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STABILITY AND EXISTENCE OF ALMOST PERIODIC SOLUTIONS OF DISCRETE VOLTERRA EQUATIONS WITH UNBOUNDED DELAY

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Abstract. We investigate the stability property and existence of almost periodic solutions of discrete Volterra equations with unbounded delay.

1. Introduction and Definitions

Difference equations are more appropriate than their continuous counterparts in cases when processes evolve stages. For example, population may grow or decline through non-overlapping generation. Besides, difference equations are well known for simulating continuous models for numerical purposes.

Indeed, many discrete models in population dynamics and neural dynamic systems are proposed and well studied with respect to their stability, permanence, bifurcation, chaotic behavior, oscillation, periodicity, etc [8].

Xia and Cheng [8] established some criteria for the existence of almost periodic solutions by assuming globally quasi-uniform asymptotic stability.

Choi et al. [2, 3, 5] studied the total stability and existence of almost periodic solutions for the discrete Volterra equations with unbounded

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delay of the form

$$x(n+1) = f(n, x(n)) + \sum_{j=-\infty}^{0} B(n, j, x(n+j), x(n)),$$

which is the discrete analogue of the integrodifferential equation

$$x'(t) = \hat{f}(t, x(t)) + \int_{-\infty}^{0} F(t, s, x(t+s), x(t)) ds.$$

See [4, 6] for the existence of periodic solutions or auotomorphic mild solutions of differential equations.

In this paper, we investigate the stability property and existence of almost periodic solutions for the above discrete Volterra equation with unbounded delay.

Consider the nonlinear difference equation

$$x(n+1) = f(n, x(n)), n \in \mathbb{Z}, \tag{1.1}$$

where $f : \mathbb{Z} \times D \to \mathbb{R}^d$ and D is an open set in $\mathbb{R}^d$, and its hull equation or limiting equation

$$x(n+1) = g(n, x(n)), n \in \mathbb{Z}, \tag{1.2}$$

where $g \in H(f)$ and

$$\begin{aligned} H(f) \;\; = \;\; & \{g \; : \; \lim_{k\to\infty} f(n+n_k, x) = g(n, x) \text{ for some integer sequence} \\ & (n_k) \subset \mathbb{Z} \text{ with } n_k \to \infty \text{ as } k \to \infty\}. \end{aligned}$$

A *solution* of (1.1) (or (1.2)) is a sequence $x = (x(n))$ which renders (1.1) (respectively (1.2)) into an identity after substitution. If $x(n) \in K$ for $n \in \mathbb{Z}$ where K is a compact set of D, then we say that x *is in* K. A solution of (1.1) (respectively (1.2)) with the initial value $x(n_0) = x_0$ is denoted by $x_f(n, n_0, x_0)$ (respectively $x_g(n, n_0, x_0)$).

Now, we list some definitions of stability for a solution of (1.1) in [8]. We will set $\mathbb{Z}[a, b] = \{a, a+1, \cdots, b-1, b\}$, and $\mathbb{Z}[a, \infty) = \{a, a+1, a+2, \cdots\}$ where $a, b \in \mathbb{Z}$.

Definition 1.1. A solution $\varphi(n)$ of (1.1) is *uniformly stable* (US) on $\mathbb{Z}[n_1, \infty)$ if given $n_0 \in \mathbb{Z}[n_1, \infty)$, for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $|x_0 - \varphi(n_0)| < \delta$ implies that

$$|x_f(n, n_0, x_0) - \varphi(n)| < \varepsilon$$

for all $n \in \mathbb{Z}[n_0, \infty)$, where $x_f(n, n_0, x_0)$ is another solution of (1.1) to the initial value problem $x(n_0) = x_0$.

