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INTEGRAL DOMAINS WHICH ARE t-LOCALLY NOETHERIAN

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Abstract. In this note, a module-theoretic characterization of t-locally Noetherian domains is given. We also give some characterizations of strong Mori domains via t-locally Noetherian domains.

1. Introduction

Strong Mori domains were introduced by Wang and McCasland in [11]. It is well known that every strong Mori domain is t-locally Noetherian ([12]). However, as Example 2.1 shows, the converse is not true in general. The aim of this note is to give a module-theoretic characterization of t-locally Noetherian domains and some characterizations of strong Mori domains via t-locally Noetherian domains. In order to do so, we first review some notions and terminologies.

Throughout, let R be an integral domain with quotient field K. Let $\mathbf{F}(R)$ be the set of nonzero fractional ideals of R. For an $I \in \mathbf{F}(R)$, define $I^{-1} = \{x \in K \mid xI \subseteq R\}$. The *$v$-operation* on R is a mapping on $\mathbf{F}(R)$ defined by $I \mapsto I_v = (I^{-1})^{-1}$. The *$t$-operation* on R is defined by $I \mapsto I_t = \bigcup\{J_v \mid J$ is a nonzero finitely generated fractional subideal of $I\}$. Recall that an ideal J of R is called a *Glaz-Vasconcelos ideal* (GV-ideal) if J is finitely generated and $J^{-1} = R$. We denote the set of GV-ideals by $GV(R)$. The *w-operation* on R is a mapping on $\mathbf{F}(R)$ defined by $I \mapsto I_w = \{x \in K \mid Jx \subseteq I$ for some $J \in GV(R)\}$. An $I \in \mathbf{F}(R)$ is said to be a *v-ideal* (resp., *t-ideal*, *w-ideal*) if $I_v = I$ (resp.,

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$I_t = I$, $I_w = I$). A t-ideal (resp., w-ideal) M of R is called a *maximal t-ideal* (resp., *maximal w-ideal*) if M is maximal among proper integral t-ideals (resp., w-ideals) of R. Let t-Max(R) (resp., w-Max(R)) be the set of maximal t-ideals (resp., maximal w-ideals) of R. It was shown that $t\text{-Max}(R) = w\text{-Max}(R)$ and the notion of t-invertibility is precisely same as that of w-invertibility. An integral domain R is a *strong Mori domain* (SM domain) if it satisfies the ascending chain condition (ACC) on integral w-ideals of R ([12]).

Let M be a module over an integral domain R. Set $\mathfrak{r}(M) := \{x \in M \mid ann_R(x)_w = R\}$. Following [8], the *$w$-envelope* of M is defined by $M_w = p^{-1}(\mathfrak{r}(E(M)/M))$, where $E(M)$ denotes the *injective envelope* (or *injective hull*) of M and $p : E(M) \to E(M)/M$ is the canonical projection. Then it is easy to see that $M_w = \{x \in E(M) \mid Jx \subseteq M$ for some $J \in GV(R)\}$. M is said to be *co-semi-divisorial* if $\mathfrak{r}(M) = 0$; equivalently, if whenever $Jx = 0$ for some $J \in GV(R)$ and $x \in M$, we have that $x = 0$. Any undefined terminology is standard, as in [4].

2. Main results

As the t-theoretic analog, an integral domain R is said to be *t-locally Noetherian* if R_P is a Noetherian domain for all maximal t-ideals P of R. It is easy to verify that R is t-locally Noetherian if and only if R_P is Noetherian for all prime w-ideals P of R. Some ring-theoretic characterizations of t-locally Noetherian domains are given in [2, Theorem 1,4]: R is t-locally Noetherian, if and only if $R[X]$ is t-locally Noetherian, if and only if $R[X]_{N_v}$ is t-locally Noetherian, if and only if $R[X]_{N_v}$ is locally Noetherian, where $N_v = \{f \in R[X] \mid C(f)_v = R\}$ and $C(f)$ is the content ideal generated by the coefficients of f.

Every SM domain is t-locally Noetherian. However, as the following example shows, the converse is not true in general. Following [7, Section IV], an integral domain R is called *t-almost Dedekind domain* if R_M is a rank-one discrete valuation domain for each maximal t-ideal M of R.

Example 2.1. Let R be a t-almost Dedekind domain which is not a Krull domain. Then R is t-locally Noetherian, but not an SM domain since a Krull domain is exactly an integrally closed SM domain.

Let M be an R-module. If M has a non-zero element c with the following condition $(*)$, then we call M a *cocyclic R-module*.

$(*)$ For every R-module N, every R-homomorphism $\phi : M \to N$ with $c \notin \ker(\phi)$ is monic.

Then it is easily seen that M is a cocyclic R-module if and only if every non-zero submodule of M contains c, i.e. Rc is the smallest submodule of M.

PROPOSITION 2.1. *A co-semi-divisorial R-module M is cocyclic if and only if M is an essential extension of R/P for some maximal w-ideal P of R.*

Proof. This follows from a slight modification of the proof of [10, Proposition 1]. □

LEMMA 2.2. *Let R be a t-locally Noetherian domain and M be a cocyclic co-semi-divisorial R-module. Then M has the structure of an R_P-module for some maximal w-ideal P of R.*

Proof. This follows from a slight modification of the proof of [10, Proposition 2] using Proposition 2.1. □

THEOREM 2.3. *The following statements are equivalent for an integral domain R.*

(1) *R is a t-locally Noetherian domain.*
(2) *Every cocyclic co-semi-divisorial R-module satisfies the DCC on submodules.*
(3) *For every maximal w-ideal P of R, $E(R/P)$ satisfies the DCC on submodules.*

Proof. (1) $\Rightarrow$ (2) Let M be a cocyclic co-semi-divisorial R-module. Then there exists a maximal w-ideal P of R such that $R/P \subseteq M \subseteq E := E(R/P)$. By Lemma 2.2, M has an R_P-module structure. Since R_P is Noetherian, by [9, Proposition 3], M satisfies the DCC on submodules as an R_P-module, and hence as an R-module.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) Let P be a maximal w-ideal of R. By [10, Lemma 1], $E(R/P)$ has an R_P-module structure and hence $E(R/P)$ satisfies the DCC on submodules as an R_P-module. It follows from [6, Corollary 3.2] that $E(R/P)$ is faithfully injective as an R_P-module. