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ON THE STABILITY OF A GENERAL QUADRATIC FUNCTIONAL EQUATION AND ITS APPLICATIONS

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Abstract. The aim of this paper is to solve the general solution of a quadratic functional equation

$$f(x+2y)+2f(x-y)=f(x-2y)+2f(x+y)$$

in the class of functions between real vector spaces and to obtain the generalized Hyers-Ulam stability problem for the equation.

1. Introduction

In 1940, S.M. Ulam [18] raised a question concerning the stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta>0$ such that if a function $h:G_1\to G_2$ satisfies the inequality $d(h(xy),h(x)h(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $H:G_1\to G_2$ with $d(h(x),H(x))<\epsilon$ for all $x\in G_1$?

In 1978, P.M. Gruber [6] imposed the following more general problem: "Suppose a mathematical object satisfies a certain property approximately. Is it then possible to approximate this objects by objects satisfying the property exactly?" This problem is of particular interest in probability theory and in the case of functional equations of different types. First, Ulam's question for approximately additive mappings was solved by D.H. Hyers [7] and then generalized by Th.M.

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Rassias [16]. During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors [2, 5, 8–10, 12, 17]. A stability problem for the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

was solved by a lot of authors [4, 13, 15]. Recently, Jun and Lee [11] proved the generalized Hyers-Ulam stability problem for a pexiderized quadratic equation of (1.1).

In this paper, we investigate the general solution of the following quadratic functional equation

$$f(x+2y) + 2f(x-y) = f(x-2y) + 2f(x+y) \tag{1.2}$$

in the class of functions between real vector spaces. In addition, we establish the generalized Hyers-Ulam stability problem for the equation (1.2) by the direct method of Hyers, Ulam and Rassias's theory.

2. General solution of Eq.(1.2)

We here present the general solution of (1.2). Let both E_1 and E_2 be real vector spaces throughout this paper.

THEOREM 2.1. *A function $f : E_1 \to E_2$ satisfies the functional equation (1.2) if and only if there exist functions $B : E_1 \times E_1 \to E_2$, $A : E_1 \to E_2$ and a constant c in E_2 such that $f(x) = B(x,x) + A(x) + c$ for all $x \in E_1$, where B is symmetric biadditive, and A is additive.*

Proof. Let $f : E_1 \to E_2$ satisfy the functional equation (1.2). If we put $g(x) = f(x) - f(0)$, we obtain that g is also a solution of (1.2) and $g(0) = 0$. So we may assume without loss of generality that f is a solution of (1.2) and $f(0) = 0$. Let $f_e(x) = \frac{f(x)+f(-x)}{2}$, $f_o(x) = \frac{f(x)-f(-x)}{2}$ for all $x \in E_1$. Then $f_e(0) = 0 = f_o(0)$ and $f(x) = f_e(x) + f_o(x)$, f_e is even and f_o is odd. Since f is a solution of (1.2), f_e and f_o also satisfy the equation (1.2).

Thus, we may assume that f is a solution of the functional equation (1.2) and f is odd, $f(0) = 0$. Putting $x = 0$ and $y = x$ in (1.2) separately, we get

$$f(2y) = 2f(y), \qquad f(3y) = 3f(y)$$

for all $y \in E_1$. Thus the equation (1.2) can be written by

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y), \tag{2.1}$$

$$f(x+2y) + f(-x+2y) = f(2x+2y) - f(2x-2y), \tag{2.2}$$

which yield

$$f(u) + f(v) = f(\frac{3u-v}{2}) - f(\frac{u-3v}{2}) \tag{2.3}$$

for all $u,\ v \in E_1$. Replacing x by $x-y$ in (1.2) and using the oddness of f, one arrives at

$$f(x+y) + 2f(x-2y) = f(x-3y) + 2f(x). \tag{2.4}$$

Setting y by $x-y$, $x+y$ in (2.1), separately, we have two equations

$$f(3x-y) + f(x+y) = 2f(2x-y) + 2f(y), \tag{2.5}$$

$$f(3x+y) + f(x-y) = 2f(2x+y) - 2f(y). \tag{2.6}$$

Using (2.5) and (2.4), one obtains that

$$\begin{aligned} f(3x-y) &+ f(x-3y) \\ &= 2f(2x-y) + 2f(x-2y) + 2f(y) - 2f(x). \end{aligned} \tag{2.7}$$

