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SOME PROPERTIES INVOLVING THE HIGHER ORDER q-GENOCCHI NUMBERS AND POLYNOMIALS WITH WEIGHT (α, β) VIA THE p-ADIC q-INTEGRAL ON $\mathbb{Z}_p$

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Abstract. The main properties of this paper is to describe the higher order q-Genocchi polynomials with weight (α, β). However, we derive some interesting properties concerning this type of polynomials.

1. Introduction, definitions and notations

The main motivations of this paper are the papers [8], [14] by *Kim et al. and Hwang et al.,* in which they introduced and studied on higher order q-Euler numbers and polynomials with weight α and higher order q-Bernoulli numbers and polynomials with weight α, respectively.

By using q-Volkenborn integral, *Kim* introduced the q-Bernoulli numbers and polynomials with weight α which are derived some interesting properties of q-Bernoulli numbers and polynomials with weight α. However, several mathematicians have studied on the special numbers and poynomials with weight α(see for details [3],[4], [8], [9], [10], [11], [14],[17], [22]).

Assume that p is a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$, will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let v_p be the normalized exponential

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valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = \frac{1}{p}$ (see [1-6,8-17]). If $q \in \mathbb{C}$, then we always assume that $|q| < 1$. If $q \in \mathbb{C}_p$, then we usually assume that $|1-q|_p < p^{-\frac{1}{p-1}}$, which yields $q^x = \exp(\log q)$ for $|x|_p \leq 1$.

The q-basic naturel numbers are defined by $[x]_q = \frac{q^x-1}{q-1}$ $(x \in \mathbb{N})$. Hence, $\lim_{q\to 1}[x]_q = x$ for any x with $|x|_p \leq 1$ in the present p-adic case. The q-factorial is defined as $[n]_q! = [n]_q \cdot [n-1]_q \cdot [n-2]_q \cdot \cdots \cdot [2]_q \cdot [1]_q$, and the Gaussian binomial coefficients is defined by $\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}$. Note that $\lim_{q\to 1}\binom{n}{k}_q = \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$ (see [5,8-16]).

In [5,10,11,16], the q-binomial formulas are known,

$$
\begin{aligned}
(b;q)_n &= (1-b)(1-qb)\cdots\left(1-q^{n-1}b\right) \\
&= \sum_{k=0}^{n}\binom{n}{k}_q q^{\binom{k}{2}}(-1)^k b^k,
\end{aligned} \tag{1.1}
$$

$$
\begin{aligned}
(b;q)_n^{-1} &= \frac{1}{(1-b)(1-qb)\cdots(1-q^{n-1}b)} \\
&= \sum_{k=0}^{\infty}\binom{n+k-1}{k}_q b^k.
\end{aligned} \tag{1.2}
$$

We say that f (and write $f \in UD(\mathbb{Z}_p)$) is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, if the quotients $F_f(x,y) = \frac{f(x)-f(y)}{x-y}$ have a limit $f'(a)$ as $(x,y) \to (a,a)$. For $f \in UD(\mathbb{Z}_p)$, *T. Kim defined the fermionic p-adic q-integral* on $\mathbb{Z}_p$ (or q-Volkenborn integral in the sense of fermionic) is defined as

$$
\begin{aligned}
I_{-q}(f) &= \int_{\mathbb{Z}_p} f(x)\,d\mu_{-q}(x) \\
&= \frac{[2]_q}{2}\lim_{N\to\infty}\sum_{x=0}^{p^N-1}(-1)^x f(x)\,q^x, \text{ (see [1-5,8-17])}.
\end{aligned} \tag{1.3}
$$

The Genocchi polynomials are defined by

$$
\frac{2te^{xt}}{e^t+1} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \quad |t| < \pi \text{ (see [6,7,10,11])}. \tag{1.4}
$$

The q-extension of Genocchi numbers are defined by

$$
G_{0,q} = 0, \; q(qG_q+1)^n + G_{n,q} = \begin{cases} [2]_q, & n = 1 \\ 0, & n \neq 1 \end{cases} \tag{1.5}
$$

with usual the convention about replacing $(G_q)^n$ by $G_{n,q}$ (see [1]).

The (h,q)-extension of Genocchi numbers are defined by

$$G_{0,q}^{(h)}=0,\ q^{h-2}\left(qG_q^{(h)}+1\right)+G_{n,q}^{(h)}=\begin{cases}[2]_q, & n=1\\ 0, & n\neq 1\end{cases} \tag{1.6}$$

with usual convention about replacing $\left(G_q^{(h)}\right)^n$ by $G_{n,q}^{(h)}$ (see[2]).

