JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **24**, No. 2, June 2011

THE TENSION FIELD OF THE ENERGY FUNCTIONAL ON RIEMANNIAN SUBMERSION

Boo-Yong Choi*

Abstract. In this paper, we will study the tension field of the function related to a Riemannain submersion $\pi : N \to M$ with totally geodesic fibres. In case that the Riemannain submersion $\pi : N \to M$ particularly has a smooth map $f : M \to N$ which happens to be a section, we will show that tension field $\tau(f)$ of the energy functional can be decomposed into the horizontal and vertical parts.

1. Introduction

Let M and N be complete Riemannian manifolds. Asumme M is compact. A smooth map $f : M \to N$ is called harmonic if it is a critical point of the energy functional. This critical point of the energy functional is written and characterized in terms of some differential equation (called the Euler-Lagrange equation). And we can now calculate the tension field to obtain the Euler-Lagrange equation. We consider the case when N is a fibre bundle over M, and $f : M \to N$ is a smooth map which happens to be a section of this fibration. We will consider the case when the fibres are totally geodesic compact submanifolds, and hence N is also a compact Riemannian manifold. In this case, the Euler-Lagrange equation for such a section is formulated([1], [2], [3], [4]). In this paper, we will obtain both horizontal and vertical parts of the tension field.
In section 2, we are primarily devoted to a summary of known results of Riemannian submersion([5]). And in section 3, we will set up the tension field of $f : M \to N$ which happens to be a section of this fibration. In main result, we will decompose $\tau(f)$ as horizontal parts $\tau^{\mathcal{H}}(f)$ and vertical parts $\tau^{\mathcal{V}}(f)$.

Received February 28, 2011; Accepted May 16, 2011.
2010 Mathematics Subject Classification: Primary 53C30; Secondary 53C43, 58E20.
Key words and phrases: tension field, Riemannian submersion, section, euler-lagrange equation.

2. Riemannian submersion with totally geodesic fibres

We will study a particular case of Riemannian submersion with totally geodesic fibres, so we need some properties and formulas about Riemannian submersion. We will use the terminology of O'Neill([5]).

DEFINITION 2.1. A Riemannian submersion $\pi : N \to M$ is a submersion of Riemannian manifolds such that:
(1) The fibre $\pi^{-1}(x)$, $x \in M$, are Riemannian submanifolds of N,
(2) $d\pi$ preserves scalar products of vectors normal to fibres.

Given a Riemannian submersion π from N to M, we denote by $\mathcal{V}$ the vector subbundle of TN defined by the foliation of N by the fibres of π. $\mathcal{H}$ will denote the complementary distribution of $\mathcal{V}$ in TN determined by the metric on N. Following O'Neill([5]), we define the tensor T for arbitrary vector fields E and F by $T_EF = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F$ where $\mathcal{V}E$, $\mathcal{H}F$, etc. denote the vertical and horizontal projections of vector field E and ∇ is the covariant derivative of N. O'Neill has described the following three properties of the tensor T :
1. T_E is a skew-symmetric operator on the tangent space of N reversing horizontal and vertical subspaces.
2. $T_E = T_{\mathcal{V}E}$
3. For vertical vector fields V and W, T is symmetric, i.e., $T_VW = T_WV$.

In fact, along a fibre T is the second fundamental form of the fibre provided we restrict ourselves to vertical vector fields. Next we define the integrability tensor A associated with the submersion. For arbitrary vector fields E and F,

$$A_EF = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F.$$

$1'$. At each point A_E is a skew-symmetric operator on TN reversing the horizontal and vertical subspaces.
$2'$. $A_E = A_{\mathcal{H}E}$
$3'$. For X, Y horizontal A is alternating, i.e., $A_XY = -A_YX$.
We define a vector field X on N to be basic provided X is horizontal and π-related to a vector field $\tilde{X}$ on M. Every vector field $\tilde{X}$ on M has a horizontal lift X to N and X is basic.
We recall the following standard results about Riemannian submersion.

