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LYAPUNOV FUNCTIONS FOR NONLINEAR DIFFERENCE EQUATIONS

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Abstract. In this paper we study h-stability of the solutions of nonlinear difference system via the notion of n_∞-summable similarity between its variational systems. Also, we show that two concepts of h-stability and h-stability in variation for nonlinear difference systems are equivalent. Furthermore, we characterize h-stability for nonlinear difference systems by using Lyapunov functions.

1. Introduction

Pinto [17] introduced the notion of h-stability for differential equations with the intention of obtaining results about for weakly stable differential systems under some perturbations. Also, Medina and Pinto [15] applied h-stability to obtain a uniform treatment for the various stability notions in difference systems and extended the study of exponential stability to a variety of reasonable systems called h-systems.

Choi et al. [2] investigated h-stability for the nonlinear differential systems using the notions of Lyapunov functions and t_∞-similarity introduced by Conti [10]. Trench [18] introduced summable similarity as a discrete analog of Conti's definition of t_∞-similarity and investigated the various stabilities of linear difference systems by using summable similarity. Choi and Koo [3] studied the variational stability for nonlinear difference systems by means of n_∞-similarity. Also, see [4, 5, 6] for the asymptotic property and h-stability of difference systems via discrete similarities and comparison principle.

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In this paper we characterize h-stability of the solutions of the nonlinear difference systems by using the notions of n_∞-summable similarity between variational systems and Lyapunov functions.

2. Main results

Let $\mathbb{N}(n_0) = \{n_0, n_0+1, \cdots, n_0+k, \cdots\}$, where n_0 is a nonnegative integer and $\mathbb{R}^s$ the s-dimensional real Euclidean space. We consider the nonlinear difference system

$$\Delta x(n) = f(n, x(n)),\ x(n_0) = x_0, \tag{2.1}$$

where $f : \mathbb{N}(n_0) \times \mathbb{R}^s \to \mathbb{R}^s$, $f(n, 0) = 0$, and Δ is the forward difference operator with unit spacing; i.e., $\Delta x(n) = x(n+1) - x(n)$. We assume that $f_x = \frac{\partial f}{\partial x}$ exists and is continuous on $\mathbb{N}(n_0) \times \mathbb{R}^s$. Let $x(n) = x(n, n_0, x_0)$ be the unique solution of (2.1) satisfying the initial condition $x(n_0, n_0, x_0) = x_0$. Also, we consider its associated variational systems

$$\Delta v(n) = f_x(n, 0)v(n) \tag{2.2}$$

and

$$\Delta z(n) = f_x(n, x(n, n_0, x_0))z(n), \tag{2.3}$$

where $I + f_x(n, x(n))$ is invertible on $\mathbb{N}(n_0) \times \mathbb{R}^s$ and I denotes the identity matrix.

To establish our main results we will use the following lemma:

Lemma 2.1. [16, Lemma 2.1] *Assume that $x(n, n_0, x_0)$ and $x(n, n_0, y_0)$ are the solutions of system (2.1) through (n_0, x_0) and (n_0, y_0), respectively, which exist for $n \geq n_0$ and such that x_0 and y_0 belong to a convex subset D of $\mathbb{R}^s$. Then for $n \geq n_0$,*

$$x(n, n_0, x_0) - x(n, n_0, y_0) = \int_0^1 \Phi(n, n_0, x_0 + \tau(y_0 - x_0))d\tau \cdot (y_0 - x_0).$$

To prove the discrete version of variation of parameters formula, we need the following result on differentiability of solutions with respect to initial values.

Lemma 2.2. *Assume that $f : \mathbb{N}(n_0) \times \mathbb{R}^s \to \mathbb{R}^s$ possesses partial derivatives on $\mathbb{N}(n_0) \times \mathbb{R}^s$ and $f_x(t, x(t, t_0, x_0))$ is regressive on $\mathbb{N}(n_0)$. Let $x(n) = x(n, n_0, x_0)$ be the solution of (2.1), which exists for $n \geq n_0$, and let*

$$H(n, n_0, x_0) = \frac{\partial f(n, x(n, n_0, x_0))}{\partial x}. \tag{2.4}$$

Then

$$\Phi(n, n_0, x_0) = \frac{\partial x(n, n_0, x_0)}{\partial x_0} \tag{2.5}$$

exists and is the solution of

$$\Delta\Phi(n, n_0, x_0) = H(n, n_0, x_0)\Phi(n, n_0, x_0), \ n \geq n_0, \tag{2.6}$$
$$\Phi(n_0, n_0, x_0) = I. \tag{2.7}$$

Proof. By differentiating (2.1) with respect to x_0 we have

$$\begin{aligned} \frac{\partial \Delta x(n, n_0, x_0)}{\partial x_0} &= \frac{\partial x(n+1, n_0, x_0)}{\partial x_0} - \frac{\partial x(n, n_0, x_0)}{\partial x_0} \\ &= \frac{\partial f(n, x(n))}{\partial x(n)} \frac{\partial x(n, n_0, x_0)}{\partial x_0}. \end{aligned}$$

It follows from the definition of Φ that Φ satisfies (2.6). This completes the proof. □

It follows from Lemma 2.2 that the fundamental matrix solution $\Phi(n, n_0, 0)$ of (2.2) is given by

$$\Phi(n, n_0, 0) = \frac{\partial x(n, n_0, 0)}{\partial x_0}$$

and the fundamental matrix solution $\Phi(n, n_0, x_0)$ of (2.3) is given by

$$\Phi(n, n_0, x_0) = \frac{\partial x(n, n_0, x_0)}{\partial x_0}$$

or equivalently

$$x(n, n_0, x_0) = \left[\int_0^1 \Phi(n, n_0, sx_0) ds\right] x_0. \tag{2.8}$$

Consider the quasilinear difference system

$$\Delta y(n) = A(n)y(n) + f(n, y(n)), \ y(n_0) = y_0, \tag{2.9}$$

where $I + A(n)$ is an $s \times s$ nonsingular matrix and $f : \mathbb{N}(n_0) \times \mathbb{R}^s \to \mathbb{R}^s$.

We obtain the following result which is a slight modification of variation of constants formula in [11, Theorem 4.6.1].

THEOREM 2.3. *The solution $y(n, n_0, y_0)$ of (2.9) satisfies the equation*

$$y(n) = \Phi(n, n_0)y_0 + \sum_{j=n_0}^{n-1} \Phi(n, j+1) f(j, y(j)), \ n \geq n_0, \tag{2.10}$$

where $\Phi(n, n_0)$ satisfies the matrix equation

$$\Delta\Phi(n, n_0) = A(n)\Phi(n, n_0). \tag{2.11}$$

Proof. Let $y(n, n_0, y_0) = \Phi(n, n_0)c(n)$ with $c(n_0) = y_0$. Substituting in (2.9), we obtain

$$\begin{aligned}\Delta y(n) &= \Phi(n+1, n_0)c(n+1) - \Phi(n, n_0)c(n)\\ &= A(n)\Phi(n, n_0)c(n) + f(n, y(n))\\ &= \Delta\Phi(n, n_0)c(n) + f(n, y(n))\\ &= \Phi(n+1, n_0)c(n) - \Phi(n, n_0)c(n) + f(n, y(n)),\end{aligned}$$

since $\Phi(n, n_0)$ is the solution of (2.11). Therefore

$$\Delta c(n) = \Phi^{-1}(n+1, n_0)f(n, y(n)),$$

and

$$c(n) = y_0 + \sum_{j=n_0}^{n-1} \Phi^{-1}(j+1, n_0)f(j, y(j)).$$

It follows that

$$y(n) = \Phi(n, n_0)y_0 + \sum_{j=n_0}^{n-1} \Phi(n, j+1)f(j, y(j)),\ n \geq n_0,$$

since $\Phi(n, n_0)\Phi^{-1}(j+1, n_0) = \Phi(n, j+1)$. This completes the proof. □

The symbol $|\cdot|$ will be used to denote any convenient vector norm in $\mathbb{R}^s$. Let $V : N(n_0) \times \mathbb{R}^s \to \mathbb{R}_+$ be a function with $V(n, 0) = 0$ for all $n \geq n_0$. We denote the total difference of the function V along the solutions x of (2.1) by

$$\begin{aligned}\Delta V_{(2.1)}(n, x) &= V(n+1, x(n+1, n, x))\\ &- V(n, x(n, n, x)).\end{aligned}$$

Conti [10] defined two $m \times m$ matrix functions A and B on $\mathbb{R}_+$ to be t_∞*-similar* if there is an $m \times m$ matrix function S defined on $\mathbb{R}_+$ such that $S'(t)$ is continuous, $S(t)$ and $S^{-1}(t)$ are bounded on $\mathbb{R}_+$, and

$$\int_0^\infty |S' + SB - AS|dt < \infty.$$

Now, we introduce the notion of n_∞-summable similarity which is the corresponding t_∞-similarity for the discrete case.

