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SOME INVARIANT SUBSPACES FOR BOUNDED LINEAR OPERATORS

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Abstract. A bounded linear operator T on a complex Banach space X is said to have *property (I)* provided that T has Bishop's property (β) and there exists an integer $p > 0$ such that for a closed subset F of $\mathbb{C}$

$$X_T(F) = E_T(F) = \bigcap_{\lambda \in \mathbb{C} \setminus F} (T - \lambda)^p X \quad \text{for all closed sets} \quad F \subseteq \mathbb{C},$$

where $X_T(F)$ denote the analytic spectral subspace and $E_T(F)$ denote the algebraic spectral subspace of T. Easy examples are provided by normal operators and hyponormal operators in Hilbert spaces, and more generally, generalized scalar operators and subscalar operators in Banach spaces.

In this paper, we prove that if T has property (I), then the quasi-nilpotent part $H_0(T)$ of T is given by

$$KerT^p = \{x \in X : r_T(x) = 0\} = \bigcap_{\lambda \neq 0} (T - \lambda)^p X$$

for all sufficiently large integers p, where $r_T(x) = \limsup_{n \to \infty} ||T^n x||^{\frac{1}{n}}$. We also prove that if T has property (I) and the spectrum $\sigma(T)$ is finite, then T is algebraic. Finally, we prove that if $T \in L(X)$ has *property* (I) and has decomposition property (δ), then T has a non-trivial invariant closed linear subspace.

1. Introduction

Let X be a complex Banach space and $L(X)$ denotes the Banach algebra of all bounded linear operators on X. For $T \in L(X)$, let, as usual, $\sigma(T)$, $\rho(T)$ and $r(T)$ denote the spectrum, resolvent set, spectral radius of T, respectively and let $Lat(T)$ stand for the collection of all $T-$invariant closed linear subspaces of X. For $T \in L(X)$ and $Y \in Lat(T)$, $T|Y$ denote the restriction of T on Y.

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Given an operator $T \in L(X)$ and a linear subspace Y of X, we say that Y is an *invariant subspace* of T if $TY \subseteq Y$. Obviously $\{0\}$ and X are invariant subspaces and M invariant implies $\overline{M}$ invariant. So the interesting closed invariant subspaces are the non-trivial ones. The invariant subspace problem asks whether every operator on a complex separable Hilbert space has a non-trivial invariant subspace. This problem has its origins approximately in 1935 when (according to [11]) J. von Neumann proved that every compact operator on a separable infinite dimensional complex Hilbert space has a non-trivial subspace. In 1954, Aronszajn and Smith [11] proved that if X is an infinite dimensional complex Banach space and $T \in L(X)$ is completely continuous then T has a non-trivial invariant subspace. In 1966, Bernstein and Robinson [17] proved that if $\mathcal{H}$ is a complex Hilbert space, and $T \in L(\mathcal{H})$ is a polynomially compact operator, i.e. for some non-zero polynomial p, $p(T)$ is compact, then T has a non-trivial invariant subspace. The proof uses non-standard analysis as well as techniques similar to [11]. In 1966, Halmos [33] gave a proof of the same result by a similar method but avoiding the non-standard tools.

In 1968, Arveson and Feldman [12] proved that if $\mathcal{H}$ is a Hilbert space and $T \in L(\mathcal{H})$ satisfy $\|TP_n - P_nTP_n\| \to 0$ for some sequence (P_n) of orthogonal projection operators which converges strongly to the identity operator. Assume that the norm closed algebra generated by T and I contains a non-zero compact operator, then T has a non-trivial invariant subspace.

In 1973, Pearcy and Salinas [48] proved that if $T \in L(\mathcal{H})$ is a quasitriangular operator on a Hilbert space $\mathcal{H}$ and $\mathcal{R}(T)$(the norm closure of the rational functions of T) contains a non-zero compact operator, then there exists a non-trivial subspace invariant under all operators in $\mathcal{R}(T)$. In 1973, Lomonosov [42] proved that if X is a complex Banach space, and $T \in L(X)$ is an operator which is not a multiple of the identity and commutes with some non-zero compact operator, then T has a non-trivial invariant subspace. Lomonosov's result was extended to real Banach spaces by Hooker in [35].