LEMMA 2.2. *If X and Y are basic vector fields on N, then*

(a) $g(X,Y) = h(\tilde{X},\tilde{Y})\circ\pi$ *where* g *is the metric on* N *and* h *the metric on* M,

(b) $\mathcal{H}[X,Y]$ *is basic and is* π*-related to* $[\tilde{X},\tilde{Y}]$

(c) $\mathcal{H}\nabla_X Y$ *is basic and is* π*-related to* $\tilde{\nabla}_{\tilde{X}}\tilde{Y}$ *where* $\tilde{\nabla}$ *is the Riemannian connection on* M.

Lemma 2.3. *If* X *and* Y *are horizontal vector fields, then*

$$A_X Y = \frac{1}{2}\mathcal{V}[X,Y].$$

Lemma 2.4. *Let* X *and* Y *be horizontal vector fields, and* V *and* W *vertical vector fields. Then*

(a) $\nabla_V W = T_V W + \mathcal{V}\nabla_V W$

(b) $\nabla_V X = T_V X + \mathcal{H}\nabla_V X$

(c) $\nabla_X V = A_X V + \mathcal{V}\nabla_X V$

(d) $\nabla_X Y = A_X Y + \mathcal{H}\nabla_X Y$

Furthermore, if X *is basic,* $\mathcal{H}\nabla_V X = A_X V$.

Lemma 2.5. *Let* X *be a horizontal vector field and* W *a vertical vector field. Then*

(a) $(\nabla_X A)_W = -A_{A_X W}$,

(b) $(\nabla_W T)_X = -T_{T_W X}$.

The next result gives a geometric characterization of the parallelism of the fundamental tensors T and A.

(a) If A is parallel, then A is identically zero, i.e., $\nabla_E A = 0$ implies $A = 0$.

(b) If T is parallel, then T is identically zero, i.e., $\nabla_E T = 0$ implies $T = 0$.

Thus Riemannian submersions with parallel integrability tensors A are characterized as those whose horizontal distributions are integrable, and Riemannian submersions with parallel tensors T as those fibres are totally geodesic.

Assume $\pi : N \to M$ has the structure of a fibred space; as usual, assume π is a Riemannian submersion and, in addition, N is complete. Let γ be a smooth curve in M with $\gamma(0) = p$ and $\gamma(t_0) = q$. Then the family of unique horizontal lifts of γ to N denoted by $\{\tilde{\gamma}_x\}$ with $\tilde{\gamma}_x(0) = x \in \pi^{-1}(p)$, we have $F_\gamma(x) = \tilde{\gamma}_x(t_0)$ and therefore, the mapping F_γ are diffeomorphisms between the fibres. Moreover, a necessary and sufficient condition for the mapping F_γ to be isometries is that the fibres are totally geodesic.

3. Tension field of sections

Let M and N be complete Riemannian manifolds. Assume M is compact. A smooth map $\pi : N \to M$ is called a Riemannian submersion if π is a submersion and if for each $x \in N$, the horizontal subspace of T_xN (orthogonal to the fibre over $\pi(x)$ in N) is mapped isometrically by $d\pi|_x$ to $T_{\pi(x)}M$. We denote by $\mathcal{H}$, $\mathcal{V}$ the horizontal and the vertical distribution, respectively. Then we can decompose the tangent bundle $TN = TN^{\mathcal{H}} \oplus TN^{\mathcal{V}}$, where we denote by $TN^{\mathcal{H}}$, $TN^{\mathcal{V}}$ the horizontal and the vertical subbundle, respectively.