Let $\mathfrak{M}$ denote the set of all $s \times s$ invertible matrices defined on $\mathbb{N}(n_0)$ and $\mathfrak{S}$ be the subset of $\mathfrak{M}$ consisting of those nonsingular bounded matrices $S(n)$ such that $S^{-1}(n)$ is also bounded.

Definition 2.4. [6, Definition 2.5] A matrix function $A \in \mathfrak{M}$ is n_∞-*summably similar* to a matrix function $B \in \mathfrak{M}$ if there exists an $s \times s$ matrix $F(n)$ absolutely summable over $\mathbb{N}(n_0)$, that is,

$$\sum_{l=n_0}^{\infty} |F(l)| < \infty$$

such that

$$\Delta S(n) + S(n+1)B(n) - A(n)S(n) = F(n) \tag{2.12}$$

for some $S \in \mathfrak{S}$.

For the example of n_∞-summable similarity, see [6].

Remark 2.5. We can easily show that the n_∞-summable similarity is an equivalence relation by the similar manner of Trench in [18]. Also, if A and B are n_∞-summably similar with $F(n) = 0$, then we say that they are *kinematically similar*.

We recall some notions of h-stability for nonlinear difference systems in [13, 15] that are needed in the sequel.

Definition 2.6. System (2.1) is called an *h-system* if there exist a positive function $h : \mathbb{N}(n_0) \to \mathbb{R}$ and a constant $c \geq 1$ such that

$$|x(n, n_0, x_0)| \leq c|x_0|h(n)h(n_0)^{-1}, \ \ n \geq n_0$$

for $|x_0|$ small enough (here $h(n)^{-1} = \frac{1}{h(n)}$).

Moreover, system (2.1) is said to be

(hS) *h-stable* if h is a bounded function in the definition of h-system,

(GhS) *globally h-stable* if system (2.1) is hS for every $x_0 \in D$, where $D \subset \mathbb{R}^s$ is a region which includes the origin,

(hSV) *h-stable in variation* if system (2.3) is hS,

(GhSV) *globally h-stable in variation* if system (2.3) is GhS.

The various notions about h-stability given by Definition 2.6 include several types of known stability properties such as uniform stability, uniform Lipschitz stability and exponential asymptotic stability. See [3, 12, 14, 15] for stability of nonlinear difference systems.

The next diagram illustrates the possible known implications among various types of stability notions for nonlinear difference systems [15].

$$\begin{tikzcd}
G\mathrm{h}SV \arrow[r] \arrow[d] & \mathrm{h}SV \arrow[r, hook] \arrow[d] & EASV \arrow[r] \arrow[d] & ULSV \arrow[d] & & \\
G\mathrm{h}S \arrow[r] & \mathrm{h}S \arrow[r, hook] & EAS \arrow[r] & ULS \arrow[r] & US \arrow[r] & S
\end{tikzcd}$$

Here "$\hookrightarrow$" means an inclusion.

For the linear difference systems, Medina and Pinto [15] showed that

$$\mathrm{G}h\mathrm{SV} \iff \mathrm{G}h\mathrm{S} \iff h\mathrm{S} \iff h\mathrm{SV}.$$

Also, the associated variational system inherits the property of hS from the original nonlinear system. That is, the zero solution $v = 0$ of (2.2) is hS when the zero solution $x = 0$ of (2.1) is hS in [15, Theorem 2].

Our purpose is to characterize the global stability in variation via n_∞-summable similarity and Lyapunov functions. To do this, we need the following lemmas.

LEMMA 2.7. [15] *The linear difference system*

$$\Delta y(n) = A(n)y(n),\ y(n_0) = y_0, \tag{2.13}$$

where $A(n)$ is an $s \times s$ matrix, is an h-system if and only if there exist a constant $c \geq 1$ and a positive function h defined on $\mathbb{N}(n_0)$ such that for every $x_0 \in \mathbb{R}^s$,

$$|\Phi(n, n_0, x_0)| \leq ch(n)h(n_0)^{-1},\ n \geq n_0,$$

where Φ is a fundamental matrix solution of (2.13).

LEMMA 2.8. [9, Corollary 3.10] *If two matrix functions A and B in the set $\mathfrak{M}$ are n_∞- summably similar, then for $n \geq n_0$, we have*

$$\Phi_B(n, n_0) = S^{-1}(n)[\Phi_A(n, n_0)S(n_0) + \sum_{s=n_0}^{n-1} \Phi_A(n, s+1)F(s)\Phi_B(s, n_0)],$$

where $\Phi_A(n, n_0)$ and $\Phi_B(n, n_0)$ are fundamental matrix solutions of the system (2.13) with the coefficient matrix functions $A(n)$ and $B(n)$, respectively.