In 1978, Brown [19] proved that every subnormal operator has a non-trivial invariant subspace. In 1987, Brown proved [20] that every hyponormal operator has a non-trivial invariant subspace whenever $C(\sigma(T)) \neq R(\sigma(T))$ where for a compact $K \subseteq \mathbb{C}$, $C(K)$ denotes the continuous functions on K and $R(K)$ denotes the closure in the $C(K)$ norm of the rational functions on K with poles outside of K. Using the technique of minimal vectors, a special case of the 1973 of Lomonosov is proved in [8]. If $T \in L(\mathcal{H})$ is a non-zero compact operator on a Hilbert space $\mathcal{H}$, then T has a non-trivial hyperinvariant subspace. In 1984, Putinar [50] proved that all hyponormal operators have property (β). The preceding result subsumes, in particular, Brown's celebrated invariant subspace theorem for hyponormal operators with thick spectrum.

In 1984, C. J. Read [52] proved that there exist quasi-nilpotent operators(and hence decomposable) on Banach spaces without non-trivial closed invariant subspaces. It is clear that the condition of thick spectrum cannot be dropped in general; see [41] for more details.

In 1990, Eschmeier and Prunaru [30] established that Lat(T) is non-trivial provided that $\sigma(T)$ is thick, and that Lat(T) is rich in the sense that it contains the lattice of all closed subspaces of some infinite-dimensional Banach space provided that the essential spectrum $\sigma_e(T)$ is thick. Here we skip the formal definition of thick subsets of the complex plane, but note that all compact sets with non-empty interior are thick. The invariant subspace problem has motivated enormous literature in operator theory, see [9], [10], [14], [16], [27], [33], [48] and [51] for more informations.

In this note, we show that if $T \in L(X)$ has property (I) on a Banach space X, then $\|T^n x_0\|^{\frac{1}{n}} \to 0$ as $n \to \infty$ if and only if $T^p x_0 = 0$ for some integer $p > 0$. Moreover, $KerT^p = X_T(\{0\})$ is the quasi-nilpotent part of T. We also prove that if T has property (I) and the spectrum $\sigma(T)$ is finite, then T is algebraic. Finally, we prove that if $T \in L(X)$ has property (I) and T has decomposition property (δ), then T has a non-trivial invariant closed linear subspace. This results are exemplified in the case of subscalar(spectral, generalized scalar, normal, subnormal, hyponormal, $\omega-$hyponormal, $log-$hyponormal, k-quasihyponormal, isometries)operators with property (δ).

2. Local spectral theory and property (I)

An operator $T \in L(X)$ is said to be *decomposable,* provided that, for each open cover $\mathbb{C} = U \cup V$ of the complex plane $\mathbb{C}$, there exist $Y, Z \in Lat(T)$ for which

$$X = Y + Z, \quad \sigma(T|Y) \subseteq U \text{ and } \sigma(T|Z) \subseteq V.$$

This simple definition is equivalent to the original notion of decomposability, as introduced by Foias in 1963 and discussed in the classical books by Colojoarvă and Foias [22], and [41]. The theory of decomposable operators is now richly developed with many interesting applications and connections. Evidently, the class of decomposable operators contains all normal operators on Hilbert spaces and more generally, all spectral operators in the sense of Dunford on Banach spaces. Moreover, a simple application of the Riesz functional calculus shows that all operators with totally disconnected spectrum are decomposable. In particular, all compact and algebraic operators are decomposable.

We now say that an operator $T \in L(X)$ on a complex Banach space X has *Bishop's property* (β) provided that for every open subset U of $\mathbb{C}$ and for every sequence of analytic functions $f_n : U \to X$ for which $(T - \lambda)f_n(\lambda)$ converges uniformly to zero on each compact subset of U, it follows that also $f_n(\lambda) \to 0$ as $n \to \infty$, locally uniformly on U. Obviously, property (β) implies that $T \in L(X)$ has the *single-valued extension property(SVEP),* if for every open $U \subseteq \mathbb{C}$, the only analytic solution $f : U \to X$ of the equation $(T - \lambda)f(\lambda) = 0$ for all $\lambda \in U$ is the constant $f \equiv 0$.