Recently, For $\alpha\in\mathbb{N}$, the weighted q-Genocchi numbers are defined by

$$\widetilde{G}_{0,q}^{(\alpha)}=0,\quad q^{1-\alpha}\left(q^{\alpha}\widetilde{G}_q^{(\alpha)}+1\right)^n+\widetilde{G}_{n,q}^{(\alpha)}=\begin{cases}[2]_q, & \text{if } n=0\\ 0, & \text{if } n\neq 0,\end{cases} \tag{1.7}$$

with usual convention abut replacing $\left(\widetilde{G}_q^{(\alpha)}\right)^n$ by $\widetilde{G}_{n,q}^{(\alpha)}$ (see[4]).

By same the motivation of [8], [14], we consider on the higher order q-Genocchi polynomials with weight (α,β). However, we derive some interesting properties concerning this type of polynomials.

2. Higher Order q-Genocchi numbers and polyomials with weight (α,β)

DEFINITION 1. *Let be $h\in\mathbb{Z}$, $\alpha,k,\beta\in\mathbb{N}$ and $n\in\mathbb{Z}_+$. Then we defined higher-order q-Genocchi polynomials with weight (α,β) as follows:*

$$\frac{\widetilde{G}_{n+k,q}^{(\alpha,\beta)}(h,k\mid x)}{k!\binom{n+k}{k}} = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_k]_{q^{\alpha}}^n\, q^{\sum_{j=1}^{k}x_j(h-j)}\, d\mu_{-q^{\beta}}(x_1)\cdots d\mu_{-q^{\beta}}(x_k) \tag{2.1}$$

By (1.3) and (2.1), we arrive at the following theorem:

THEOREM 1. *For $h\in\mathbb{Z}$, $\alpha,k,\beta\in\mathbb{N}$ and $n\in\mathbb{Z}_+$. We obtain*

$$\frac{\widetilde{G}_{n+k,q}^{(\alpha,\beta)}(h,k\mid x)}{k!\binom{n+k}{k}}=\frac{[2]_{q^{\beta}}^k}{(1-q^{\alpha})^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{q^{\alpha l x}}{(-q^{\alpha l+h-k+\beta};q)_k} \tag{2.2}$$

We simply get,

$$\begin{aligned}
& [x_1+x_2+\cdots+x_k]_{q^\alpha}^n \, q^{x_1(h-1)+\cdots+x_k(h-k)} \\
= & [x_1+x_2+\cdots+x_k]_{q^\alpha}^n \, q^{x_1(h-\alpha-1)+\cdots+x_k(h-\alpha-k)} q^{\alpha(x_1+x_2+\cdots+x_k)} \\
= & [x_1+x_2+\cdots+x_k]_{q^\alpha}^n \, q^{x_1(h-\alpha-1)+\cdots+x_k(h-\alpha-k)}((q^\alpha-1)\,\frac{q^{\alpha(x_1+x_2+\cdots+x_k)}-1}{q^\alpha-1}+1) \\
= & [x_1+x_2+\cdots+x_k]_{q^\alpha}^n \, q^{x_1(h-\alpha-1)+\cdots+x_k(h-\alpha-k)}((q^\alpha-1)\,[x_1+x_2+\cdots+x_k]_{q^\alpha}+1)
\end{aligned}$$

Namely, we have Corollary 1 as follows:

Corollary 1. *For $\alpha, k \in \mathbb{N}$. Then we get*

$$\begin{aligned}
(2.3) \qquad & [x_1+x_2+\cdots+x_k]_{q^\alpha}^n \, q^{x_1(h-1)+\cdots+x_k(h-k)} \\
= & \left(\begin{array}{c} (q^\alpha-1)\,[x_1+x_2+\cdots+x_k]_{q^\alpha}^{n+1} \, q^{x_1(h-\alpha-1)+\cdots+x_k(h-\alpha-k)} \\ +\,[x_1+x_2+\cdots+x_k]_{q^\alpha}^n \, q^{x_1(h-\alpha-1)+\cdots+x_k(h-\alpha-k)} \end{array} \right)
\end{aligned}$$

With regard to (2.3) and by using fermionic p-adic q-integral, we obtain the following theorem:

Theorem 2. *For $h \in \mathbb{Z}$, $\alpha, k, \beta \in \mathbb{N}$ and $n \in \mathbb{Z}_+$. We obtain*

$$\begin{aligned}
(2.4) \qquad & \widetilde{G}_{n+k,q}^{(\alpha,\beta)}(h,k) \\
& = (q-1)\,[\alpha]_q \, \frac{\binom{n+k}{k}}{\binom{n+k+1}{k}} \widetilde{G}_{n+k+1,q}^{(\alpha,\beta)}(h-\alpha,k) + \widetilde{G}_{n+k}^{(\alpha,\beta)}(h-\alpha,k)
\end{aligned}$$

By (2.1), we have,

$$\begin{aligned}
& \frac{\widetilde{G}_{k,q}^{(\alpha,\beta)}(m\alpha,k+1)}{k!} \\
& = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{j=1}^{k+1}(m\alpha-j)x_j} d\mu_{-q^\beta}(x_1)\cdots d\mu_{-q^\beta}(x_{k+1}) \\
& = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(\frac{q^{\alpha(x_1+\cdots+x_{k+1})}-1}{q^\alpha-1}(q^\alpha-1)+1 \right)^m q^{-\sum_{j=1}^{k+1} jx_j} \\
& \qquad d\mu_{-q^\beta}(x_1)\cdots d\mu_{-q^\beta}(x_{k+1}) \\
& = \sum_{l=0}^m \binom{m}{l} (q^\alpha-1)^l \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1+\cdots+x_{k+1}]_{q^\alpha}^l \, q^{-\sum_{j=1}^{k+1} jx_j} \\
& \qquad d\mu_{-q^\beta}(x_1)\cdots d\mu_{-q^\beta}(x_{k+1}) \\
& = \sum_{l=0}^m \binom{m}{l} (q^\alpha-1)^l \, \frac{\widetilde{G}_{l+k}^{(\alpha,\beta)}(0,k+1)}{k!\binom{l+k}{k}}
\end{aligned}$$

Therefore, we obtain the following theorem:

THEOREM 3. *For $h\in\mathbb{Z}$, $\alpha,k,\beta\in\mathbb{N}$ and $m\in\mathbb{Z}_+$. We obtain*

$$\widetilde{G}_{k,q}^{(\alpha,\beta)}(m\alpha,k+1)=\sum_{l=0}^{m}\frac{\binom{m}{l}}{\binom{l+k}{k}}(q^{\alpha}-1)^{l}\,\widetilde{G}_{l+k}^{(\alpha,\beta)}(0,k+1) \tag{2.5}$$

Moreover, we readily see,

$$\widetilde{G}_{k,q}^{(\alpha,\beta)}(m\alpha,k+1)=\frac{k!\,[2]_{q^{\beta}}^{k+1}}{(-q^{m\alpha-k+\beta};q)_{k}} \tag{2.6}$$

By (2.5) and (2.6), we get following theorem:

THEOREM 4. *For $h\in\mathbb{Z}$, $\alpha,k,\beta\in\mathbb{N}$ and $m\in\mathbb{Z}_+$. We obtain*

$$\sum_{l=0}^{m}\frac{\binom{m}{l}}{\binom{l+k}{k}}(q^{\alpha}-1)^{l}\,\widetilde{G}_{l+k}^{(\alpha,\beta)}(0,k+1)=\frac{k!\,[2]_{q^{\beta}}^{k+1}}{(-q^{m\alpha-k+\beta};q)_{k}} \tag{2.7}$$

From (2.1), we can derive the following equation:

$$\begin{aligned}
&\sum_{j=0}^{i}\binom{i}{j}(q^{\alpha}-1)^{j}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_k]_{q^{\alpha}}^{n-i+j}\,q^{\sum_{l=1}^{k}(h-\alpha-l)x_l}\\
&\qquad\qquad d\mu_{-q^{\beta}}(x_1)\cdots d\mu_{-q^{\beta}}(x_k)\\
=&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_k]_{q^{\alpha}}^{n-i}\,q^{\sum_{l=1}^{k}(h-l)x_l}q^{\alpha(x_1+\cdots+x_k)(i-1)}\\
&\qquad\qquad d\mu_{-q^{\beta}}(x_1)\cdots d\mu_{-q^{\beta}}(x_k)\\
=&\sum_{j=0}^{i-1}\binom{i-1}{j}(q^{\alpha}-1)^{j}\,\frac{\widetilde{G}_{n-i+j+k}^{(\alpha,\beta)}(h,k)}{k!\binom{n-i+j+k}{k}}.
\end{aligned}$$

Thus, we obtain the following theorem:

THEOREM 5. *For $h\in\mathbb{Z}$, $\alpha,k,\beta\in\mathbb{N}$ and $n,i\in\mathbb{Z}_+$. We obtain*

$$\begin{aligned}
&\sum_{j=0}^{i}\frac{\binom{i}{j}}{\binom{n-i+j+k}{k}}(q^{\alpha}-1)^{j}\,\widetilde{G}_{n-i+j+k}^{(\alpha,\beta)}(h-\alpha,k)\\
&=\sum_{j=0}^{i-1}\frac{\binom{i-1}{j}}{\binom{n-i+j+k}{k}}(q^{\alpha}-1)^{j}\,\widetilde{G}_{n-i+j+k}^{(\alpha,\beta)}(h,k).
\end{aligned} \tag{2.8}$$

3. Polynomials $\widetilde{G}_{n,q}^{(\alpha,\beta)}(0,k \mid x)$

In this section, we discuss the polynomials $\widetilde{G}_{n,q}^{(\alpha,\beta)}(0,k \mid x)$ by

$$\begin{aligned}\frac{\widetilde{G}_{n+k,q}^{(\alpha,\beta)}(0,k \mid x)}{k!\binom{n+k}{k}} &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_k]_{q^\alpha}^n q^{-\sum_{j=1}^k jx_j} \\ &\qquad d\mu_{-q^\beta}(x_1)\cdots d\mu_{-q^\beta}(x_k).\end{aligned} \tag{3.1}$$

From (3.1), we get

$$\frac{\widetilde{G}_{n+k,q}^{(\alpha,\beta)}(0,k \mid x)}{k!\binom{n+k}{k}} = \frac{[2]_{q^\beta}^k}{(1-q^\alpha)^n}\sum_{l=0}^n\binom{n}{l}(-1)^l\frac{q^{lx}}{(-q^{\alpha l-k+\beta};q)_k} \tag{3.2}$$

By (3.1), we can easily derive the equation.

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{j=1}^k(\alpha n-j)x_j+\alpha nx}d\mu_{-q^\beta}(x_1)\cdots d\mu_{-q^\beta}(x_k) \\ &\qquad = \sum_{l=0}^n(q^\alpha-1)^l\frac{\binom{n}{l}}{k!\binom{l+k}{k}}\widetilde{G}_{l+k}^{(\alpha,\beta)}(0,k \mid x)\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{j=1}^k(\alpha n-j)x_j+\alpha nx}d\mu_{-q^\beta}(x_1)\cdots d\mu_{-q^\beta}(x_k) \\ &\qquad = \frac{[2]_{q^\beta}^k q^{\alpha nx}}{(-q^{\alpha n-k+\beta};q)_k}\end{aligned} \tag{3.4}$$

Therefore, we get the following theorem.

Theorem 6. *For $\alpha, k, \beta \in \mathbb{N}$ and $n \in \mathbb{Z}_+$. We get*

$$\frac{\widetilde{G}_{n+k,q}^{(\alpha,\beta)}(0,k \mid x)}{k!\binom{n+k}{k}} = \frac{[2]_{q^\beta}^k}{(1-q^\alpha)^n}\sum_{l=0}^n\binom{n}{l}(-1)^l\frac{q^{\alpha lx}}{(-q^{\alpha l-k+\beta};q)_k}$$

and

$$\sum_{l=0}^n(q^\alpha-1)^l\frac{\binom{n}{l}}{k!\binom{l+k}{k}}\widetilde{G}_{l+k,q}^{(\alpha,\beta)}(0,k \mid x) = \frac{[2]_{q^\beta}^k q^{\alpha nx}}{(-q^{\alpha n-k+\beta};q)_k}.$$

Assume that $d\in\mathbb{N}$ with $d\equiv 1\,(\mathrm{mod}\,2)$. We readily see, namely,

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[x+\sum_{j=1}^{k}x_j\right]_{q^{\alpha}}^{n}q^{-\sum_{j=1}^{k}jx_j}d\mu_{-q^{\beta}}(x_1)\cdots d\mu_{-q^{\beta}}(x_k)$$

$$=\frac{[d]_{q^{\alpha}}^{n}}{[d]_{-q^{\beta}}}\sum_{a_1,\cdots,a_k=0}^{d-1}q^{-\sum_{j=2}^{k}(j-1)a_j}(-1)^{\sum_{j=1}^{k}a_j}\times$$

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[\frac{x+\sum_{j=1}^{k}a_j}{d}+\sum_{j=1}^{k}x_j\right]_{q^{\alpha d}}^{n}q^{-d\sum_{j=1}^{k}jx_j}$$

$$d\mu_{\left(-q^{\beta}\right)^{d}}(x_1)\cdots d\mu_{\left(-q^{\beta}\right)^{d}}(x_k)$$

As well as, we obtain the following theorem.