On the other hand, utilizing the equation (2.3) we lead to

$$f(3x-y) + f(x-3y) = f(4x) - f(4y) = 4f(x) - 4f(y), \tag{2.8}$$

which induces by (2.7)

$$f(2x-y) + f(x-2y) = 3f(x) - 3f(y). \tag{2.9}$$

Putting $-x+y$ instead of y in (2.9), one has by the oddness of f

$$f(3x-y) + f(3x-2y) = 3f(x) + 3f(x-y). \tag{2.10}$$

Replacing y in (2.10) by $-y$ and then adding the resulting relation to (2.10), we have that

$$\begin{aligned} f(3x+y)+f(3x-y) &+ f(3x+2y)+f(3x-2y) \\ &= 6f(x)+3f(x+y)+3f(x-y). \end{aligned} \tag{2.11}$$

In turn, it follows from (2.5), (2.6) and (2.1) that

$$f(3x+y)+f(3x-y)=3f(x+y)+3f(x-y), \tag{2.12}$$

from which we deduce the following relation together with (2.11)

$$f(3x+2y)+f(3x-2y)=6f(x)=2f(3x). \tag{2.13}$$

Here the equation (2.13) is equivalent to $f(X+Y)+f(X-Y)=2f(X)$, which is in fact the Cauchy-Jensen equation. Hence in this case $f(x)=A(x)$ for some additive mapping A.

Now we second assume that f is a solution of the functional equation (1.2) and f is even, $f(0)=0$. Thus the equation (1.2) is written by

$$f(x+2y)+2f(x-y)=f(x-2y)+2f(x+y) \tag{2.14}$$

for all $x,\ y\in E_1$. Putting $y=x$ and $y=\frac{x}{2}$ in (2.14) separately, we get

$$\begin{aligned} f(3x) &= f(x)+2f(2x), \\ f(2x)+2f(\frac{x}{2}) &= 2f(\frac{3x}{2})=2f(\frac{x}{2})+4f(x), \end{aligned}$$

which implies that $f(2x)=4f(x)$, $f(3x)=9f(x)$ for all $x\in E_1$. Interchange x and y in (2.14) to get the relation

$$f(2x+y)+2f(x-y)=f(2x-y)+2f(x+y) \tag{2.15}$$

for all $x,\ y\in E_1$. Setting x by $x+y$ in (2.14), one obtains that

$$f(x+3y)+2f(x)=f(x-y)+2f(x+2y) \tag{2.16}$$

for all $x,\ y\in E_1$. Replacing y by $-y$ in (2.16), we obtain that

$$f(x-3y)+2f(x)=f(x+y)+2f(x-2y) \tag{2.17}$$

for all $x,\ y \in E_1$. Setting y by $x-y$ in (2.14) and then putting y by $\frac{3y}{2}$ in the resulting relation, we have

$$9f(x-y)+\frac{9}{2}f(y)=f(x-3y)+\frac{1}{2}f(4x-3y). \tag{2.18}$$

Adding the equation (2.17) to (2.18) side by side, one leads to

$$\begin{aligned} 9f(x-y) &+ \frac{9}{2}f(y)+2f(x) \\ &= f(x+y)+2f(x-2y)+\frac{1}{2}f(4x-3y) \end{aligned} \tag{2.19}$$

for all $x,\ y \in E_1$. Exchanging x with y in (2.19) and then subtracting the resulting relation from (2.19), we have

$$\begin{aligned} \frac{5}{2}f(y)-\frac{5}{2}f(x) &= 2f(x-2y)-2f(2x-y) \\ &+ \frac{1}{2}f(4x-3y)-\frac{1}{2}f(3x-4y) \end{aligned} \tag{2.20}$$

for all $x,\ y \in E_1$. Replacing x by $4x$ in (2.17), we get

$$f(4x-3y)+32f(x)=f(4x+y)+8f(2x-y) \tag{2.21}$$

for all $x,\ y \in E_1$. Interchanging x and y in (2.21), we have by the evenness of f

$$f(3x-4y)+32f(y)=f(x+4y)+8f(x-2y) \tag{2.22}$$

for all $x,\ y \in E_1$. Subtracting (2.22) from (2.21) and dividing it by 2, we arrive at the equation

$$\begin{aligned} \frac{1}{2}f(4x-3y) &- \frac{1}{2}f(3x-4y)+16f(x)-16f(y) \\ &= \frac{1}{2}f(4x+y)-\frac{1}{2}f(x+4y)+4f(2x-y)-4f(x-2y). \end{aligned} \tag{2.23}$$

