{R}^+$ such that for every $x_0 \in \mathbb{R}^n$,*

$$|X(t,t_0)| \leq ch(t)h(t_0)^{-1},\ t \geq t_0 \geq 0, \tag{2.9}$$

where $X(t,t_0)$ is the Cauchy matrix of (2.2).

THEOREM 2.7. *If the zero solution $x = 0$ of (2.2) is h-stable and there exists a constant $M > 0$ such that*

$$\int_0^\infty |D(s)|ds + \sum_{0 \leq \tau_k \leq \infty} h(\tau_k)h(\tau_k^+)^{-1}|D_k| \leq M, \tag{2.10}$$

then the zero solution $y = 0$ of (2.5) is h-stable.

Proof. It follows from Lemma 2.2 that the solution $y(t,t_0,y_0)$ of (2.5) satisfies the corresponding impulsive integral equation

$$\begin{aligned} y(t) &= X(t,t_0)y(t_0) + \int_{t_0}^t X(t,\tau)D(\tau)y(\tau)d\tau \\ &\quad + \sum_{t_0 \leq \tau_k < t} X(t,\tau_k^+)D_k y(\tau_k),\ t \geq t_0, \end{aligned} \tag{2.11}$$

where $X(t,t_0)$ is the Cauchy matrix for linear impulsive equation (2.2). Then, from Lemma 2.6, there exist a constant $c \geq 1$ and a positive bounded left continuous function $h : \mathbb{R}^+ \to \mathbb{R}$ such that

$$|X(t,t_0)| \leq ch(t)h(t_0)^{-1},\ t \geq t_0,$$

where $X(t,t_0)$ is the Cauchy matrix for (2.2). Thus we obtain

$$\begin{aligned} |y(t)| &\leq |X(t,t_0)||y(t_0)| + \int_{t_0}^t |X(t,\tau)||D(\tau)||y(\tau)|d\tau \\ &\quad + \sum_{t_0 \leq \tau_k < t} |X(t,\tau_k^+)||D_k||y(\tau_k)| \\ &\leq h(t)h(t_0)^{-1}|y(t_0)| + \int_{t_0}^t h(t)h(\tau)^{-1}|D(\tau)||y(\tau)|d\tau \\ &\quad + \sum_{t_0 \leq \tau_k < t} h(t)h(\tau_k^+)^{-1}|D_k||y(\tau_k)|,\ t \geq t_0. \end{aligned}$$

Letting $u(t) = \frac{|y(t)|}{h(t)}$, we have

$$\begin{aligned} u(t) &\leq u(t_0) + \int_{t_0}^{t} |D(s)|u(s)ds \\ &\quad + \sum_{t_0 \leq \tau_k < t} h(\tau_k)h(\tau_k^+)^{-1}|D_k|u(\tau_k),\ t \geq t_0. \end{aligned}$$

By the Gronwall impulsive integral inequality [1], we obtain

$$\begin{aligned} &|y(t)| \\ &\leq h(t)h(t_0)^{-1}|y(t_0)| \exp(\int_{t_0}^{t} |D(s)|ds + \sum_{t_0 \leq \tau_k < t} h(\tau_k)h(\tau_k^+)^{-1}|D_k|) \\ &\leq h(t)h(t_0)^{-1}|y(t_0)| \exp(\int_{t_0}^{\infty} |D(s)|ds + \sum_{t_0 \leq \tau_k < \infty} h(\tau_k)h(\tau_k^+)^{-1}|D_k|) \\ &\leq c|y(t_0)|h(t)h(t_0)^{-1},\ t \geq t_0, \end{aligned}$$

where $c = exp(M)$. Hence the zero solution $y = 0$ of (2.5) is h-stable. The proof is complete. □

Remark 2.8. We note that if $h(t)$ is continuous in Theorem 2.7, then we have $\frac{h(\tau_k)}{h(\tau_k^+)} = 1$ for each $k \in \mathbb{N}$.

We can obtain the following results in Theorem 2.3 [1] as the corollary of Theorem 2.7.

Corollary 2.9. *Suppose that the assumptions of Theorem 2.7 hold.*

(1) *If we set $h(t) = c$ for each $t \in \mathbb{R}^+$ in Theorem 2.7, then the zero solution $y = 0$ of (2.5) is uniformly stable.*
(2) *If $h(t) \to 0$ as $t \to \infty$, then the zero solution $y = 0$ of (2.5) is asymptotically stable.*

Example 2.10. *To illustrate Lemma 2.6, consider the linear impulsive differential equation*

$$\begin{aligned} x'(t) &= a(t)x,\ t \neq \tau_k, \\ \Delta x &= a_k x, \quad t = \tau_k,\ k \in \mathbb{Z}, \end{aligned} \tag{2.12}$$

where $a(t) \in PC(\mathbb{R}, \mathbb{R}), a_k \in \mathbb{R}$, and $\det(1 + a_k) \neq 0$ for each $k \in \mathbb{Z}$. Suppose that $\int_{t_0}^{\infty} |a(s)|ds < \infty$ for each $t_0 \in \mathbb{R}$ and $\sum_{t_0 \leq \tau_k \leq \infty} |a_k| < \infty$. Then the zero solution $x = 0$ of (2.12) is h-stable.