Theorem 7. *For $\alpha,k,\beta\in\mathbb{N}$, $n\in\mathbb{Z}_{+}$ and $d\equiv 1\,(\mathrm{mod}\,2)$. We get*

$$\widetilde{G}_{n+k,q}^{(\alpha,\beta)}(0,k\mid x)$$

$$=\frac{[d]_{q^{\alpha}}^{n}}{[d]_{-q^{\beta}}}\sum_{a_1,\cdots,a_k=0}^{d-1}q^{-\sum_{j=2}^{k}(j-1)a_j}(-1)^{\sum_{j=1}^{k}a_j}$$

$$\times\widetilde{G}_{n+k,q^{d}}^{(\alpha,\beta)}\left(0,k\mid\frac{x+a_1+a_2+\cdots+a_k}{d}\right)$$

From (3.1), we have

$$\widetilde{G}_{n+k,q}^{(\alpha,\beta)}(0,k\mid x)=\sum_{l=0}^{n}\binom{n+k}{l+k}q^{\alpha lx}[x]_{q^{\alpha}}^{n-l}\,\widetilde{G}_{l+k,q}^{(\alpha,\beta)}(0,k).$$

Thus, we note that

$$\widetilde{G}_{n+k,q}^{(\alpha,\beta)}(0,k\mid x+y)=\sum_{l=0}^{n}\binom{n+k}{l+k}q^{\alpha ly}[y]_{q^{\alpha}}^{n-l}\,\widetilde{G}_{l+k,q}^{(\alpha,\beta)}(0,k\mid x).$$

4. Polynomials $\widetilde{G}_{n,q}^{(\alpha,\beta)}(h,1\mid x)$

We now consider polynomials $\widetilde{G}_{n,q}^{(h)}(h,1\mid x)$ as follows:

$$\frac{\widetilde{G}_{n+1,q}^{(\alpha,\beta)}(h,1\mid x)}{n+1}=\int_{\mathbb{Z}_p}[x+x_1]_{q^{\alpha}}^{n}q^{(h-1)x_1}d\mu_{-q^{\beta}}(x_1).\tag{4.1}$$

By (4.1), we get

$$\frac{\widetilde{G}_{n+1,q}^{(\alpha,\beta)}(h,1\mid x)}{n+1}=\frac{[2]_{q^{\beta}}}{(1-q^{\alpha})^{n}}\sum_{l=0}^{n}\binom{n}{l}(-1)^{l}\frac{q^{\alpha lx}}{1+q^{\alpha l+h-1+\beta}} \tag{4.2}$$

By applying *fermionic p-adic q-integral* on $\mathbb{Z}_p$, we see that

$$\begin{aligned} & q^{\alpha x}\int_{\mathbb{Z}_p}[x+x_1]_{q^{\alpha}}^{n}\,d\mu_{-q^{\beta}}(x_1) \\ = \ & (q-1)[\alpha]_q\int_{\mathbb{Z}_p}[x+x_1]_{q^{\alpha}}^{n+1}\,q^{x_1(h-\alpha-1)}d\mu_{-q^{\beta}}(x_1) \\ & +\int_{\mathbb{Z}_p}[x+x_1]_{q^{\alpha}}^{n}\,q^{x_1(h-\alpha-1)}d\mu_{-q^{\beta}}(x_1) \end{aligned}$$

Therefore, we obtain the following theorem:

THEOREM 8. *For $h\in\mathbb{Z}$, $\alpha,\beta\in\mathbb{N}$ and $n\in\mathbb{Z}_{+}$. We get*

$$\begin{aligned} & q^{\alpha x}\widetilde{G}_{n+1,q}^{(\alpha,\beta)}(h,1\mid x) \\ & =\frac{n+1}{n+2}(q-1)[\alpha]_q\,\widetilde{G}_{n+2,q}^{(\alpha,\beta)}(h-\alpha,1\mid x)+\widetilde{G}_{n+1,q}^{(\alpha,\beta)}(h-\alpha,1\mid x) \end{aligned} \tag{4.3}$$