Medina and Pinto showed that hSV implies hS [15, Theorem 3]. Also, they proved the converse when the condition

$$\sum_{l=n_0}^{\infty} \frac{h(l)}{h(l+1)} |f_x(l, n_0, x_0) - f_x(l, 0)| < \infty,\ n_0 \geq 0 \tag{2.14}$$

for $|x_0| \leq \delta$, holds [15, Theorem 14].

In order to establish our main results, we will introduce the following condition:

(H) $f_x(n, 0)$ and $f_x(n, x(n, n_0, x_0))$ are n_∞-summably similar for $n \geq n_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$ and $\sum_{n=n_0}^{\infty} \frac{h(n)}{h(n+1)}|F(n)| < \infty$ with the positive function $h(n)$ defined on $\mathbb{N}(n_0)$.

In the following theorem, we can show that hS implies hSV by assuming (H), instead of the above condition (2.14).

LEMMA 2.9. [8, Theorem 3.4] *Assume that condition (H) is satisfied. Then the variational system (2.2) is also an h-system if and only if the variational system (2.3) is an h-system.*

Proof. Suppose that $v = 0$ of (2.2) is an h-system. It follows from Lemma 2.7 that there exist a constant $c_1 \geq 1$ and a positive function h defined on $\mathbb{N}(n_0)$ such that for every $x_0 \in \mathbb{R}^s$,

$$|\Phi(n, n_0, 0)| \leq c_1 h(n) h(n_0)^{-1},\ n \geq n_0 \geq 0, \tag{2.15}$$

where $\Phi(n, n_0, 0)$ is a fundamental matrix solution of (2.2).

Since $f_x(n, 0)$ and $f_x(n, x(n, n_0, x_0))$ are n_∞-summably similar, from Lemma 2.8, we have

$$\begin{aligned} \Phi(n, n_0, x_0) &= S^{-1}(n)[\Phi(n, n_0, 0)S(n_0) \\ &+ \sum_{s=n_0}^{n-1} \Phi(n, s+1, 0)F(s)\Phi(s, n_0, x_0)],\ n \geq n_0, \end{aligned}$$

where $\Phi(n, n_0, x_0)$ denotes a fundamental matrix solution of (2.3).

In view of (2.15) and the boundedness of $S(n)$ and $S^{-1}(n)$, there is a positive constant c_2 such that

$$\begin{aligned} |\Phi(n, n_0, x_0)| &\leq c_1 c_2 h(n) h(n_0)^{-1} \\ &+ c_1 c_2 \sum_{l=n_0}^{n-1} \frac{h(n)}{h(l+1)} |F(l)||\Phi(l, n_0, x_0)|. \end{aligned}$$

It follows that

$$\frac{|\Phi(n, n_0, x_0)|}{h(n)} \leq \frac{c_1 c_2}{h(n_0)} + c_1 c_2 \sum_{l=n_0}^{n-1} \frac{h(l)}{h(l+1)} |F(l)| \frac{|\Phi(l, n_0, x_0)|}{h(l)}.$$

Applying the discrete Bellman's inequality [1], we have

$$\begin{aligned} |\Phi(n, n_0, x_0)| &\leq d h(n) h(n_0)^{-1} \prod_{l=n_0}^{n-1} \left(1 + \frac{h(l)}{h(l+1)} |F(l)|\right) \\ &\leq d h(n) h(n_0)^{-1} \exp\left(\sum_{l=n_0}^{n-1} \frac{h(l)}{h(l+1)} |F(l)|\right) \\ &\leq c h(n) h(n_0)^{-1}, \end{aligned}$$

where $c = d \exp\left(\sum_{l=n_0}^{\infty} \frac{h(l)}{h(l+1)} |F(l)|\right)$ and $d = c_1 c_2$.

Therefore we have

$$|\Phi(n, n_0, x_0)| \le ch(n)h^{-1}(n_0), \quad n \ge n_0 \ge 0,$$

for some positive constant $c \ge 1$. Hence system (2.3) is an h-system by Lemma 2.7.

The converse holds by the similar method. This completes the proof. □

For nonlinear difference system (2.1), we can show that

$$\mathrm{G}h\mathrm{SV} \Leftrightarrow \mathrm{G}h\mathrm{S}, \;\; h\mathrm{S} \Leftrightarrow h\mathrm{SV}$$

by using the concept of n_∞-summable similarity.

