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THE EQUIVALENCE OF THE AP–HENSTOCK AND AP–DENJOY INTEGRALS

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Abstract. In this paper, we define the ap-Denjoy integral and investigate some properties of the ap-Denjoy integral.

1. Introduction

Let E be a measurable set and let c be a real number. The *density* of E at c is defined by

$$d_c E = \lim_{h \to 0^+} \frac{\mu(E \cap (c-h, c+h))}{2h} \ ,$$

provided the limit exists. The point c is called a *point of density* of E if $d_c E = 1$. The set E^d represents the set of all points $x \in E$ such that x is a point of density of E.

A function $F : [a,b] \to \mathbb{R}$ is said to be *approximately differentiable* at $c \in [a,b]$ if there exists a measurable set $E \subseteq [a,b]$ such that $c \in E^d$ and $\lim_{\substack{x \to c \\ x \in E}} \frac{F(x)-F(c)}{x-c}$ exists. The approximate derivative of F at c is denoted by $F'_{ap}(c)$.

An *approximate neighborhood*(or ap-nbd) of $x \in [a,b]$ is a measurable set $S_x \subseteq [a,b]$ containing x as a point of density. For every $x \in E \subseteq [a,b]$, choose an ap-nbd $S_x \subseteq [a,b]$ of x. Then we say that $S = \{S_x : x \in E\}$ is a *choice* on E. A tagged interval $(x, [c,d])$

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is said to be *subordinate* to the choice $S = \{S_x\}$ if $c, d \in S_x$. Let $\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \leq i \leq n\}$ be a finite collection of non-overlapping tagged intervals. If $(x_i, [c_i, d_i])$ is subordinate to a choice S for each i, then we say that $\mathcal{P}$ is subordinate to S. If $\mathcal{P}$ is subordinate to S and $[a, b] = \bigcup_{i=1}^{n} [c_i, d_i]$, then we say that $\mathcal{P}$ is a tagged partition of $[a, b]$ that is subordinate to S.

2. The ap-Denjoy and ap-Henstock integrals

We introduce the notion of the approximate Lusin function. This function is used to define the ap-Denjoy integral.

DEFINITION 2.1. Let $F : [a, b] \to \mathbb{R}$ be a function. The function F is an approximate Lusin function(or F is an AL function) on $[a, b]$ if for every measurable set $E \subseteq [a, b]$ of measure zero and for every $\varepsilon > 0$ there exists a choice S on E such that $|(\mathcal{P}) \sum F(I)| < \varepsilon$ for every finite collection $\mathcal{P}$ of non-overlapping tagged intervals that is subordinate to S.

Recall that $F : [a, b] \to \mathbb{R}$ is AC_s on a measurable set $E \subseteq [a, b]$ if for each $\varepsilon > 0$ there exist a positive number η and a choice S on E such that $|(\mathcal{P}) \sum F(I)| < \varepsilon$ for every finite collection $\mathcal{P}$ of non-overlapping tagged intervals that is subordinate to S and satisfies $(\mathcal{P}) \sum |I| < \eta$, where $|I|$ is the Lebesgue measure of an interval I. The function F is ACG_s on E if E can be expressed as a countable union of measurable sets on each of which F is AC_s.

LEMMA 2.1. *If $F : [a, b] \to \mathbb{R}$ is ACG_s on $[a, b]$, then F is an AL function on $[a, b]$.*

Proof. Suppose that $E \subseteq [a, b]$ is a measurable set of measure zero. Let $\epsilon > 0$ and $E = \cup_{n=1}^{\infty} E_n$, where $\{E_n\}$ is a sequence of disjoint measurable sets and F is AC_s on each E_n. For each n, there exists a choice $S^n = \{S_x^n : x \in E_n\}$ on E_n and a positive number η_n such that $|(\mathcal{P}) \sum F(I)| < \epsilon / 2^n$ whenever $\mathcal{P}$ is subordinate to S^n and