Combining (2.20) with (2.23), we easily see that

$$\begin{aligned}\frac{27}{2}f(x) &- \frac{27}{2}f(y) \\ &= \frac{1}{2}f(4x+y) - \frac{1}{2}f(x+4y) + 2f(2x-y) - 2f(x-2y)\end{aligned} \tag{2.24}$$

for all $x,\ y \in E_1$. Putting $x+y$ instead of x in (2.16), we obtain that

$$f(x+4y) + 2f(x+y) = f(x) + 2f(x+3y) \tag{2.25}$$

for all $x,\ y \in E_1$. Replacing y by $2x+y$ in (2.15), one obtains that

$$f(4x+y) + 2f(x+y) = f(y) + 2f(3x+y) \tag{2.26}$$

for all $x,\ y \in E_1$. Subtraction the equation (2.25) from (2.26) to yield the relation

$$\begin{aligned}f(4x+y) &- f(x+4y) \\ &= 2f(3x+y) - 2f(x+3y) + f(y) - f(x)\end{aligned} \tag{2.27}$$

for all $x,\ y \in E_1$. Multiplying the equation (2.27) by $\frac{1}{2}$, and then adding the resulting relation to (2.24), we have

$$\begin{aligned}14f(x) &- 14f(y) \\ &= 2f(2x-y) - 2f(x-2y) + f(3x+y) - f(x+3y)\end{aligned} \tag{2.28}$$

for all $x,\ y \in E_1$.

In turn, interchanging x and y in (2.16) and then subtracting (2.16) from the resulting relation, one obtains that

$$\begin{aligned}f(3x+y) &- f(x+3y) + 2f(y) - 2f(x) \\ &= 2f(2x+y) - 2f(x+2y)\end{aligned} \tag{2.29}$$

for all $x,\ y \in E_1$. Adding the relation (2.29) to (2.28) side by side and dividing it by 2, we arrive at the equation

$$\begin{aligned}6f(x) &- 6f(y) \\ &= f(2x+y) - f(x+2y) + f(2x-y) - f(x-2y).\end{aligned} \tag{2.30}$$

Applying the relations (2.14) and (2.15) to (2.30), we have the following crucial equation

$$3f(x) - 3f(y) = f(2x + y) - f(x + 2y) \tag{2.31}$$

for all $x,\ y \in E_1$.

Now utilizing (2.31) one obtains the following two relations

$$\begin{aligned} f(x+y) - f(x - \frac{y}{2}) &= \frac{1}{3} f\left(3(x + \frac{y}{2})\right) - \frac{1}{3} f(3x), \\ f(x-y) - f(x + \frac{y}{2}) &= \frac{1}{3} f\left(3(x - \frac{y}{2})\right) - \frac{1}{3} f(3x). \end{aligned}$$

Since $f(2x) = 4f(x), f(3x) = 9f(x)$ for all $x \in E_1$, adding the above two relations we get the equation

$$f(x + y) + f(x - y) + 6f(x) = f(2x + y) + f(2x - y), \tag{2.32}$$

which is equivalent to the original quadratic functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ [3]. Therefore $f(x) = Q(x, x)$, where Q is a symmetric biadditive function.

That is, if $f : E_1 \to E_2$ satisfies the functional equation (1.2), then $f(x) = f_e(x) + f_o(x) = B(x, x) + A(x)$ for all $x \in E_1$, where B, A are mappings stated in the theorem. Since we regard $f(x)$ as $f(x) - f(0)$, we get $f(x) = B(x, x) + A(x) + f(0)$ for all $x \in E_1$ and we obtain the desired results.

Conversely, if there exist function $B : E_1 \times E_1 \to E_2$, $A : E_1 \to E_2$ and a constant c in E_2 such that $f(x) = B(x, x) + A(x) + c$ for all $x \in E_1$, where A is additive and B is symmetric biadditive, then it is obvious that f satisfies the equation (1.2). □

3. Generalized Hyers-Ulam stability for (1.2)

We now investigate the Hyers-Ulam stability problem for the equation (1.2). Thus we find the condition that there exists a true solution function near an approximate solution function for (1.2). From now on, let X be a real vector space and let Y be a Banach space unless we give any specific reference. Let $\mathbb{R}^+$ denote the set of all nonnegative real numbers and $\mathbb{N}$ the set of all positive integers.

THEOREM 3.1. *Let $\phi : X^2 \to \mathbb{R}^+$ be a function such that the series*

$$\bar{\Phi}(x,y) := \sum_{i=0}^{\infty} \frac{\phi(2^i x, 2^i y)}{4^i}$$

converges for all $x, y \in X$. Suppose that an even function $f : X \to Y$ satisfies

$$\begin{aligned} \|f(x+2y) + 2f(x-y) &- f(x-2y) - 2f(x+y)\| \\ &\leq \phi(x,y) \end{aligned} \tag{3.1}$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ which satisfies the equation (1.2) and the inequality

$$\|f(x) - f(0) - Q(x)\| \leq \frac{1}{4}\bar{\Phi}(x, \frac{x}{2}) + \frac{1}{2}\bar{\Phi}(\frac{x}{2}, \frac{x}{2}) \tag{3.2}$$

for all $x \in X$. The function Q is given by

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n} \tag{3.3}$$

for all $x \in X$. If further, either f is measurable or for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to Y is continuous, then $Q(rx) = r^2 Q(x)$ for all $r \in \mathbb{R}$.