Given an arbitrary operator $T \in L(X)$, let $\sigma_T(x) \subseteq \mathbb{C}$ denote the *local spectrum* of T at the point $x \in X$, i.e. the complement of the set $\rho_T(x)$ of all $\lambda \in \mathbb{C}$ for which there exist an open neighborhood U of λ in $\mathbb{C}$ and an analytic function $f : U \to X$ such that $(T - \mu)f(\mu) = x$ holds for all $\mu \in U$. For every closed subset F of $\mathbb{C}$, let $X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}$ denote the corresponding *analytic spectral subspace* of T. It is easy to see that $X_T(F)$ is a $T-$invariant linear subspace of X and also hyperinvariant for T. This space doesn't have to be closed in general. These spectral subspaces date back to early work of E. Bishop [18] and have been fundamental in the recent progress of local spectral theory, for instance in connection with functional models and invariant subspaces [7], [41].

An operator $T \in L(X)$ is said to have *Dunford's property (C)* if $X_T(F)$ is closed for every closed $F \subseteq \mathbb{C}$. Finally, an operator $T \in L(X)$ is said to have the *decomposition property* (δ) if, given an arbitrary open covering $\{U_1, U_2\}$ of $\mathbb{C}$, every $x \in X$ has a decomposition $x = u_1 + u_2$ where $u_1, u_2 \in X$ satisfy

$$u_k = (T - \lambda) f_k(\lambda) \text{ for all } \lambda \in \mathbb{C} \setminus \overline{U_k}$$

and some analytic function $f_k : \mathbb{C} \setminus \overline{U_k} \to X$ for $k = 1, 2$. It is not difficult to see that property (δ) is inherited by quotients, and that T is decomposable precisely when T has both properties (β) and (δ). It is clear that Bishop's property (β) implies Dunford property (C) and property (C) implies SVEP. Note that neither of the implications may be reversed in general, [22], [45].

Associated with the operator T and each closed subset F of $\mathbb{C}$ is also an *algebraic spectral subspace* $E_T(F)$, defined to be the linear span of the collection of all (not necessarily closed) linear subspaces Y of X for which

$$(T - \lambda)Y = Y \text{ for each } \lambda \in \mathbb{C} \setminus F.$$

Evidently, $E_T(F)$ is the largest linear subspace Y for which $(T-\lambda)Y = Y$ for all $\lambda \in \mathbb{C} \setminus F$. It follows from Proposition 1.2.16 of [41] that $X_T(F) \subseteq E_T(F)$ for every $T \in L(X)$ and all closed set $F \subseteq \mathbb{C}$. Thus if T has no non-trivial divisible subspace in the sense that $E_T(\phi) = \{0\}$, then clearly T has SVEP. By the open mapping theorem, we observe, for a closed set $F \subseteq \mathbb{C}$ that if $E_T(F)$ is closed, then $E_T(F) = X_T(F)$, see [41].

Definition 2.1. *An operator $T \in L(X)$ has property (I) provided that T has Bishop's property (β) and there exists an integer $p > 0$ such that*

$$X_T(F) = E_T(F) = \bigcap_{\lambda \in \mathbb{C} \setminus F} (T - \lambda)^p X \quad \textit{for all closed sets } F \subseteq \mathbb{C}.$$

We say that a linear subspace Y of X is said to be $T-$*divisible* if $(T-\lambda)Y = Y$ for all $\lambda \in \mathbb{C}$. Evidently, $E_T(\phi)$ is exactly the largest $T-$divisible linear subspace. It is clear that if $T \in L(X)$ has property (I) then T is admissible, i.e. for each closed set $F \subseteq \mathbb{C}$, the algebraic spectral subspace $E_T(F)$ is closed. In particular, T has no divisible subspace different from zero.

Example 2.2. *Let T be a normal operator on a complex Hilbert space $\mathcal{H}$. Pták and Vrbová proved in [49] that the ranges of the spectral projections can be represented in the form*

$$\mathcal{H}_T(F) = \mathcal{E}(F)\mathcal{H} = \bigcap_{\lambda \in \mathbb{C}\setminus F} (T-\lambda)\mathcal{H}$$

for all closed sets $F \subseteq \mathbb{C}$, where $\mathcal{E}$ denotes the spectral measure associated with T. Evidently, T has Bishop's property (β), and hence T has property (I).