It is simple to indicate

$$\begin{aligned} \widetilde{G}_{n+1,q}^{(\alpha,\beta)}(h,1\mid x) &= (n+1)\int_{\mathbb{Z}_p}[x+x_1]_{q^{\alpha}}^{n}\,d\mu_{-q^{\beta}}(x_1) \\ &= q^{-\alpha x}\sum_{l=0}^{n}\binom{n+1}{l+1}[x]_{q^{\alpha}}^{n+1-(l+1)}\,q^{\alpha(l+1)x}\widetilde{G}_{l+1,q}^{(\alpha,\beta)}(h,1) \\ &= q^{-\alpha x}\left(q^{\alpha x}\widetilde{G}_{q}^{(\alpha,\beta)}(h,1)+[x]_{q^{\alpha}}\right)^{n+1},\ \text{for } n\geq 0, \end{aligned} \tag{4.4}$$

with usual the convention about replacing $\left(\widetilde{G}_{q}^{(\alpha,\beta)}(h,1)\right)^{n}$ by $\widetilde{G}_{n,q}^{(\alpha,\beta)}(h,1)$.

By $qI_{-q}(f_1)+I_{-q}(f)=[2]_q f(0)$ where $f_1(x)=f(x+1)$. We get,

$$\begin{aligned} & q^{\beta+h-1}\int_{\mathbb{Z}_p}[x+x_1+1]_{q^{\alpha}}^{n}\,q^{(h-1)x_1}d\mu_{-q^{\beta}}(x_1) \\ & +\int_{\mathbb{Z}_p}[x+x_1]_{q^{\alpha}}^{n}\,q^{(h-1)x_1}d\mu_{-q^{\beta}}(x_1) \\ & =[2]_{q^{\beta}}[x]_{q^{\alpha}}^{n}. \end{aligned} \tag{4.5}$$

From (4.1) and (4.5), we simply see,

$$\frac{q^{\beta+h-1}}{n+1}\widetilde{G}_{n+1}^{(\alpha,\beta)}(h,1\mid x)+\frac{1}{n+1}\widetilde{G}_{n+1}^{(\alpha,\beta)}(h,1\mid x)=[2]_{q^{\beta}}[x]_{q^{\alpha}}^{n}. \tag{4.6}$$

For $x=0$ into (4.6), we obtain

$$\frac{q^{\beta+h-1}}{n+1}\widetilde{G}_{n+1}^{(\alpha,\beta)}(h,1\mid x)+\frac{1}{n+1}\widetilde{G}_{n+1}^{(\alpha,\beta)}(h,1\mid x)=\begin{cases}[2]_{q^{\beta}}, & \text{if } n=0\\ 0, & \text{if } n\neq 0.\end{cases}$$

From the *fermionic p-adic q-integral* on $\mathbb{Z}_p$, we simply get,

$$\widetilde{G}_{1,q}^{(\alpha,\beta)}(h,1\mid x)=\int_{\mathbb{Z}_p}q^{x_1(h-1)}d\mu_{-q^{\beta}}(x_1)=\frac{[2]_{q^{\beta}}}{[2]_{q^{h+\beta-1}}}.$$

By (4.1), we see that,

$$\begin{aligned}\frac{\widetilde{G}_{n+1,q^{-1}}^{(\alpha,\beta)}(h,1\mid 1-x)}{n+1} &= \int_{\mathbb{Z}_p}[1-x+x_1]_{q^{-\alpha}}^{n}\,q^{-(h-1)x_1}d\mu_{-q^{-\beta}}(x_1)\\ &= (-1)^n q^{n\alpha+h-1}\left(\frac{[2]_{q^{\beta}}}{(1-q^{\alpha})^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{q^{\alpha lx}}{1+q^{\alpha l+h-1+\beta}}\right)\\ &= (-1)^n q^{n\alpha+h-1}\frac{\widetilde{G}_{n+1,q}^{(\alpha,\beta)}(h,1\mid x)}{n+1}\end{aligned}$$

Therefore, we obtain the following theorem,

THEOREM 9. *For $h\in\mathbb{Z}$, $\alpha,\beta\in\mathbb{N}$ and $n\in\mathbb{Z}_+$. We get*

$$\widetilde{G}_{n+1,q^{-1}}^{(\alpha,\beta)}(h,1\mid 1-x)=(-1)^n q^{n\alpha+h-1}\widetilde{G}_{n+1,q}^{(\alpha,\beta)}(h,1\mid x)$$