Proof. Let $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}$. Then we have

$$X(t, t_0) = \prod_{t_0 \le \tau_k < t} (1 + a_k) \exp\Big(\int_{t_0}^{t} a(s) ds\Big),\ t \ge t_0,$$

and furthermore

$$\begin{aligned} |X(t, t_0)| &= \Big|\prod_{t_0 \le \tau_k < t} (1 + a_k) \exp\Big(\int_{t_0}^{t} a(s) ds\Big)\Big| \\ &\le \exp\Big(\int_{t_0}^{t} |a(s)| + \sum_{t_0 \le \tau_k \le \infty} |a_k|\Big)\Big| \\ &\le ch(t)h(t_0)^{-1},\ t \ge t_0, \end{aligned}$$

where $h(t) = \exp(\int_0^t |a(s)| ds)$ is a bounded positive function for $t \ge t_0$ and

$$1 \le c = \exp\Big(\sum_{t_0 \le \tau_k \le \infty} |a_k|\Big) < \infty.$$

Hence the zero solution $x = 0$ of (2.12) is h-stable by Lemma 2.6. □

Denote by $\mathcal{S}$ the set of all matrix functions $S : \mathbb{R}^+ \to \mathbb{R}^n \times \mathbb{R}^n$ which belong to $PC(\mathbb{R}^+, \mathbb{R}^n \times \mathbb{R}^n)$ and are bounded in $\mathbb{R}^+$ together with inverse $S^{-1}(t)$. Let $\mathcal{M}$ be denoted by

$$\begin{aligned} \mathcal{M} = \{(A, A_k) \ \mid \ & A \in PC(\mathbb{R}^+, \mathbb{R}^n \times \mathbb{R}^n), A_k \in \mathbb{R}^n \times \mathbb{R}^n, \\ & \det(E + A_k) \ne 0, k \in \mathbb{N}\}. \end{aligned}$$

Definition 2.11. We say that $(A, A_k) \in \mathcal{M}$ is *kinematically similar* to $(B, B_k) \in \mathcal{M}$ if there exists a matrix function $S \in \mathcal{S}$ such that

$$\begin{aligned} & S'(t) - A(t)S(t) + S(t)B(t) = 0,\ t \ne \tau_k, \\ & \Delta S(\tau_k) - A_k S(\tau_k) + S(\tau_k^+)B_k = 0,\ t = \tau_k, k \in \mathbb{N}, \end{aligned}$$

where $\Delta S(\tau_k) = S(\tau_k^+) - S(\tau_k)$. We say that equations (2.2) and (2.4) are *kinematically similar* if $(A, A_k) \in \mathcal{M}$ is kinematically similar to $(B, B_k) \in \mathcal{M}$.

Remark 2.12. The notion of kinematical similarity is an equivalence relation in the set $\mathcal{M}$, and it preserves various stability concepts: stability, uniform stability, uniform asymptotic stability, strict stability [1, Theorem 10.2].

The results of various stabilities in Theorem 10.2 [1] are generalized for impulsive linear equations using the kinematical similarity.

THEOREM 2.13. *Suppose that (2.2) and (2.4) are kinematically similar. Then (2.2) is h-stable if and only if (2.4) is also h-stable.*

Proof. Suppose that (2.2) is h-stable. Then there exist a constant $c \geq 1$ and a positive bounded left continuous function h defined on $\mathbb{R}^+$ such that

$$|X(t,t_0)| \leq ch(t)h(t_0)^{-1},\ t \geq t_0 \geq 0, \tag{2.13}$$

where $X(t,t_0)$ is the Cauchy matrix of (2.2). Since (2.2) and (2.4) are kinematically similar, we see that the change of variables

$$x = S(t)y$$

transforms equation (2.2) into (2.4). We obtain the following relation:

$$X(t)X^{-1}(\tau) = S(t)Y(t)Y^{-1}(\tau)S^{-1}(\tau),\ t,\tau \in \mathbb{R}^+,$$

where $X(t)$ and $Y(t)$ are fundamental matrices of (2.2) and (2.4), respectively. Thus we obtain

$$\begin{aligned} |Y(t,t_0)| &= |Y(t)Y^{-1}(t_0)| = |S^{-1}(t)X(t)X^{-1}(t_0)S(t_0)| \\ &\leq |S^{-1}(t)||X(t)X^{-1}(t_0)||S(t_0)| \\ &\leq c_1c_2ch(t)h(t_0)^{-1} \\ &\leq dh(t)h(t_0)^{-1},\ t \geq t_0 \geq 0, \end{aligned}$$

where $d = cc_1c_2$ is a constant. Hence (2.4) is h-stable by Lemma 2.6.

The converse also holds. Hence the proof is complete. □

COROLLARY 2.14. *Suppose that (2.2) and (2.4) are kinematically similar.*

(1) *If we set $h(t) = c$ for a some positive constant c, then (2.2) is uniformly stable if and only if (2.4) is also uniformly stable.*

(2) *If we set $h(t) = e^{-\lambda t}$ for a some positive constant λ, then (2.2) is uniformly exponentially stable if and only if (2.4) is also uniformly exponentially stable.*

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Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: sgchoi@cnu.ac.kr

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Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: njkoo@cnu.ac.kr

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Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: chmiry@yahoo.co.kr