An important generalization of normal operators to the setting of Banach spaces is the class of generalized scalar operators. Recall that an operator $T \in L(X)$ is said to be *generalized scalar operator* if there exists a continuous algebra homomorphism $\Phi : C^\infty(\mathbb{C}) \to L(X)$ such that

$$\Phi(1) = I \quad \text{and} \quad \Phi(z) = T,$$

where $C^\infty(\mathbb{C})$ denote the Fréchet algebra of all infinitely differentiable complex valued functions on $\mathbb{C}$, and z denotes the identity function on $\mathbb{C}$. The class of generalized scalar operators was introduced by Colojoară and Foiaş [22]. Every linear operator on a finite dimensional space as well as every spectral operator of finite type are generalized scalar operators.

Example 2.3. *Let $T \in L(X)$ be a generalized scalar operator on a complex Banach space X. In [55], Vrbová proved that*

$$X_T(F) = E_T(F) = \bigcap_{\lambda \in \mathbb{C}\setminus F} (T-\lambda)^p X$$

for all sufficiently large integers p *and closed sets* $F \subseteq \mathbb{C}$*. Since every generalized scalar operator is decomposable, and hence* T *has Bishop's property. Vrbová's result shows that*

$$E_T(\phi) = \bigcap_{\lambda \in \mathbb{C}} (T-\lambda)^p X = \{0\},$$

i.e. generalized scalar operators have no divisible subspace different from zero and there exists an integer $p > 0$ *such that the intersection of the ranges* $(T-\lambda)^p X$ *over all* $\lambda \in \mathbb{C}$ *is trivial. Hence* T *has property* (I)*.*

An operator $T \in L(X)$ is said to be a *spectral operator* if there exists a spectral measure $\mathcal{E}$ on $\sigma-$algebra $\mathcal{B}$ which satisfies $\mathcal{E}(B)T = T\mathcal{E}(B)$ and $\sigma(T|\mathcal{E}(B)X) \subseteq \overline{B}$ for all $B \in \mathcal{B}$. Dunford and Schwartz [26] proved that if $T \in L(X)$ is a spectral operator with spectral measure $\mathcal{E}$ on a Banach space X, then $X_T(F) = \mathcal{E}(F)X$ for all closed $F \subseteq \mathbb{C}$. Later Curtis and Neumann (Theorem 3.1, [23]) proved that if $T \in L(X)$ is a spectral operator of type k on a Banach space X, then

$$X_T(F) = E_T(F) = \bigcap_{\lambda \in \mathbb{C}\setminus F} (T-\lambda)^p X$$

for all closed $F \subseteq \mathbb{C}$ and all $p \in \mathbb{N}$ with $p \geq k+2$, and hence spectral operator has property (I).

An operator $T \in L(X)$ is said to be *subscalar*, if T is similar to the restriction of a generalized scalar operator to one of its closed invariant subspaces. It is well known that hyponormal operators, isometries, $k-$quasihyponormal, $\omega-$hyponormal, $log-$hyponormal operators are subscalar, see [15], [37] and [50]. It is well known [41] that if there exist an operator $S \in L(X)$ and a constant $c > 0$ for which

$$\|(S-\overline{\lambda})x\| \leq c\|(T-\lambda)x\| \text{ for all } \lambda \in \mathbb{C} \text{ and } x \in X,$$

then T is subscalar. In particular, if $T \in L(\mathcal{H})$ on a complex Hilbert space $\mathcal{H}$ is $M-$hyponormal, i.e. there exists a constant $M > 0$ such that

$$\|(T^*-\overline{\lambda})x\| \leq M\|(T-\lambda)x\| \text{ for all } x \in \mathcal{H} \text{ and } \lambda \in \mathbb{C},$$

where T^* denotes the Hilbert space adjoint of T, then T is subscalar. The class of subscalar operators is strictly larger than the class of generalized scalar operators. For instance, the unilateral right shift T on the space $\ell^2(\mathbb{N})$ is hyponormal, and hence T is subscalar. But T is not generalized scalar operator, since T^* does not have the SVEP, we have that T is not decomposable. It follows from [40] that T is not generalized scalar.

