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MODULE-THEORETIC CHARACTERIZATIONS OF GENERALIZED GCD DOMAINS, II

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Abstract. It is shown that generalized GCD domains satisfy a certain property of injectivity and conversely this property characterizes generalized GCD domains.

1. Introduction

Generalized GCD domains (for short, GGCD domains) were first introduced in [1], further investigated in [2], and since then, have played important roles in multiplicative ideal theory. Several ring- or ideal-theoretic characterizations of GGCD domains were given in the literature. The purpose of this note is to give another module-theoretic characterizations of GGCD domains, as a continuation of the study of module-theoretic characterizations of certain integral domains ([6, 7, 8]).

We first introduce some definitions and notations. Let R be an integral domain with quotient field K. Let I be a nonzero fractional ideal I of R. Then $I^{-1} := \{x \in K | xI \subseteq R\}$, $I_v := (I^{-1})^{-1}$, $I_t := \bigcup\{J_v \mid J$ is a finitely generated (f.g.) subideal of $I\}$, and an ideal J of R is called a *GV-ideal*, denoted by $J \in GV(R)$, if J is a f.g. ideal of R with $J^{-1} = R$. A fractional ideal I of R is said to be *invertible* (resp., *t-invertible*) if $II^{-1} = R$ (resp., $(II^{-1})_t = R$).

For a torsion-free R-module M, Wang and McCasland defined the w-envelope of M as $M_w := \{x \in M \otimes_R K \mid Jx \subseteq M$ for some $J \in GV(R)\}$ ([10], cf., [6]). A torsion-free R-module is called a *w-module* (or

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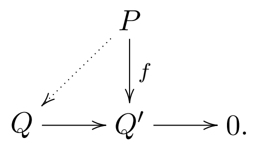
semidivisorial) if $M_w = M$. We say that a torsion-free R-module M is *w-finite* if $M = N_w$, for some f.g. submodule N of M.

Following [1], an integral domain R is called a *generalized GCD domain* if the intersection of two (integral) invertible ideals is invertible. It is well known that R is a GGCD domain if and only if A_v (equivalently A^{-1}) is invertible for every f.g. ideal A of R ([2, Theorem 1]). Recall that an integral domain R is called a *Prüfer v-multiplication domain* (for short, *PvMD*) if A_v (equivalently A^{-1}) is t-invertible for every f.g. ideal A of R. Thus the class of GGCD domains is contained in the class of PvMDs. It is also well known that in a PvMD, $t = w$. Therefore, R is a GGCD domain if and only if every w-finite w-ideal is invertible. Any undefined terminology is standard, as in [3] or [9].

2. Main results

In this section, it is shown that the class of GGCD domains contains the class of integral domains with a certain property of projectivity and that GGCD domains satisfy a certain property of injectivity and conversely this property characterizes GGCD domains. To do so, we need the following lemma.

LEMMA 2.1. ([9, Lemma 4.18]) *An R-module P is projective if and only if every diagram with exact row and with Q injective can be completed to a commutative diagram; that is, every R-homomorphism $f : P \to Q'$ can be lifted. The dual is also true.*



Recall that a module M is said to be *quasi-projective* if for every epimorphism $\beta : M \to N$, $Hom(M, \beta) : Hom(M, M) \to Hom(M, N)$ is also an epimorphism.

THEOREM 2.2. *Consider the following conditions for an integral domain R:*

1. *R is a GGCD domain;*
2. *Every w-finite w-submodule of a projective R-module is projective;*
3. *Every w-finite w-submodule of a projective R-module is quasi-projective;*

(4) *Every diagram*

$$\begin{array}{ccccc} 0 & \longrightarrow & P & \xrightarrow{i} & P' \\ & & \Big\downarrow{\scriptstyle f} & & \\ Q & \xrightarrow{\pi} & Q' & \longrightarrow & 0. \end{array}$$

with P' projective, Q injective and P a w-finite w-submodule of P' and exact rows can be embedded in a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & P & \xrightarrow{i} & P' \\ & & \Big\downarrow{\scriptstyle f} & \swarrow & \\ Q & \xrightarrow{\pi} & Q' & \longrightarrow & 0. \end{array}$$