Proof. If we replace y by x in (3.1), we have

$$\|f(3x) + 2f(0) - 2f(2x) - f(x)\| \leq \phi(x,x) \tag{3.4}$$

for all $x \in X$. Substituting x for $2y$ in (3.1) and then replacing y by x in the resulting inequality, one obtains that

$$\|f(4x) + 2f(x) - f(0) - 2f(3x)\| \leq \phi(2x, x) \tag{3.5}$$

for all $x \in X$. Multiplying the relation (3.4) by 2 and then adding it to (3.5), we have the following relation

$$\|[f(4x) - f(0)] - 4[f(2x) - f(0)]\| \leq \phi(2x,x) + 2\phi(x,x), \tag{3.6}$$

which can be written by

$$\left\|\frac{[f(2x)-f(0)]}{4}-[f(x)-f(0)]\right\|\le\frac{\phi(x,\frac{x}{2})+2\phi(\frac{x}{2},\frac{x}{2})}{4} \tag{3.7}$$

for all $x\in X$.

By induction arguments, it is not difficult to show from (3.7) that

$$\begin{aligned}&\left\|[f(x)-f(0)]-\frac{[f(2^nx)-f(0)]}{4^n}\right\|\\ &\le\sum_{i=0}^{n-1}\frac{\phi(2^ix,2^{i-1}x)+2\phi(2^{i-1}x,2^{i-1}x)}{4^{i+1}}\end{aligned} \tag{3.8}$$

for all $x\in X$. Note that the right hand side of (3.8) is a convergent series by assumption.

In order to prove the convergence of the sequence $\{\frac{[f(2^nx)-f(0)]}{4^n}\}$, we show that the sequence is a Cauchy sequence in Y. By (3.8), we obtain that for $n>m>0$,

$$\begin{aligned}&\left\|\frac{[f(2^n)-f(0)]}{4^n}-\frac{[f(2^mx)-f(0)]}{4^m}\right\|\\ &=\frac{1}{4^m}\left\|\frac{[f(2^{n-m}2^mx)-f(0)]}{4^{n-m}}-[f(2^mx)-f(0)]\right\|\\ &\le\sum_{i=0}^{n-m-1}\frac{\phi(2^i2^mx,2^{i-1}2^mx)+2\phi(2^{i-1}2^mx,2^{i-1}2^mx)}{4^{m+i+1}}\end{aligned} \tag{3.9}$$

Since the series $\sum_{i=0}^{\infty}\frac{\phi(2^ix,2^iy)}{4^i}$ converges for all $x,y\in X$, the right hand side of the inequality (3.9) tends to 0 as m tends to infinity and hence the sequence $\{\frac{[f(2^nx)-f(0)]}{4^n}\}$ is a Cauchy sequence as desired. Therefore, we may define

$$Q(x)=\lim_{n\to\infty}2^{-2n}[f(2^nx)-f(0)]=\lim_{n\to\infty}2^{-2n}f(2^nx)$$

for all $x\in X$. By letting $n\to\infty$ in (3.8), we arrive at the formula (3.2).

To show that Q satisfies the equation (1.2), replace x, y by $2^n x, 2^n y$, respectively, in (3.1) and divide by 4^n, then it follows that

$$\begin{aligned} 4^{-n}\|f(2^n(x+2y)) + 2f(2^n(x-y)) \\ -f(2^n(x-2y)) - 2f(2^n(x+y))\| \leq 4^{-n}\phi(2^n x, 2^n y). \end{aligned}$$

Taking the limit as $n \to \infty$, we find that Q satisfies (1.2) for all $x, y \in X$. Obviously, it follows that Q is even since f is even, and hence Q is a quadratic function by Theorem 2.1.

The proof of the uniqueness of Q with the stated property in the theorem goes through in the similar way as that of [12]. The proof of the last assertion in the theorem follows by the same way as that of [4]. This completes the proof of the theorem. □

In the next part, we investigate the Hyers-Ulam stability problem for the equation (1.2) satisfied by an odd function.