Definition 1.2. A solution $\varphi(n)$ of (1.1) is *globally uniformly attractive* (GUA) on $\mathbb{Z}[n_1,\infty)$ if, for any $n_0 \in \mathbb{Z}[n_1,\infty)$ and $\varepsilon > 0$, there exists a positive integer $T(\varepsilon) > 0$ such that for any $x_0 \in \mathbb{R}^d$, we have

$$|x_f(n, n_0, x_0) - \varphi(n)| < \varepsilon$$

for all $n \geq n_0 + T(\varepsilon)$.

Definition 1.3. A solution $\varphi(n)$ of (1.1) is *globally uniformly asymptotically stable* (GUAS) on $\mathbb{Z}[n_1,\infty)$ if $\varphi(n)$ is not only US on $\mathbb{Z}[n_1,\infty)$, but also GUA on $\mathbb{Z}[n_1,\infty)$.

Definition 1.4. A solution $\varphi(n)$ of (1.1) is *globally quasi-uniformly asymptotically stable* (GQUAS) on $\mathbb{Z}[n_1,\infty)$ if for any $\varepsilon > 0$ and $r > 0$, there exist positive integers $T(r,\varepsilon) > 0$ and $M(r) > 0$ so that when $n_0 \in \mathbb{Z}[n_1,\infty)$ and $|x_0 - \varphi(n_0)| < r$, we have

$$|x_f(n, n_0, x_0) - \varphi(n)| < M(r),\ n \geq n_0$$

and

$$|x_f(n, n_0, x_0) - \varphi(n)| < \varepsilon,\ n \geq n_0 + T(r,\varepsilon).$$

Remark 1.5. GQUAS is much weaker than GUAS. If we consider

$$x(n+1) = Bx(n),\ n \in \mathbb{Z}, \tag{1.3}$$

where B is a positive constant with $0 < B < 1$, then the zero solution of (1.3) is GQUAS, but the zero solution is not GUAS [7].

Definition 1.6. A sequence $x : \mathbb{Z} \to \mathbb{R}^d$ is called *almost periodic* if for any given $\varepsilon > 0$, there exists an integer $l = l(\varepsilon) > 0$ such that each discrete interval of length l contains an integer τ for which

$$|x(n+\tau) - x(n)| < \varepsilon,\ n \in \mathbb{Z}.$$

2. Main results

We consider the discrete Volterra equation with unbounded delay

$$\begin{aligned} x(n+1) &= f(n, x(n)) + \sum_{j=-\infty}^{n} B(n, j, x(j), x(n)), x(n_0) = x_0 \quad (2.1) \\ &= f(n, x(n)) + \sum_{j=-\infty}^{0} B(n, n+j, x(n+j), x(n)), \end{aligned}$$

where $f : \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous in $x \in \mathbb{R}^d$ for every $n \in \mathbb{Z}$ and is almost periodic in $n \in \mathbb{Z}$ uniformly for $x \in \mathbb{R}^d$, $B : \mathbb{Z}\times\mathbb{Z}\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d$ is continuous in $x, y \in \mathbb{R}^d$ for any $j \le n \in \mathbb{Z}$ and is almost periodic in n uniformly for $(j, x, y) \in \mathbb{Z}^* = \mathbb{Z}^- \times \mathbb{R}^d \times \mathbb{R}^d$, that is, for any $\varepsilon > 0$ and any compact set $K^* \subset \mathbb{Z}^*$, there exists a number $l = l(\varepsilon, K^*) > 0$ such that any discrete interval of length l contains an integer τ for which

$$|B(n+\tau, j, x, y) - B(n, j, x, y)| \le \varepsilon,\ n \in \mathbb{Z}, (j, x, y) \in K^*. \qquad (2.2)$$

Here the compactness of K^* means that there exist a finite integer set $\Delta \subset \mathbb{Z}^-$ and compact set $\Theta \subset \mathbb{R}^d \times \mathbb{R}^d$ such that $K^* = \Delta \times \Theta$.