We now consider a Riemannian submersion with totally geodesic fibre F, that is, for each x in N with $p = \pi(x)$, $\pi^{-1}(p) = F_x$ is a totally geodesic submanifold of N. Then all the fibres are isometric to each other and π is a Riemannian fibration. Furthermore, the horizontal distribution defines a connection on this fibre bundle. Let $f : M \to N$ be a smooth map which happens to be a section. The energy functional of the section f is $E(f) = \int_M e(f)dv$, where $e(f) = \frac{1}{2}\|df\|^2$ is the energy density of f. The differential map df is a differential 1-form with values in the pull-back bundle $f^{-1}(TN)$ and hence a section of $T^*M \otimes f^{-1}(TN)$. Decompose $f^{-1}(TN)$ as $f^{-1}(TN^{\mathcal{H}}) \oplus f^{-1}(TN^{\mathcal{V}})$, and then we have $df = df^{\mathcal{H}} + df^{\mathcal{V}}$, where $df^{\mathcal{H}} \in \Gamma\big(T^*M \otimes f^{-1}(TN^{\mathcal{H}})\big)$, $df^{\mathcal{V}} \in \Gamma\big(T^*M \otimes f^{-1}(TN^{\mathcal{V}})\big)$, and $\Gamma(\cdot)$ denotes the set of all smooth sections of the corresponding bundle. Then the energy $E(f)$ is given by

$$E(f) = E^{\mathcal{H}}(f) + E^{\mathcal{V}}(f) = \frac{1}{2}\int_M \|df^{\mathcal{H}}\|^2 dv + \frac{1}{2}\int_M \|df^{\mathcal{V}}\|^2 dv.$$

Since f is a section of a Riemannian fibration, the linear map $df_p^{\mathcal{H}} : T_pM \to (T_xN)^{\mathcal{H}}$ is an isometry for each $p = \pi(x)$, and hence we have $E^{\mathcal{H}}(f) = \frac{m}{2}\,\mathrm{Vol}(M)$ (dim $M = m$).

For $f : M \to N$ we now consider the Euler-Lagrange equation of the energy functional. Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connection on M and N, respectively, and let $\bar{\nabla}$ be the induced connection on the pull-back bundle. Then we have $\bar{\nabla}(df) \in \Gamma((S^2M \otimes f^{-1}(TN^{\mathcal{H}})) \oplus (S^2M \otimes f^{-1}(TN^{\mathcal{V}})))$, where S^2M is the space of symmetric covariant 2-tensors. Taking trace of the second fundamental form gives the tension field,

$$\tau(f) = -\bar{\nabla}^*(\bar{\nabla}df) = Tr(\bar{\nabla}df) \in \Gamma(f^{-1}(TN)).$$

4. Main result

In this section, we will show both horizontal and vertical parts of the tension field. Let $\pi : N \to M$ be a Riemannian submersion with totally geodesic fibre and $f : M \to N$ be a smooth section as section 3. In section 3,

$$df \in \Gamma\big\{(T^*M \otimes f^{-1}(TN^{\mathcal{H}}) \oplus (T^*M \otimes f^{-1}(TN^{\mathcal{V}})\big\},$$

we have $df = df^{\mathcal{H}} + df^{\mathcal{V}}$. And

$$\bar{\nabla}(df) \in \Gamma((S^2M \otimes f^{-1}(TN^{\mathcal{H}})) \oplus (S^2M \otimes f^{-1}(TN^{\mathcal{V}}))).$$

Since $\tau(f) = -\bar{\nabla}^*(\bar{\nabla}df) = Tr(\bar{\nabla}df)$, we decompose $\tau(f)$ as $\tau(f) = \tau^{\mathcal{H}}(f)+\tau^{\mathcal{V}}(f)$, where $\tau^{\mathcal{H}}(f) \in \Gamma\big(f^{-1}(TN^{\mathcal{H}})\big)$ and $\tau^{\mathcal{V}}(f) \in \Gamma\big(f^{-1}(TN^{\mathcal{V}})\big)$. For a vector field X on M let $\tilde{X}$ denote the basic vector field which is a horizontal lift of X. Then for a local orthonormal frame field $\{e_i\}$ of M, $df^{\mathcal{H}}(e_i) = \tilde{e_i}$.

$$\begin{aligned} \tau(f) &= Tr(\bar{\nabla}df) \\ &= \sum_i(\bar{\nabla}df)(e_i, e_i) = \sum_i(\bar{\nabla}_{e_i}df)(e_i) \\ &= \sum_i \big(\bar{\nabla}_{e_i}(df(e_i)) - df(\nabla_{e_i}e_i)\big) \\ &= \sum_i \big\{\bar{\nabla}_{e_i}\big(df^{\mathcal{H}}(e_i) + df^{\mathcal{V}}(e_i)\big) - \big(df^{\mathcal{H}}(\nabla_{e_i}e_i) + df^{\mathcal{V}}(\nabla_{e_i}e_i)\big)\big\}. \qquad (4.1) \end{aligned}$$

Now, we can calculate the horizontal and vertical parts of the tension field. Since $(\bar{\nabla}_{e_i}df^{\mathcal{H}})(e_i) = \bar{\nabla}_{e_i}\big(df^{\mathcal{H}}(e_i)\big) - df^{\mathcal{H}}(\nabla_{e_i}e_i)$ and $\tilde{\nabla}_{\tilde{e_i}}\tilde{e_i} = \widetilde{(\nabla_{e_i}e_i)}$ by O'Neill's formular in [5].

$$\begin{aligned} (\bar{\nabla}_{e_i}df^{\mathcal{H}})(e_i) &= (\bar{\nabla}_{e_i}\tilde{e_i}) - \widetilde{(\nabla_{e_i}e_i)} \\ &= \big(\tilde{\nabla}_{\tilde{e_i}}\tilde{e_i} + \tilde{\nabla}_{df^{\mathcal{V}}(e_i)}\tilde{e_i}\big) - \widetilde{(\nabla_{e_i}e_i)} \\ &= \tilde{\nabla}_{df^{\mathcal{V}}(e_i)}\tilde{e_i}. \qquad (4.2) \end{aligned}$$

And since fibres are totally geodesic, we have $\tilde{\nabla}_{df^{\mathcal{V}}(e_i)}\tilde{e_i} \in \mathcal{H}$ i.e., $(\bar{\nabla}_{e_i}df^{\mathcal{H}})^{\mathcal{V}}(e_i) = 0$. For the vertical component, we locally extend $df^{\mathcal{V}}(e_i)$, a vector field along f, to a vertical vector field on N which we also denote by $df^{\mathcal{V}}(e_i)$. We then have

$$\begin{aligned}(\bar{\nabla}_{e_i} df^{\mathcal{V}})(e_i) &= \bar{\nabla}_{e_i} df^{\mathcal{V}}(e_i) - df^{\mathcal{V}}(\nabla_{e_i} e_i) \\ &= \big(\tilde{\nabla}_{\tilde{e_i}} df^{\mathcal{V}}(e_i) + \tilde{\nabla}_{df^{\mathcal{V}}(e_i)} df^{\mathcal{V}}(e_i)\big) - df^{\mathcal{V}}(\nabla_{e_i} e_i)\end{aligned}$$

where $df^{\mathcal{V}}(\nabla_{e_i} e_i)$ and $\tilde{\nabla}_{df^{\mathcal{V}}(e_i)} df^{\mathcal{V}}(e_i)$ are in $\mathcal{V}$ because the fibres are totally geodesic. Furthermore $[\tilde{e_i}, df^{\mathcal{V}}(e_i)] \in \mathcal{V}$. Therefore

$$[\tilde{e_i}, df^{\mathcal{V}}(e_i)]^{\mathcal{H}} = \big(\tilde{\nabla}_{\tilde{e_i}} df^{\mathcal{V}}(e_i)\big)^{\mathcal{H}} - \big(\tilde{\nabla}_{df^{\mathcal{V}}(e_i)} \tilde{e_i}\big)^{\mathcal{H}} = 0,$$