THEOREM 3.2. *Let $\phi : X^2 \to \mathbb{R}^+$ be a function such that the series*

$$\Phi(x, y) := \sum_{i=0}^{\infty} \frac{\phi(2^i x, 2^i y)}{2^i}$$

converges for all $x, y \in X$. Suppose that an odd function $f : X \to Y$ satisfies

$$\begin{aligned} \|f(x+2y) + 2f(x-y) &- f(x-2y) - 2f(x+y)\| \\ &\leq \phi(x, y) \end{aligned} \tag{3.10}$$

for all $x, y \in X$. Then there exists a unique additive function $A : X \to Y$ which satisfies the equation (1.2) and the inequality

$$\|f(x) - A(x)\| \leq \frac{1}{4}\Phi(0, x) \tag{3.11}$$

for all $x \in X$. The function A is given by

$$A(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{3.12}$$

for all $x \in X$. If further, for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to Y is continuous, then $A(rx) = rA(x)$ for all $r \in \mathbb{R}$.

Proof. Substituting 0 for x and then replacing y by x, one obtains that

$$\|2f(2x) - 4f(x)\| \le \phi(0, x), \tag{3.13}$$

which can be written by

$$\|\frac{f(2x)}{2} - f(x)\| \le \frac{\phi(0, x)}{4} \tag{3.14}$$

for all $x \in X$.

By induction, it follows from (3.14) that

$$\left\| f(x) - \frac{f(2^n x)}{2^n} \right\| \le \frac{1}{4} \sum_{i=0}^{n-1} \frac{\phi(0, 2^i x)}{2^i} \tag{3.15}$$

for all $x \in X$. Note that the right hand side of (3.15) is a convergent series by assumption.

Now using the same argument as that of Theorem 3.1, we obtain that the sequence $\{\frac{f(2^n x)}{2^n}\}$ is a Cauchy sequence in Y, and hence there exists a unique function $A : X \to Y$, defined by $A(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$, which satisfies the equation (1.2) and the inequality (3.11). It is clear that A is odd since f is odd, and thus A is additive by Theorem 2.1. This completes the proof. □

Combining Theorem 3.1 and Theorem 3.2, we arrive at the following Hyers-Ulam stability of the equation (1.2).

THEOREM 3.3. *Let $\phi : X^2 \to \mathbb{R}^+$ be a function such that the series*

$$\Phi(x, y) := \sum_{i=0}^{\infty} \frac{\phi(2^i x, 2^i y)}{2^i}$$

converges for all $x, y \in X$. *Suppose that a function* $f : X \to Y$ *satisfies*

$$\begin{aligned} \|f(x+2y)+2f(x-y) &- f(x-2y)-2f(x+y)\| \\ &\le \phi(x,y) \end{aligned} \tag{3.16}$$

for all $x, y \in X$. *Then there exist a unique quadratic function* $Q : X \to Y$, *a unique additive function* $A : X \to Y$ *which satisfy the equation (1.2) and the inequality*

$$\|\frac{f(x)-f(-x)}{2} - A(x)\| \le \frac{1}{8}\Phi(0,x) + \frac{1}{8}\Phi(0,-x), \tag{3.17}$$

$$\begin{aligned} \|\frac{f(x)+f(-x)}{2} - f(0) - Q(x)\| &\le \frac{1}{8}\bar{\Phi}(x,\frac{x}{2}) + \frac{1}{4}\bar{\Phi}(\frac{x}{2},\frac{x}{2}) \\ &\quad + \frac{1}{8}\bar{\Phi}(-x,-\frac{x}{2}) + \frac{1}{4}\bar{\Phi}(-\frac{x}{2},-\frac{x}{2}), \\ \|f(x)-f(0)-A(x)-Q(x)\| &\le \frac{1}{8}\Phi(0,x) + \frac{1}{8}\Phi(0,-x) \\ &\quad + \frac{1}{8}\bar{\Phi}(x,\frac{x}{2}) + \frac{1}{4}\bar{\Phi}(\frac{x}{2},\frac{x}{2}) \\ &\quad + \frac{1}{8}\bar{\Phi}(-x,-\frac{x}{2}) + \frac{1}{4}\bar{\Phi}(-\frac{x}{2},-\frac{x}{2}) \end{aligned}$$

for all $x \in X$. *The functions* Q, A *are given by*

$$\begin{aligned} Q(x) &= \lim_{n\to\infty} \frac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n}, \\ A(x) &= \lim_{n\to\infty} \frac{f(2^n x) - f(-2^n x)}{2^{n+1}} \end{aligned} \tag{3.18}$$

for all $x \in X$. *If further, for each fixed* $x \in X$ *the mapping* $t \mapsto f(tx)$ *from* $\mathbb{R}$ *to* Y *is continuous, then* $Q(rx) = r^2 Q(x)$, $A(rx) = rA(x)$ *for all* $r \in \mathbb{R}$.