We study the relation between h-stability of the zero solution of system (2.1) and the zero solutions of its variational systems (2.2) and (2.3) by assuming condition (H) is satisfied.

Theorem 2.10. [15, Theorem 2] *Suppose that condition (H) is satisfied. If $x = 0$ of (2.1) is h-stable, then $v = 0$ of (2.2) is h-stable.*

Note that the converse of Theorem 2.10 does not hold in general. We give the following example.

Example 2.11. [6, Example 5.2] *We consider the nonlinear difference equation*

$$\Delta x(n) = f(n, x(n)) = -\frac{1}{2}x(n) + x^2(n), \; x(n_0) = x_0 = 1 \qquad (2.16)$$

and its variational difference equation

$$\Delta v(n) = f_x(n, 0)v(n) = -\frac{1}{2}v(n), \; v(n_0) = v_0 \ne 0, \qquad (2.17)$$

where $f_x(n, x) = -\frac{1}{2} + 2x$. Then $v = 0$ of (2.17) is h-stable, but $x = 0$ of (2.16) is not h-stable.

Proof. Since the fundamental solution is $\phi(n) = (\frac{1}{2})^{n-n_0}v_0$ for each $n \ge n_0$, (2.17) is h-stable with a positive bounded function $h(n) = \frac{1}{2^n}$. But (2.16) is not h-stable because there exists a unbounded solution $x(n, 0, 1)$ of (2.16) satisfying

$$x(n, 0, 1) = x(n) > n, \; n = 1, 2, \cdots .$$

□

We obtain the following result from (2.8).

Theorem 2.12. *If $z = 0$ of (2.3) is h-stable, then $x = 0$ of (2.1) is h-stable.*

We can obtain the following result by using Lemma 2.9 and Theorem 2.10.

THEOREM 2.13. *Assume that $x = 0$ of (2.1) is h-stable. If condition (H) holds, then $z = 0$ of (2.1) is h-stable in variation.*

REMARK 2.14. For the nonlinear difference system (2.1), we show that two concepts of h-stability and h-stability in variation are equivalent under the condition that two variational systems (2.2) and (2.3) are n_∞-summably similar.

Choi et al. investigated Massera type converse theorems for nonlinear difference system $x(n+1) = f(n, x(n))$ via n_∞-similarity in [3, Theorem 5] and [4, Theorem 2.1]. Furthermore, they characterized h-stability in variation for nonlinear difference system by using the notion of n_∞-summable similarity in [6].

We obtain the following result that characterize h-stability for nonlinear difference system (2.1) via the notions of Lyapunov functions and n_∞-summable similarity.

THEOREM 2.15. *Assume that condition (H) is satisfied. Then (2.1) is GhS if and only if there exists a function $V(n,x)$ defined on $\mathbb{N}(n_0) \times \mathbb{R}^s$ such that the following properties hold:*

(i) *$V(n,x)$ is defined on $\mathbb{N}(n_0) \times \mathbb{R}^s$ and continuous with respect to the second argument;*
(ii) *$|x| \leq V(n,x)| \leq c|x|$ for $(n,x) \in \mathbb{N}(n_0) \times \mathbb{R}^s$;*
(iii) *$|V(n,x_1) - V(n,x_2)| \leq c|x_1 - x_2|$ for $n \in \mathbb{N}(n_0)$ and $x_1, x_2 \in \mathbb{R}^s$;*
(iv) *$\Delta V(n,x) \leq \frac{\Delta h(n)}{h(n)} V(n,x)$ for $(n,x) \in \mathbb{N}(n_0) \times \mathbb{R}^s$.*

Proof. Necessity: Suppose that (2.1) is GhS. Then (2.1) is GhSV by Theorem 2.13; i.e., there exist a constant $c \geq 1$ and a positive bounded function h defined on $\mathbb{N}(n_0)$ such that

$$|\Phi(n, n_0, x_0)| \leq c h(n) h(n_0)^{-1}, \ n \geq n_0,$$

where Φ is a fundamental matrix solution of (2.3). Define the function $V : \mathbb{N}(n_0) \times \mathbb{R}^s \to \mathbb{R}_+$ by

$$V(n,x) = \sup_{\tau \in \mathbb{N}(0)} |x(n+\tau, n, x)| h(n+\tau)^{-1} h(n).$$

Then, the rest of proof can be proved in a similar manner as that of Theorem 2.1 of [4], so we omit the detail. $\square$

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