Also, for any $(g, D) \in H(f, B)$, we consider the limiting equation of (2.1):

$$x(n+1) = g(n, x(n)) + \sum_{j=-\infty}^{n} D(n, j, x(j), x(n)), x(n_0) = x_0. \qquad (2.3)$$

Theorem 2.1. *Suppose that (2.1) has a bounded GQUAS solution u on $\mathbb{Z}[n_1, \infty)$ and there exists an integer sequence (n_k) with $\lim n_k = \infty$ as $k \to \infty$. If the limiting equation (2.3) has a unique solution $x_g(n, n_0, x_0)$ to the initial value problems in K, then $v(n) = \lim_{k\to\infty} u(n + n_k)$ is a GQUAS bounded solution of (2.3) on $\mathbb{Z}$.*

Proof. Let $\varepsilon > 0$ and $r > 0$. Without loss of generality, we may assume that $\varepsilon < 1$. Note that v is defined on $\mathbb{Z}$, so we take an integer $n_0 \in \mathbb{Z}$ such that

$$|x(n_0) - u(n_0)| = |x_0 - u(n_0)| < r.$$

Then there is a sufficiently large integer $N > 0$ such that

$$n_0 + n_k \ge n_1$$

and

$$|x_0 - u(n_0 + n_k)| < r + 1 \equiv R$$

when $k \ge N$.

From GQUAS, there exist integers $M(R) > 0$ and $T(R, \frac{\varepsilon}{3}) > 0$ such that

$$|x_f(n + n_k, n_0 + n_k, x_0) - u(n + n_k)| < M(R), n \ge n_0, \qquad (2.4)$$

and

$$|x_f(n + n_k, n_0 + n_k, x_0) - u(n + n_k)| < \frac{\varepsilon}{3}, n \ge n_0 + T(R, \frac{\varepsilon}{3}). \qquad (2.5)$$

In view of (2.4), we have

$$|x_f(n+n_k, n_0+n_k, x_0)| < M(r) + |v|,\ n \geq n_1 - n_k.$$

Letting $k \to \infty$, we obtain

$$\lim_{k\to\infty} x_f(n+n_k, n_0+n_k, x_0) = x_g(n, n_0, x_0).$$

Thus

$$|x_f(n+n_k, n_0+n_k, x_0) - x_g(n, n_0, x_0)| < \frac{\varepsilon}{3}.$$

Also we have

$$|v(n) - u(n+n_k)| < \frac{\varepsilon}{3},\ k \geq N_1(\varepsilon).$$

Hence we have the following estimation:

$$\begin{aligned} |x_g(n, n_0, x_0) - v(n)| &\leq |x_g(n, n_0, x_0) - x_f(n+n_k, n_0+n_k, x_0)| \\ &+ |x_f(n+n_k, n_0+n_k, x_0) - u(n+n_k)| \\ &+ |u(n+n_k) - v(n)| \\ &< \frac{\varepsilon}{3} + M(R) + \frac{\varepsilon}{3} \\ &< \varepsilon + M(R) \equiv M_1(r),\ n \geq n_0, \end{aligned}$$

and

$$\begin{aligned} |x_g(n, n_0, x_0) - v(n)| &\leq |x_g(n, n_0, x_0) - x_f(n+n_k, n_0+n_k, x_0)| \\ &+ |x_f(n+n_k, n_0+n_k, x_0) - u(n+n_k)| \\ &+ |u(n+n_k) - v(n)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,\ n \geq n_0 + T(R, \frac{\varepsilon}{3}). \end{aligned}$$

This implies that v is GQUAS. □

THEOREM 2.2. *If $u(n)$ is a GQUAS bounded solution of (2.1) on $\mathbb{Z}$, then $u(n)$ is unique.*