Proof. If we put $g(x) = f(x) - f(0)$, we obtain that g also satisfies the inequality (3.16) and $g(0) = 0$. Let $g_e(x) = \frac{g(x)+g(-x)}{2}$, $g_o(x) = \frac{g(x)-g(-x)}{2}$ for all $x \in E_1$. Then $g_e(0) = 0 = g_o(0)$ and $g(x) = g_e(x) +$

$g_o(x)$, g_e is even and g_o is odd. Since g satisfies the inequality (3.16), both g_e and g_o also satisfy the inequalities

$$\|g_e(x+2y)+2g_e(x-y)-g_e(x-2y)-2g_e(x+y)\| \le \frac{\phi(x,y)+\phi(-x,-y)}{2}, \tag{3.19}$$

$$\|g_o(x+2y)+2g_o(x-y)-g_o(x-2y)-2g_o(x+y)\| \le \frac{\phi(x,y)+\phi(-x,-y)}{2} \tag{3.20}$$

for all $x, y \in X$.

In view of Theorem 3.1, we obtain by (3.19) that Q, given by $Q(x) = \lim_{n\to\infty} \frac{f(2^n x)+f(-2^n x)}{2\cdot 4^n}$, satisfies the equation (1.2) and the inequality

$$\begin{aligned} &\|\frac{f(x)+f(-x)}{2} - f(0) - Q(x)\| = \|g_e(x) - Q(x)\| \\ &\le \sum_{i=0}^{\infty}\left[\frac{\phi(2^i x, 2^{i-1}x) + 2\phi(2^{i-1}x, 2^{i-1}x)}{2\cdot 4^{i+1}}\right] \\ &+ \sum_{i=0}^{\infty}\left[\frac{\phi(-2^i x, -2^{i-1}x) + 2\phi(-2^{i-1}x, -2^{i-1}x)}{2\cdot 4^{i+1}}\right]. \end{aligned} \tag{3.21}$$

Similarly, it follows from (3.20) and Theorem 3.2 that A, defined by $A(x) = \lim_{n\to\infty} \frac{f(2^n x)-f(-2^n x)}{2^{n+1}}$, satisfies the equation (1.2) and the inequality

$$\begin{aligned} &\|\frac{f(x)-f(-x)}{2} - A(x)\| = \|g_o(x) - A(x)\| \\ &\le \frac{1}{8}\sum_{i=0}^{\infty}\left[\frac{\phi(0, 2^i x)}{2^i} + \frac{\phi(0, -2^i x)}{2^i}\right]. \end{aligned} \tag{3.22}$$

The rest proof of the theorem follows by the same way as that of Theorem 3.1, Theorem 3.2. □

From the main theorem 3.3, we obtain the following corollary concerning the stability of the equation

$$\begin{aligned} f(x+y+z) &+ f(x-y)+f(x-z)-f(x-y-z) \\ &- f(x+y)-f(x+z)=0. \end{aligned} \tag{3.23}$$

Corollary 3.4. *Let $\varphi : X^3 \to \mathbb{R}^+$ be a function such that the series*

$$\Lambda(x,y,z) := \sum_{i=0}^{\infty} \frac{\phi(2^i x, 2^i y, 2^i z)}{2^i}$$

converges for all $x, y, z \in X$. Suppose that a function $f : X \to Y$ satisfies

$$\begin{aligned} &\|f(x+y+z)+f(x-y)+f(x-z)-f(x-y-z) \\ &-f(x+y)-f(x+z)\| \le \varphi(x,y,z) \end{aligned}$$

for all $x, y, z \in X$. Then there exist a unique quadratic function $Q : X \to Y$, a unique additive function $A : X \to Y$ which satisfy both the equations (1.2), (3.23) and the inequality