EXAMPLE 2.4. *Let $T \in L(X)$ be a subscalar on a Banach space X. Then, by Theorem 4, [46], there exists an integer $p \in \mathbb{N}$ such that*

$$X_T(F) = E_T(F) = \bigcap_{\lambda \in \mathbb{C} \setminus F} (T - \lambda)^p X \qquad \text{for all closed sets } F \subseteq \mathbb{C}.$$

It is clear that the restriction of an operator T with property (β) to a closed invariant subspace certainly inherits this property. Hence T has property (I).

Let $(\omega_n)_{n \in \mathbb{N}_0}$ be a bounded sequence of strictly positive real numbers. Let us consider the corresponding weighted unilateral right shift on the sequence space $X := \ell^q(\mathbb{N}_0)$ for some $1 \leq q < \infty$, defined by

$$Tx := \sum_{n=0}^{\infty} \omega_n x_n e_{n+1} \qquad \text{for all } \; x = (x_n)_{n \in \mathbb{N}_0} \in \ell^p(\mathbb{N}_0),$$

where (e_n) stands for the canonical basis of $\ell^p(\mathbb{N}_0)$. It follows from Corollary 7 [46] that if there exist constants $c, s > 0$ such that

$$\frac{1}{cn^s} \leq \inf_{k \geq 0} \omega_k \cdot \omega_{k+1} \cdots \omega_{n+k-1} \leq \sup_{k \geq 0} \omega_k \cdot \omega_{k+1} \cdots \omega_{n+k-1} \leq cn^s$$

for all $n \in \mathbb{N}$, then T is subscalar and for all sufficiently large integers p,

$$X_T(F) = E_T(F) = \bigcap_{\lambda \in \mathbb{C} \setminus F} (T - \lambda)^p X \;\; \text{for all closed } F \subseteq \mathbb{C}.$$

Thus T has property (I).

We recall [40] that an operator $T \in L(X)$ is said to be *super-decomposable,* provided that, for pair of open sets $U, V \subseteq \mathbb{C}$ such that $U \cup V = \mathbb{C}$

there exists some $R \in L(X)$ such that $RT = TR$, $\sigma(T|\overline{R(X)}) \subseteq U$, and $\sigma(T|\overline{(I-R)(X)}) \subseteq V$. Super-decomposable operators are obviously decomposable. Evidently, the class of super-decomposable operators contains all normal operators on Hilbert spaces and more generally, all spectral operators, all generalized scalar operators as well as all operators with totally disconnected spectrum are super-decomposable; for more information we refer to [40], [41].

Example 2.5. *Let $T \in L(X)$ be a super-decomposable operator on a Banach space X, and suppose that*

$$\bigcap_{\lambda \in \mathbb{C}} (T-\lambda)^p X = \{0\}$$

for some integer $p \geq 1$. Then, by Proposition 1.4.15 [41], for every closed set $F \subseteq \mathbb{C}$, we have the identities

$$X_T(F) = E_T(F) = \bigcap_{\lambda \in \mathbb{C}\setminus F} (T-\lambda)^p X.$$

Clearly, T has Bishop's property (β) and hence T has property (I).

Colojoarvă and Foias [22] proved that every multiplication operator on a regular semi-simple commutative Banach algebra is decomposable. Later it was observed by Frunză [32] that the decomposability of all multiplication operators actually characterizes the regularity of a semi-simple commutative Banach algebra $\mathcal{A}$, i.e. for each $a \in \mathcal{A}$, T_a is decomposable if and only if $\mathcal{A}$ is regular.

Example 2.6. *Let $\mathcal{A}$ be a regular semi-simple commutative Banach algebra, and let $T := T_a \in L(\mathcal{A})$ is a multiplication operator on $\mathcal{A}$. Then, by Theorem 1.4 [47], T is decomposable and hence T has Bishop's property (β), and the spectral maximal spaces of T are given by*

$$\mathcal{A}_T(F) = E_T(F) = \bigcap_{\lambda \in \mathbb{C}\setminus F} (T-\lambda)\mathcal{A} \quad \textit{for all closed sets} \quad F \subseteq \mathbb{C}.$$

Hence T has property (I).

EXAMPLE 2.7. *Consider the Banach space $X := C([0,1])$ and the Volterra operator $T \in L(X)$ given by*

$$(Tf)(s) = \int_0^s f(t)dt \qquad \text{for all } f \in C([0,1]) \text{ and } s \in [0,1].$$

Then T is both compact and quasi-nilpotent, and hence decomposable. It is clear that $X_T(\phi) = \{0\}$ and has the following non-trivial divisible subspace

$$E_T(\phi) = \bigcap_{n=1}^{\infty} T^n X = \{f \in C^\infty([0,1]) : f^{(k)}(0) = 0 \ \text{ for all integer } \ k \geq 0\}.$$

It follows that T does not have the property (I).