so $\big(\tilde{\nabla}_{\tilde{e_i}} df^{\mathcal{V}}(e_i)\big)^{\mathcal{H}} = \big(\tilde{\nabla}_{df^{\mathcal{V}}(e_i)} \tilde{e_i}\big)^{\mathcal{H}}$. But since $\tilde{\nabla}_{df^{\mathcal{V}}(e_i)} \tilde{e_i} \in \mathcal{H}$, $\big(\tilde{\nabla}_{df^{\mathcal{V}}(e_i)} \tilde{e_i}\big)^{\mathcal{H}} = \tilde{\nabla}_{df^{\mathcal{V}}(e_i)} \tilde{e_i}$. Thus we conclude that

$$\big(\bar{\nabla}_{e_i} df^{\mathcal{V}}(e_i)\big)^{\mathcal{H}} = \big(\tilde{\nabla}_{\tilde{e_i}} df^{\mathcal{V}}(e_i)\big)^{\mathcal{H}} = \tilde{\nabla}_{df^{\mathcal{V}}(e_i)} \tilde{e_i}. \tag{4.3}$$

Now, we can decompose the horizontal and vertical parts of tension field by (4.2) and (4.3).

$$\begin{aligned}\tau(f) &= \sum_i \big\{\bar{\nabla}_{e_i}\big(df^{\mathcal{H}}(e_i) + df^{\mathcal{V}}(e_i)\big) - \big(df^{\mathcal{H}}(\nabla_{e_i} e_i) + df^{\mathcal{V}}(\nabla_{e_i} e_i)\big)\big\} \\ &= \sum_i \big\{\tilde{\nabla}_{df^{\mathcal{V}}(e_i)} \tilde{e_i}\big\} \\ &\quad + \sum_i \big\{\big(\bar{\nabla}_{e_i} df^{\mathcal{V}}(e_i)\big)^{\mathcal{H}} + \big(\bar{\nabla}_{e_i} df^{\mathcal{V}}(e_i)\big)^{\mathcal{V}} - df^{\mathcal{V}}(\nabla_{e_i} e_i)\big\} \\ &= 2\sum_i \big\{\tilde{\nabla}_{df^{\mathcal{V}}(e_i)} \tilde{e_i}\big\} + \sum_i \big\{\big(\bar{\nabla}_{e_i} df^{\mathcal{V}}(e_i)\big)^{\mathcal{V}} - df^{\mathcal{V}}(\nabla_{e_i} e_i)\big\}.\end{aligned}$$

Therefore we now have the following.

THEOREM 4.1. *For $f : M \to N$ be a smooth section as section 3, we can decompose $\tau(f)$ as horizontal and vertical parts i.e.,*

$$\tau(f) = \tau^{\mathcal{H}}(f) + \tau^{\mathcal{V}}(f)$$

where $\tau^{\mathcal{H}}(f) = 2\sum_i \big(\tilde{\nabla}_{df^{\mathcal{V}}(e_i)} \tilde{e_i}\big)$ and $\tau^{\mathcal{V}}(f) = \sum_i \big\{\big(\bar{\nabla}_{e_i} df^{\mathcal{V}}(e_i)\big)^{\mathcal{V}} - df^{\mathcal{V}}(\nabla_{e_i} e_i)\big\}$.

References

[1] B. Y. Choi and J. W. Yim, *Distributions on Riemannian Manifolds, which are Harmonic maps*, Tohoku Math. J. **55** (2003), 175-188.

[2] T. Ishihara, *Harmonic section of tangent bundles*, J. Math. Tokushima Univ. **13** (1979), 23-27.

[3] G. Jensen and M. Rigoli, *Harmonic Gauss maps*, Pacific J. Math. 1**36** (1989), 261-282.

[4] J. J. Konderak, *On harmonic vector field*, Publ. Mat. **36** (1992), 217-228.

[5] B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459-469.

*

Department of Mathematics
Air Force Academy
Cheongwon 363-849, Republic of Korea
E-mail: byc8734@hanmail.net