Assume that $d\in\mathbb{N}$ with $d\equiv 1 \pmod 2$. Then we have

$$\begin{aligned}(4.7)\quad &\int_{\mathbb{Z}_p}[dx+x_1]_{q^{\alpha}}^{n}\,q^{x_1(h-1)}d\mu_{-q^{\beta}}(x_1)\\ &= \frac{[d]_{q^{\alpha}}^{n}}{[d]_{-q^{\beta}}}\sum_{a=0}^{d-1}q^{ha}(-1)^a\int_{\mathbb{Z}_p}\left[x+\frac{a}{d}+x_1\right]_{q^{\alpha d}}^{n}q^{d(h-1)x_1}d\mu_{(-q^{\beta})^d}(x_1)\end{aligned}$$

Thus, from (4.7) we obtain the following theorem,

THEOREM 10. *For $h\in\mathbb{Z}$, $d\equiv 1 \pmod 2$, $\alpha,\beta\in\mathbb{N}$ and $n\in\mathbb{Z}_+$. We get*

$$\widetilde{G}_{n+1,q}^{(\alpha,\beta)}(h,1\mid dx)=\frac{[d]_{q^{\alpha}}^{n}}{[d]_{-q^{\beta}}}\sum_{a=0}^{d-1}q^{ha}(-1)^a\,\widetilde{G}_{n+1,q^{d}}^{(\alpha,\beta)}\left(h,1\mid x+\frac{a}{d}\right).$$

5. Polynomials $\widetilde{G}_{n,q}^{(\alpha,\beta)}(k,k \mid x)$

In (2.1), we know that

$$\frac{\widetilde{G}_{n+k,q}^{(\alpha,\beta)}(h,k \mid x)}{k!\binom{n+k}{k}} = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_k]_{q^\alpha}^n q^{\sum_{j=1}^k (h-j)x_j} d\mu_{-q^\beta}(x_1)\cdots d\mu_{-q^\beta}(x_k) \tag{5.1}$$

As well as we get,

$$\frac{\widetilde{G}_{n+k,q}^{(\alpha,\beta)}(h,k \mid x)}{k!\binom{n+k}{k}} = [2]_{q^\beta}^k\left(\frac{1}{1-q^\alpha}\right)^n\sum_{l=0}^n\binom{n}{l}(-1)^l\frac{q^{lx}}{(-q^{\alpha l+h-k+\beta};q)_k} \tag{5.2}$$

and

$$q^{h+\beta-1}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+1+x_1+\cdots+x_k]_{q^\alpha}^n \times q^{\sum_{j=1}^k (h-j)x_j} d\mu_{-q^\beta}(x_1)\cdots d\mu_{-q^\beta}(x_k) \tag{5.3}$$
$$= \left(\begin{array}{c} -\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_k]_{q^\alpha}^n q^{\sum_{j=1}^k (h-j)x_j} d\mu_{-q^\beta}(x_1)\cdots d\mu_{-q^\beta}(x_k) \\ +\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_k]_{q^\alpha}^n q^{\sum_{j=2}^k (h-j)x_j} d\mu_{-q^\beta}(x_2)\cdots d\mu_{-q^\beta}(x_k)\end{array}\right).$$

From (5.1) and (5.3), we obtain the following theorem.

THEOREM 11. *For $h \in \mathbb{Z}$, $\alpha, k, \beta \in \mathbb{N}$ and $n \in \mathbb{Z}_+$. We get*

$$q^{h+\beta-1}\widetilde{G}_{n+k}^{(\alpha,\beta)}(h,k \mid x+1)+\widetilde{G}_{n+k}^{(\alpha,\beta)}(h,k \mid x) = (n+k)\,\widetilde{G}_{n+k-1}^{(\alpha,\beta)}(h-1,k-1 \mid x)$$

Assume that $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. Then we obtain

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[x+\sum_{j=1}^k x_j\right]_{q^{\alpha d}}^n q^{\sum_{j=1}^k (h-j)x_j} d\mu_{-q^\beta}(x_1)\cdots d\mu_{-q^\beta}(x_k)$$
$$= \frac{[d]_{q^\alpha}^n}{[d]_{-q^\beta}^n}\sum_{a_1,\cdots,a_k=0}^{d-1} q^{h\sum_{j=1}^k a_j-\sum_{j=2}^k (j-1)a_j}(-1)^{\sum_{j=1}^k a_j} \times$$

$$= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[\frac{x+\sum_{j=1}^{k}a_j}{d}+\sum_{j=1}^{k}x_j\right]_{q^{\alpha d}}^{n} q^{d\sum_{j=1}^{k}(h-j)x_j} d\mu_{\left(-q^{\beta}\right)^d}(x_1)\cdots d\mu_{\left(-q^{\beta}\right)^d}(x_k)$$

Thus, we obtain the following theorem.