3. Invariant subspaces of a bounded Linear operators

Given an operator $T \in L(X)$, the *quasi-nilpotent part* of T is the set

$$H_0(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

Obviously, $H_0(T)$ is a linear subspace of X, generally not closed. It is clear that

$$Ker(T^n) \subseteq H_0(T) \subseteq \{x \in X : \sigma_T(x) \subseteq \{0\}\}$$

for all $n \in \mathbb{N}$. Furthermore, T is quasi-nilpotent if and only if $H_0(T) = X$. Moreover, if T is invertible, then $H_0(T) = \{0\}$, see more details [1] and [44]. Given an operator $T \in L(X)$ on a Banach space X and $x \in X$, the quantity

$$r_T(x) := \limsup_{n\to\infty} \|T^n x\|^{\frac{1}{n}}$$

is called the *local spectral radius* of T at x. It is clear that

$$\max\{|\lambda| : \lambda \in \sigma_T(x)\} \leq r_T(x)$$

for all $x \in X$. It follows from Proposition 3.3.13 of [41] that if T has SVEP, then the local spectral radius formula

$$r_T(x) = \max\{|\lambda| : \lambda \in \sigma_T(x)\}$$

holds for all non-zero $x \in X$, but for operators without SVEP, this inequality may well strict.

Theorem 3.1. *Assume that $T \in L(X)$ has property (I) on a Banach space X. Then $\lim_{n\to\infty} \|T^n x_0\|^{\frac{1}{n}} = 0$ if and only if $T^p x_0 = 0$ for some integer $p > 0$. Moreover, $KerT^p = X_T(\{0\})$ is the quasi-nilpotent part of T. In this case,*

$$H_0(T) = KerT^p = E_T(\{0\}) = \{x \in X : r_T(x) = 0\} = \bigcap_{\lambda \in \mathbb{C}\setminus\{0\}} (T-\lambda)^p X,$$

where $r_T(x)$ denotes the local spectral radius of T at x.

Proof. Suppose that $T \in L(X)$ has property (I). Then there exists an integer $p > 0$ such that

$$E_T(F) = X_T(F) = \bigcap_{\mu \in \mathbb{C}\setminus F} (T-\mu)^p X$$

holds for all closed set $F \subseteq \mathbb{C}$. For each $\lambda \in F$, we obtain

$$E_T(\{\lambda\}) = \bigcap_{\mu \neq \lambda} (T-\mu)^p X.$$

Thus we have

$$\begin{aligned}(T-\lambda)^p E_T(\{\lambda\}) &= (T-\lambda)^p [\bigcap_{\mu\neq\lambda} (T-\mu)^p X] \\ &\subseteq \bigcap_{\mu \in \mathbb{C}} (T-\mu)^p X = X_T(\phi) = \{0\},\end{aligned}$$

since T has single-valued extension property. It follows that $(T-\lambda)^p E_T(\{\lambda\}) = \{0\}$, this implies that $E_T(\{\lambda\}) \subseteq Ker(T-\lambda)^p$ for all $\lambda \in F$. On the other hand, by Proposition 1.2.16 of [41]

$$Ker(T-\lambda)^k \subseteq X_T(\{\lambda\}) \subseteq E_T(\{\lambda\})$$

for all $\lambda \in F$ and $k \in \mathbb{N}$ and hence

$$Ker(T-\lambda)^p = X_T(\{\lambda\}) = E_T(\{\lambda\}) \text{ for all } \lambda \in F.$$

Thus we have

$$KerT^p = H_0(T) = X_T(\{0\}) = E_T(\{0\}) = \bigcap_{\lambda \neq 0} (T-\lambda)^p X.$$

Since T has Bishop's property (β), T has SVEP and hence, by Corollary 2.4 of [39],

$$X_T(\{0\}) = \{x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

Finally, it follows from Proposition 3.3.7 of [41] that $r_T(x) = \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}}$. This completes the proof. □