$(\mathcal{P})\sum|I|<\eta_n$. For each n, choose an open set O_n such that $E_n\subseteq O_n$ and $|O_n|<\eta_n$. Let $S_x=S_x^n\cap O_n$ for each $x\in E_n$. Then $S=\{S_x: x\in E\}$ is a choice on E. Suppose that $\mathcal{P}$ is subordinate to S. Let $\mathcal{P}_n\subseteq\mathcal{P}$ that has tags in E_n and note that $(\mathcal{P})\sum|I|<|O_n|<\eta_n$. Hence

$$|(\mathcal{P})\sum F(I)|\leq\sum_{n=1}^{\infty}|(\mathcal{P}_n)\sum F(I)|<\sum_{n=1}^{\infty}\frac{\epsilon}{2^n}=\epsilon,$$

as desired. □

Definition 2.2. A function $f:[a,b]\to\mathbb{R}$ is *ap-Denjoy integrable* on $[a,b]$ if there exists an AL function F on $[a,b]$ such that F is approximately differentiable a.e. on $[a,b]$ and $F'_{ap}=f$ a.e. on $[a,b]$. The function f is ap-Denjoy integrable on a measurable set $E\subseteq[a,b]$ if $f\chi_E$ is ap-Denjoy integrable on $[a,b]$.

If we add the condition $F(a)=0$, then the function F is unique. We will denote this function $F(x)$ by $(AD)\int_a^x f$.

It is easy to show that if $f:[a,b]\to\mathbb{R}$ is ap-Denjoy integrable on $[a,b]$, then f is ap-Denjoy integrable on every subinterval of $[a,b]$. This gives rise to an interval function F such that $F(I)=(AD)\int_I f$ for every subinterval $I\subseteq[a,b]$. The function F is called the primitive of f.

Recall that $F:[a,b]\to\mathbb{R}$ is AC_* on a measurable set $E\subseteq[a,b]$ if for each $\varepsilon>0$ there exists $\delta>0$ such that $\sum_{i=1}^n\omega(F,[c_i,d_i])<\varepsilon$ whenever $\{[c_i,d_i]:1\leq i\leq n\}$ is a finite collection of non-overlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^n(d_i-c_i)<\delta$, where $\omega(F,[c_i,d_i])=\sup\{|F(y)-F(x)|: c_i\leq x<y\leq d_i\}$. The function F is ACG_* on E if $F|_E$ is continuous on E, $E=\cup_{n=1}^{\infty}E_n$ and F is AC_* on each E_n. It is easy to show that is F is ACG_* on $[a,b]$, then F is ACG_s on $[a,b]$. A function $f:[a,b]\to\mathbb{R}$ is Denjoy

integrable on $[a, b]$ if there exists an ACG_* function $F : [a, b] \to \mathbb{R}$ such that $F' = f$ almost everywhere on $[a, b]$.

The following theorem shows that the ap-Denjoy integral is an extension of the Denjoy integral.

THEOREM 2.2. *If $f : [a, b] \to \mathbb{R}$ is Denjoy integrable on $[a, b]$, then f is ap-Denjoy integrable on $[a, b]$.*

Proof. Suppose that $f : [a, b] \to \mathbb{R}$ is Denjoy integrable on $[a, b]$. Then there exists an ACG_* function $F : [a, b] \to \mathbb{R}$ such that $F' = f$ almost everywhere on $[a, b]$. Since F is ACG_s on $[a, b]$, by Lemma 2.1, F is an AL function on $[a, b]$ and $F'_{ap} = F' = f$ almost everywhere on $[a, b]$. Hence f is ap-Denjoy integrable on $[a, b]$. □

THEOREM 2.3. *Let $f : [a, b] \to \mathbb{R}$ be ap-Denjoy integrable on $[a, b]$ and let $F(x) = (AD) \int_a^x f$ for each $x \in [a, b]$. Then*

(a) the function F is approximately differentiable a.e. on $[a, b]$ and $F'_{ap} = f$ a.e. on $[a, b]$; and

(b) the functions F and f are measurable.