Then we have (1) $\Leftarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

Proof. (2) $\Rightarrow$ (1). [7, Theorem 2.2]. (2) $\Rightarrow$ (3). This is trivial. (3) $\Rightarrow$ (2). Assume (3) and let N be a w-finite w-submodule of a projective R-module P. Then there exists a f.g. projective module P' such that $P' \to N \to 0$ is exact. Thus $P' \oplus N$ is a w-finite w-submodule of a projective R-module $P' \oplus P$ by [6, Corollary 6.5], and hence is quasi-projective by hypothesis. By [4, Theorem 2,2] we have that N is projective. (2) $\Rightarrow$ (4). Let the diagram be given. By (2) P is projective. Thus there exists an R-homomorphism $g : P \to Q$ such that $\pi \circ g = f$. Now since Q is injective, there exists an R-homomorphism $h : P' \to Q$ such that $h \circ i = g$. Let $\bar{f} := \pi \circ h$. Then $\bar{f}$ is an R-homomorphism from P' to Q' such that $\bar{f} \circ i = f$. (4) $\Rightarrow$ (2). Let P be a w-finite w-submodule of a projective R-module P'. By Lemma 2.1, in order to prove that P is projective, it is sufficient that every diagram

$$\begin{array}{ccccc} & & P & & \\ & & \Big\downarrow{\scriptstyle f} & & \\ Q & \xrightarrow{\pi} & Q' & \longrightarrow & 0, \end{array}$$

in which the row is exact and Q injective can be embedded in a commutative diagram

$$\begin{array}{ccccc} & & P & & \\ & \swarrow & \Big\downarrow{\scriptstyle f} & & \\ Q & \xrightarrow{\pi} & Q' & \longrightarrow & 0. \end{array}$$

So let the diagram be given. Embedded it in the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & P & \xrightarrow{i} & P' \\ & & \big\downarrow{\scriptstyle f} & & \\ Q & \xrightarrow{\pi} & Q' & \longrightarrow & 0, \end{array}$$

where i denotes the inclusion map. By (3) there exists $\bar{f} : P' \to Q'$ such that $\bar{f} \circ i = f$. Now since P' is projective, there exists $g : P' \to Q$ such that $\pi \circ g = \bar{f}$. Let $h := g \circ i$. Then $\pi \circ h = f$. □

Let $\mathscr{F}$ be a set of ideals of the integral domain R. An R-module M is said to be *$\mathscr{F}$-injective* if for every ideal $I \in \mathscr{F}$, every R-homomorphism from I into M can be extended to an R-homomorphism from R into M. Denote by $\mathscr{F}_{w,f}(R)$ the set of all w-finite w-ideals of R. The following result is a w-theoretic analogue of [5, Theorem 3.1] in some sense.

Theorem 2.3. *The following conditions are equivalent for an integral domain R.*

(1) *R is a GGCD domain;*
(2) *Every quotient module of a $\mathscr{F}_{w,f}(R)$-injective R-module is $\mathscr{F}_{w,f}(R)$-injective;*
(3) *Every quotient module of an injective R-module is $\mathscr{F}_{w,f}(R)$-injective.*

Proof. (1) $\Rightarrow$ (2). Let M be a $\mathscr{F}_{w,f}(R)$-injective R-module, M' be a quotient module of M, and p be a homomorphism from M onto M'. Let $I \in \mathscr{F}_{w,f}(R)$ and let $f : I \to M'$ be an R-homomorphism. Thus we have the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \xrightarrow{i} & R \\ & & \big\downarrow{\scriptstyle f} & & \\ M & \xrightarrow{p} & M' & \longrightarrow & 0, \end{array}$$

where i denotes the inclusion map. Since I is a w-finite w-ideal of the GGCD domain R, I is invertible, equivalently projective. Thus there exists an R-homomorphism $g : I \to M$ such that $p \circ g = f$. Now M being $\mathscr{F}_{w,f}(R)$-injective, there exists an R-homomorphism $\bar{g} : R \to M$ such that $\bar{g} \circ i = g$. Now let $h := p \circ \bar{g}$. Then $h \circ i = f$. (2) $\Rightarrow$ (3). This is trivial. (3) $\Rightarrow$ (1). We prove that every w-finite w-ideal of R is projective, equivalently invertible. For this, we appeal to Lemma 2.1.

Consider the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \xrightarrow{i} & R \\ & & \downarrow{\scriptstyle f} & & \\ M & \xrightarrow{p} & M' & \longrightarrow & 0, \end{array}$$

where M is injective, rows are exact, and i is the inclusion map. Since M' is $\mathscr{F}_{w,f}(R)$-injective, there exists an R-homomorphism $\bar{f} : R \to M'$ such that $\bar{f} \circ i = f$. Now since R is a projective R-module, there exists an R-homomorphism $g : R \to M$ such that $p \circ g = \bar{f}$. Let $h := g \circ i$. Then $p \circ h = f$. □

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