$$\begin{aligned} \|\frac{f(x)-f(-x)}{2} - A(x)\| &\le \frac{1}{8}\Lambda(0,x,x)+\frac{1}{8}\Lambda(0,-x,-x), \\ \|\frac{f(x)+f(-x)}{2} - f(0) - Q(x)\| &\le \frac{1}{8}\bar{\Lambda}(x,\frac{x}{2},\frac{x}{2})+\frac{1}{4}\bar{\Lambda}(\frac{x}{2},\frac{x}{2},\frac{x}{2}) \\ +\frac{1}{8}\bar{\Lambda}(-x,-\frac{x}{2},-\frac{x}{2}) &+ \frac{1}{4}\bar{\Lambda}(-\frac{x}{2},-\frac{x}{2},-\frac{x}{2}), \\ \|f(x)-f(0)-A(x)-Q(x)\| &\le \frac{1}{8}\Lambda(0,x,x)+\frac{1}{8}\Lambda(0,-x,-x) \\ &+ \frac{1}{8}\bar{\Lambda}(x,\frac{x}{2},\frac{x}{2})+\frac{1}{4}\bar{\Lambda}(\frac{x}{2},\frac{x}{2},\frac{x}{2}) \\ +\frac{1}{8}\bar{\Lambda}(-x,-\frac{x}{2},-\frac{x}{2}) &+ \frac{1}{4}\bar{\Lambda}(-\frac{x}{2},-\frac{x}{2},-\frac{x}{2}) \end{aligned}$$

for all $x \in X$, where $\bar{\Lambda}(x,y,z) := \sum_{i=0}^{\infty} \frac{\phi(2^i x, 2^i y, 2^i z)}{4^i}$. The functions Q, A are given by (3.18). If further, for each fixed $x \in X$ the mapping $t \mapsto$

$f(tx)$ from $\mathbb{R}$ to Y is continuous, then $Q(rx) = r^2Q(x)$, $A(rx) = rA(x)$ for all $r \in \mathbb{R}$.

Proof. Replacing z by y in the given condition, we obtain that

$$\|f(x+2y) + 2f(x-y) - f(x-2y) - 2f(x+y)\| \le \varphi(x, y, y)$$

for all $x, y \in X$. If we consider $\phi(x, y) = \varphi(x, y, y)$, the conclusion follows from Theorem 3.3. □

Corollary 3.5. *Let X and Y be a real normed space and a Banach space, respectively, and let $\theta, \varepsilon \ge 0$, $p < 1$ be real numbers. Suppose that a function $f : X \to Y$ satisfies*

$$\begin{aligned} \|f(x+2y) &+ 2f(x-y) - f(x-2y) - 2f(x+y)\| \\ &\le \theta + \varepsilon(\|x\|^p + \|y\|^p) \end{aligned} \tag{3.24}$$

for all $x, y \in X$. Then there exist a unique quadratic function $Q : X \to Y$, a unique additive function $A : X \to Y$ which satisfy the equation (1.2) and the inequality

$$\left\|\frac{f(x) - f(-x)}{2} - A(x)\right\| \le \frac{\theta}{2} + \frac{\varepsilon\|x\|^p}{2(2-2^p)}, \tag{3.25}$$

$$\left\|\frac{f(x) + f(-x)}{2} - f(0) - Q(x)\right\| \le \theta + \frac{\varepsilon\|x\|^p}{4-2^p} + \frac{5\varepsilon\|x\|^p}{2^p(4-2^p)},$$

$$\|f(x) - f(0) - A(x) - Q(x)\| \le \frac{3\theta}{2} + \frac{\varepsilon\|x\|^p}{2(2-2^p)} + \frac{\varepsilon\|x\|^p}{4-2^p} + \frac{5\varepsilon\|x\|^p}{2^p(4-2^p)}$$

for all $x \in X$. The functions Q, A are given by (3.18). Moreover, if for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to Y is continuous, then $Q(rx) = r^2Q(x)$, $A(rx) = rA(x)$ for all $r \in \mathbb{R}$.

Proof. Considering $\phi(x, y)$ as $\theta + \varepsilon(\|x\|^p + \|y\|^p)$ in Theorem 3.3, we obtain easily the conclusions. □

By Theorem 3.3, we obtain the following Hyers-Ulam stability of the equation (1.2).

COROLLARY 3.6. *Let X and Y be a real normed space and a Banach space, respectively, and let $\theta \geq 0$ be a real number. Suppose that a function $f : X \to Y$ satisfies*

$$\|f(x+2y)+2f(x-y)-f(x-2y)-2f(x+y)\| \leq \theta \tag{3.26}$$

for all $x, y \in X$. Then there exist a unique quadratic function $Q : X \to Y$, a unique additive function $A : X \to Y$ which satisfy the equation (1.2) and the inequality

$$\begin{aligned} \|\frac{f(x)-f(-x)}{2} - A(x)\| &\leq \frac{\theta}{2}, \\ \|\frac{f(x)+f(-x)}{2} - f(0) - Q(x)\| &\leq \theta, \\ \|f(x)-f(0)-A(x)-Q(x)\| &\leq \frac{3\theta}{2} \end{aligned} \tag{3.27}$$

for all $x \in X$. Furthermore, if for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to Y is continuous, then $Q(rx) = r^2Q(x)$, $A(rx) = rA(x)$ for all $r \in \mathbb{R}$.