Proof. (a) follows from the definition of the ap-Denjoy integral. Since F is approximately continuous a.e. on $[a, b]$, F is measurable by [3, Theorem 14.7]. It follows from [3, Theorem 14.12] that f is measurable. □

THEOREM 2.4. *Let $F : [a, b] \to \mathbb{R}$ be an AL function on $[a, b]$. If F is approximately differentiable a.e. on $[a, b]$, then F'_{ap} is ap-Denjoy integrable on $[a, b]$ and $(AD) \int_a^x F'_{ap} = F(x) - F(a)$ for each $x \in [a, b]$.*

Proof. Suppose that F is an AL function on $[a, b]$ and F is approximately differentiable a.e. on $[a, b]$. Then for every constant function C, $F + C$ is also an AL function on $[a, b]$, approximately differentiable a.e. on $[a,b]$ and $(F+C)'_{ap} = F'_{ap}$ a.e. on $[a,b]$. Hence F'_{ap} is ap-Denjoy integrable on $[a,b]$ and

$$F(x) + C = (AD)\int_a^x F'_{ap} \quad \text{for each } x \in [a,b].$$

Since $F(a) + C = 0$, $C = -F(a)$ and

$$(AD)\int_a^x F'_{ap} = F(x) - F(a)$$

for each $x \in [a,b]$. □

We can easily show that if f is ap-Denjoy integrable on each of intervals $[a,c]$ and $[c,b]$, then f is ap-Denjoy integrable on $[a,b]$ and

$$(AD)\int_a^b f = (AD)\int_a^c f + (AD)\int_c^b f \ .$$

Recall that a function $f : [a,b] \to \mathbb{R}$ is *ap-Henstock integrable* on $[a,b]$ if there exists a real number A with the following property ; for each $\varepsilon > 0$ there exists a choice S on $[a,b]$ such that $|(\mathcal{P})\sum f(x)|I| - A| < \varepsilon$ whenever $\mathcal{P} = \{(x,I) : x \in [a,b]\}$ is a tagged partition of $[a,b]$ that is subordinate to S. The real number A is called the ap-Henstock integral of f on $[a,b]$ and is denoted by $(AH)\int_a^b f$. If f is ap-Henstock integrable on $[a,b]$, then f is also ap-Henstock integrable on any subinterval I of $[a,b]$. Hence an interval function F can be defined with $F(I) = (AH)\int_I f$. The function F is called the primitive of f. It is well-known [3] that the ap-Henstock integral is equivalent to the ap-Perron integral.

The following theorem shows that the ap-Denjoy integral is equivalent to the ap-Henstock integral and the integrals are equal.

THEOREM 2.5. *The function $f : [a,b] \to \mathbb{R}$ is ap-Denjoy integrable on $[a,b]$ if and only if f is ap-Henstock integrable on $[a,b]$ and the integrals are equal.*

Proof. If f is ap-Henstock integrable on $[a,b]$ with the primitive F, then F is ACG_s on $[a,b]$ and $F'_{ap} = f$ a.e. on $[a,b]$ [3, Theorem 16.18]. By Lemma 2.1, f is ap-Denjoy integrable on $[a,b]$.

Suppose that f is ap-Denjoy integrable on $[a,b]$ with the primitive F. Then F is an AL function on $[a,b]$ such that F is approximately differentiable a.e. on $[a,b]$ and $F'_{ap} = f$ a.e. on $[a,b]$. Let

$$E = \{x \in [a,b] : F'_{ap}(x) \neq f(x)\} \ .$$

Then $|E| = 0$. Let $D = [a,b] - E$ and let $\varepsilon > 0$.

For each $x \in D$, there exists a measurable set $D_x \subseteq [a,b]$ such that $x \in D_x^d$ and

$$F'_{ap}(x) = \lim_{\substack{y \to x \\ y \in D_x}} \frac{F(y) - F(x)}{y - x} \quad .$$

So there exists $\delta_x > 0$ such that for every $y \in D_x \cap (x - \delta_x, x + \delta_x) = S_x$

$$|F(y) - F(x) - F'_{ap}(x)(y - x)| \leq \varepsilon |y - x| \ .$$

If $(x, [u,v])$ is a tagged interval that is subordinate to $\{S_x\}$, then

$$\begin{aligned} |F(v) - F(u) - F'_{ap}(x)(v - u)| &\leq |F(v) - F(x) - F'_{ap}(x)(v - x)| \\ &\quad + |F(x) - F(u) - F'_{ap}(x)(x - u)| \\ &< \varepsilon(v - x) + \varepsilon(x - u) = \varepsilon(v - u) \ . \end{aligned}$$

Hence, there exists a choice S' on D such that $|(\mathcal{P})\sum f(x)|I| - (\mathcal{P})\sum F(I)| < \varepsilon(\mathcal{P})\sum |I|$, whenever $\mathcal{P}$ is a collection of tagged intervals that is subordinate to S'.