Theorem 12. *For $h\in\mathbb{Z}$, $\alpha,k,\beta\in\mathbb{N}$, $n\in\mathbb{Z}_+$ with $d\equiv 1\,(\mathrm{mod}\,2)$. We get*

$$\begin{aligned}&\widetilde{G}_{n+k,q}^{(\alpha,\beta)}(h,k\mid x)\\&=\frac{[d]_{q^{\alpha}}^{n}}{[d]_{-q^{\beta}}^{n}}\sum_{a_1,\cdots,a_k=0}^{d-1} q^{h\sum_{j=1}^{k}a_j-\sum_{j=2}^{k}(j-1)a_j}(-1)^{\sum_{j=1}^{k}a_j}\\&\qquad\times\widetilde{G}_{n+k,q^d}^{(\alpha,\beta)}\left(h,k\mid\frac{x+\sum_{j=1}^{k}a_j}{d}\right)\end{aligned}$$

Assume that $\widetilde{G}_{n,q}^{(\alpha,\beta)}(k,k\mid x)=\widetilde{G}_{n,q}^{(\alpha,\beta)}(k\mid x)$. Then we obtain

$$\frac{\widetilde{G}_{n+k,q}^{(\alpha,\beta)}(k\mid x)}{k!\binom{n+k}{k}}=\frac{[2]_{q^{\beta}}^{k}}{(1-q^{\alpha})^{n}}\sum_{l=0}^{n}\binom{n}{l}(-1)^{l}\frac{q^{lx}}{(-q^{\alpha l+\beta};q)_k} \tag{5.4}$$

and

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[k-x+x_1+x_2+\cdots+x_k]_{q^{-\alpha}}^{n}\times\\&q^{-((k-1)x_1+\cdots+(k-k)x_k)}d\mu_{-q^{-\beta}}(x_1)\cdots d\mu_{-q^{-\beta}}(x_k)\\&=(-1)^{n}q^{n\alpha+\binom{k}{2}}\frac{[2]_{q^{\beta}}^{k}}{(1-q^{\alpha})^{n}}\sum_{l=0}^{n}\binom{n}{l}(-1)^{l}\frac{q^{\alpha lx}}{(-q^{\alpha l+\beta};q)_k}\\&=(-1)^{n}q^{n\alpha+\binom{k}{2}}\frac{\widetilde{G}_{n+k,q}^{(\alpha,\beta)}(k\mid x)}{k!\binom{n+k}{k}}\end{aligned} \tag{5.5}$$

By (5.5), we get the following theorem.

Theorem 13. *For $h\in\mathbb{Z}$, $\alpha,k,\beta\in\mathbb{N}$ and $n\in\mathbb{Z}_+$. We get*

$$\widetilde{G}_{n+k,q^{-1}}^{(\alpha,\beta)}(k-x\mid x)=(-1)^{n}q^{n\alpha+\binom{k}{2}}\widetilde{G}_{n+k,q}^{(\alpha,\beta)}(k\mid x)$$

By simple calculation, we get

$$\sum_{l=0}^{n}(q-1)^{l}[\alpha]_{q}^{l}\binom{n}{l}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+\cdots+x_k]_{q^{\alpha}}^{n}q^{\sum_{l=1}^{k}(k-l)x_l}\,d\mu_{-q^{-\beta}}(x_1)\cdots d\mu_{-q^{-\beta}}(x_k)$$

$$=\frac{[2]_{q^{\beta}}^{k}}{(-q^{\alpha l+\beta};q)_k}$$

and

$$\frac{\widetilde{G}_{n+k,q}^{(\alpha,\beta)}(k\mid x)}{k!\binom{n+k}{k}}$$
$$=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_k]_{q^{\alpha}}^{n}q^{(k-1)x_1+\cdots+(k-k)x_k}\,d\mu_{-q^{\beta}}(x_1)\cdots d\mu_{-q^{\beta}}(x_k)$$
$$=\sum_{l=0}^{n}\binom{n}{l}q^{\alpha lx}[x]_{q^{\alpha}}^{n-l}\frac{\widetilde{G}_{l+k,q}^{(\alpha,\beta)}(k\mid 0)}{k!\binom{l+k}{k}}.$$

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