THEOREM 3.2. *Assume that $T \in L(X)$ has property (I) on a Banach space X. If T has finite spectrum, then T is algebraic. In particular, if T is quasi-nilpotent then T is nilpotent.*

Proof. Suppose that $T \in L(X)$ has property (I). Then there exists an integer $m > 0$ such that

$$E_T(F) = X_T(F) = \bigcap_{\mu\in\mathbb{C}\setminus F} (T-\mu)^m X$$

holds for all closed set $F \subseteq \mathbb{C}$. Assume that $\sigma(T)$ is a finite set, say $\sigma(T) = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. Then, by the first part of Theorem 3.1, there exists positive integer $m_k \in \mathbb{N}$ such that

$$E_T(\{\lambda_k\}) = X_T(\{\lambda_k\}) = Ker(T-\lambda_k)^{m_k}$$

for all $k = 1, 2, \cdots, n$. Thus we have

$$X_T(\sigma(T)) = E_T(\sigma(T)) = \bigcap_{\mu\in\mathbb{C}\setminus\sigma(T)} (T-\mu)^m X = X,$$

since for each $\mu \in \mathbb{C}\setminus\sigma(T)$, $T-\mu$ is invertible. It follows from Theorem 1 of [53] that

$$\begin{aligned} X = X_T(\sigma(T)) &= X_T(\{\lambda_1\}) \oplus X_T(\{\lambda_2\}) \oplus \cdots \oplus X_T(\{\lambda_n\}) \\ &= Ker(T-\lambda_1)^{m_1} \oplus Ker(T-\lambda_2)^{m_2} \oplus \cdots \oplus Ker(T-\lambda_n)^{m_n} \end{aligned}$$

holds as an algebraic direct sum. Consequently, if p denotes the complex polynomial given by

$$p(\lambda) := (\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_n)^{m_n} \quad \text{for all } \lambda \in \mathbb{C},$$

then we conclude that $p(T) = 0$, and hence T is algebraic. This completes the proof. □

We shall also need the following elementary lemma which may be known; see [41].

Lemma 3.3. *Every decomposable operator whose spectrum contains at least two points has a non-trivial hyperinvariant closed linear subspace.*

In 1984, C. J. Read proved [52] that there exist quasi-nilpotent operators(and hence decomposable) on Banach spaces without non-trivial closed invariant subspaces.

We can now prove the main result of this section.

Theorem 3.4. *Let $T \in L(X)$ be a bounded linear operator on a Banach space X of dimension greater than 1. If $T \in L(X)$ has both property (I) and decomposition property (δ), then T has a non-trivial invariant closed linear subspace.*

Proof. Suppose that $T \in L(X)$ has both property (I) and decomposition property (δ) on a Banach space X of dimension greater than 1. Then T is decomposable. At first, we show that if $\sigma(T)$ contains at least two points, then T has a non-trivial hyperinvariant closed linear subspace. Since T is decomposable, it follows from Lemma 3.3 that T has a non-trivial hyperinvariant closed linear subspace. It remains to consider the case of operator $T \in L(X)$ such that X is at least two-dimensional and $\sigma(T)$ is a singleton. Then it follows from Theorem 3.2 that $T = \lambda I + N$ for some $\lambda \in \mathbb{C}$ and some nilpotent operator $N \in L(X)$. Let $p \in \mathbb{N}$ be the smallest integer for which $N^p = 0$, and choose an $x \in X$ for which $N^{p-1}x \neq 0$. The linear subspace generated by $N^{p-1}x$ is a one-dimensional $T-$invariant linear subspace of X. This completes the proof. □

The previous result extends [41, Proposition 1.5.11].

Corollary 3.5. *Every subscalar operator with property (δ) on a Banach space of dimension greater than 1 has a non-trivial invariant closed linear subspace.*

The above result applies to all generalized scalar($k-$quasihyponormal, isometries, $M-$hyponormal, $\omega-$hyponormal, *log*-hyponormal or hyponormal) with property (δ).

Corollary 3.6. *Every generalized scalar operator on a Banach space of dimension greater than 1 has a non-trivial invariant closed linear subspace.*

Corollary 3.7. *Every hyponormal operator with property* (δ) *on a Hilbert space of dimension greater than 1 has a non-trivial invariant closed linear subspace.*

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