By [3, Lemma 9.15] and the fact that F is an AL function on $[a,b]$, there exists a choice S'' on E such that $|(\mathcal{P})\sum f(x)|I|| < \varepsilon$ and $|(\mathcal{P})\sum F(I)| < \varepsilon$, whenever $\mathcal{P}$ is subordinate to S''. Let $S = S' \cup S''$. Then S is a choice on $[a,b]$.

Suppose that $\mathcal{P}$ is a tagged partition of $[a,b]$ that is subordinate to S. Let $\mathcal{P}_E$ be the subset of $\mathcal{P}$ that has tags in E and let $\mathcal{P}_D = \mathcal{P} - \mathcal{P}_E$. Then we have

$$\begin{aligned}|(\mathcal{P})\sum f(x)|I| - (\mathcal{P})\sum F(I)| &\le |(\mathcal{P}_D)\sum f(x)|I| - (\mathcal{P}_D)\sum F(I)| \\ &\quad + |(\mathcal{P}_E)\sum f(x)|I|| + |(\mathcal{P}_E)\sum F(I)| \\ &< \varepsilon(b-a+2)\ .\end{aligned}$$

Hence, f is ap-Henstock integrable on $[a,b]$ and

$$(\mathrm{AH})\int_a^b f = (\mathcal{P})\sum F(I) = F(b) - F(a) = (\mathrm{AD})\int_a^b f,$$

as desired. □

References

1. P.S. Bullen, *The Burkill approximately continuous integal*, J. Austral. Math. Soc. (Series A) **35** (1983), 236–253.
2. T.S. Chew and K. Liao, *The descriptive definitions and properties of the AP-integral and their application to the problem of controlled convergence*, Real Analysis Exchange **19** (1993–94), 81–97.
3. R.A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron and Henstock*, Amer. Math. Soc., Providence, R.I., 1994.
4. R.A. Gordon, *Some comments on the McShane and Henstock integrals*, Real Analysis Exchange **23** (1997–98), 329–342.
5. J. Kurzweil, *On multiplication of Perron integrable functions*, Czechoslovak Math. J. **23** (1973), 542–566.
6. J. Kurzweil and J. Jarnik, 'Perron type integration on n-dimensional intervals as an extension of integration of step functions by strong equiconvergence', *Czechoslovak Math. J.*.
7. T.Y. Lee, *On a generalized dominated convergence theorem for the AP integral*, Real Analysis Exchange **20** (1994–95), 77–88.

8. K. Liao, *On the descriptive definition of the Burkill approximately continuous integral*, Real Analysis Exchange **18** (1992–93), 253–260.
9. Y.J. Lin, *On the equivalence of four convergence theorems for the AP-integral*, Real Analysis Exchange **19** (1993–94), 155–164.
10. J.M. Park, *Bounded convergence theorem and integral operator for operator valued measures*, Czechoslovak Math. J. **47** (1997), 425–430.
11. J.M. Park, *The Denjoy extension of the Riemann and McShane integrals*, Czechoslovak Math. J. **50** (2000), 615–625.
12. J.M. Park, C.G. Park, J.B. Kim, D.H. Lee and W.Y. Lee, *The integrals of s-Perron, sap-Perron and ap-McShane*, Czechoslovak Math. J. (to appear).
13. A.M. Russell, *Stieltjes type integrals*, J. Austral. Math. Soc. (Series A) **20** (1975), 431–448.
14. A.M. Russell, *A Banach space of functions of generalized variation*, Bull. Austral. Math. Soc. **15** (1976), 431–438.

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