Proof. Suppose that there is another bounded solution $u_1(n)$ of (2.1) on $\mathbb{Z}$. Assume that there exist n_1 and $r > 0$ such that

$$|u(n_1) - u_1(n_1)| = \varepsilon > 0$$

and

$$|u(n) - u_1(n)| < r,\ n \in \mathbb{Z}.$$

From GQUAS, there exists an integer $T(r, \frac{\varepsilon}{2}) > 0$ such that

$$|u(n) - u_1(n)| < \frac{\varepsilon}{2},\ n \geq n_0 + T(r, \frac{\varepsilon}{2})$$

when $|u(n_0) - u_1(n_0)| < r$. If we choose $n_0 = n_1 - T(r, \frac{\varepsilon}{2}) \in \mathbb{Z}$, then we have

$$|u(n_1) - u_1(n_1)| < \frac{\varepsilon}{2},\ n_1 \geq n_0$$

which is a contradiction. Hence $u(n) = u_1(n)$. □

To show that the existence of almost periodic solutions of (2.1) we need the following lemma.

LEMMA 2.3. *[7] (i) $x : \mathbb{Z} \to \mathbb{R}^d$ is an almost periodic sequence if and only if for any integer sequence $\{k_i'\}$, there exists a subsequence $\{k_i\} \subset \{k_i'\}$ such that $x(n + k_i)$ converges uniformly on $\mathbb{Z}$ as $i \to \infty$. Furthermore, the limit sequence is also almost periodic.*

(ii) $f : \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}^d$ is almost periodic in $n \in \mathbb{Z}$ uniformly for $x \in \mathbb{R}^d$ if and only if for any sequences (α_n') and (β_n') in $\mathbb{Z}$, there exist subsequences $(\alpha_n) \subset (\alpha_n')$ and $(\beta_n) \subset (\beta_n')$ such that

$$\lim_{n\to\infty} f(n + \alpha_n + \beta_n, x) = \lim_{n\to\infty}\{\lim_{l\to\infty} f(n + \alpha_n + \beta_l, x)\}. \tag{2.6}$$

THEOREM 2.4. *If the limiting equation (2.3) of (2.1) has a unique solution in a compact set $K \subset \mathbb{R}^d$, then the solutions of (2.1) are almost periodic.*

Proof. Let u be any solution of (2.1) in K. From Lemma 2.3, for given sequences (m_k') and (n_k'), there exist subsequences $(m_k) \subset (m_k')$ and $(n_k) \subset (n_k')$ such that

$$\lim_{k\to\infty} f(n + m_k + n_k, x) = \lim_{k\to\infty} \hat{f}(n + m_k, x),$$

where $\hat{f}(n, x) = \lim_{k\to\infty} f(n + n_k, x)$, and

$$\begin{aligned}&\lim_{k\to\infty} B(n + m_k + n_k, n + l + m_k + n_k, x, y)\\ &= \lim_{k\to\infty} \hat{B}(n + m_k, n + l + m_k, x, y),\end{aligned}$$

where $\hat{B}(n, n + l, x, y) = \lim_{k\to\infty} B(n + n_k, n + l + n_k, x, y)$, uniformly on $\mathbb{Z} \times K$. Also, there exist

$$\lim_{k\to\infty} u(n + m_k + n_k) = v(n) \text{ and } \lim_{k\to\infty} \hat{u}(n + m_k) = \hat{v}(n),$$

where $\hat{u}(n) = \lim_{k\to\infty} u(n + n_k)$, uniformly on the compact subset of $\mathbb{Z}$. Note that $v(n)$ and $\hat{v}(n)$ are solutions in K of the same limiting equation

$$\begin{aligned} x(n+1) &= g(n + m_k + n_k, x) \\ &+ \sum_{j=-\infty}^{n} D(n + m_k + n_k, j, x(j), x(n + m_k + n_k)). \end{aligned}$$

By the uniqueness of solutions, we obtain that $v(n) = \hat{v}(n)$ for all $n \in \mathbb{Z}$. It follows from Lemma 2.3 that v is almost periodic. This completes the proof. □

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