In the last part of this paper, let B be a unital Banach algebra with norm $|\cdot|$, and let ${}_B\mathbb{B}_1$ and ${}_B\mathbb{B}_2$ be left Banach B-modules with norms $\|\cdot\|$ and $||\cdot||$, respectively. A quadratic function $Q : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ is called B-quadratic if

$$Q(ax) = a^2Q(x), \quad \forall a \in B, \forall x \in {}_B\mathbb{B}_1.$$

THEOREM 3.7. *Suppose that a function $f : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfies*

$$||f(\alpha x+2\alpha y)+2f(\alpha x-\alpha y)-\alpha^2 f(x-2y)-2\alpha^2 f(x+y)|| \leq \phi(x,y)$$

for all $\alpha \in B$ ($|\alpha| = 1$), and for all $x, y \in {}_B\mathbb{B}_1$, where ϕ is given as in Theorem 3.3. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, then there exists a unique B-quadratic function $Q : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$, defined by (3.18), which satisfies the equation (1.2) and the inequality (3.17) with $A = 0$.

Proof. By Theorem 3.3, it follows from the inequality of the statement for $\alpha = 1$ that there exist a unique quadratic function $Q : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ and a unique additive function $A : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$, which satisfy the equation (1.2) and the inequality (3.17). Under the assumption that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, the quadratic function $Q : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfies

$$Q(tx) = t^2 Q(x), \quad \forall x \in {}_B\mathbb{B}_1, \forall t \in \mathbb{R}.$$

That is, Q is $\mathbb{R}$-quadratic. For each fixed $\alpha \in B$ ($|\alpha| = 1$), putting $y = 0$ in the given inequality of the statement we have

$$Q(\alpha x) = \alpha^2 Q(x), \quad A(\alpha x) = \alpha^2 A(x)$$

for all $x \in {}_B\mathbb{B}_1$. The last relation is also true for $\alpha = 0$. Since A is odd, we obtain from the last equation that for $\alpha = -1$, $-A(x) = A(-x) = A(x)$ and hence $A = 0$ identically. Since Q is $\mathbb{R}$-quadratic and $Q(\alpha x) = \alpha^2 Q(x)$ for each element $\alpha \in B(|\alpha| = 1)$, for each element $a \in B$ ($a \neq 0$), $a = |a| \cdot \frac{a}{|a|}$ and thus

$$\begin{aligned} Q(ax) = Q(|a| \cdot \frac{a}{|a|} x) &= |a|^2 \cdot Q(\frac{a}{|a|} x) = |a|^2 \cdot \frac{a^2}{|a|^2} \cdot Q(x) \\ &= a^2 Q(x), \quad \forall a \in B(a \neq 0), \forall x \in {}_B\mathbb{B}_1. \end{aligned}$$

So the unique $\mathbb{R}$-quadratic function $Q : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ is also B-quadratic as desired. This completes the proof of the theorem. □

THEOREM 3.8. *Suppose that a function $f : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfies*

$$||f(\alpha x + 2\alpha y) + 2f(\alpha x - \alpha y) - \alpha f(x - 2y) - 2\alpha f(x + y)|| \leq \phi(x, y)$$

for all $\alpha \in B$ ($|\alpha| = 1$), and for all $x, y \in {}_B\mathbb{B}_1$, where ϕ is given as in Theorem 3.3. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, then there exists a unique B-linear function $A : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$, defined by (3.18), which satisfies the equation (1.2) and the inequality (3.17) with $Q = 0$.

Proof. The proof of the theorem follows by the same way as that of Theorem 3.7. □

Since $\mathbb{C}$ is a Banach algebra, the Banach spaces E_1 and E_2 are considered as Banach modules over $\mathbb{C}$. Thus we have the following corollary.

COROLLARY 3.9. *Suppose that a function $f : E_1 \to E_2$ satisfies*

$$||f(\alpha x + 2\alpha y) + 2f(\alpha x - \alpha y) - \alpha^2 f(x - 2y) - 2\alpha^2 f(x + y)|| \leq \theta$$

for all $\alpha \in \mathbb{C}$ ($|\alpha| = 1$), and for all $x, y \in E_1$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$, then there exists a unique $\mathbb{C}$-quadratic function $Q : E_1 \to E_2$, defined by (3.18), which satisfies the equation (1.2) and the inequality

$$\|f(x) - f(0) - Q(x)\| \leq \frac{3\theta}{2}$$

for all $x \in E_1$.

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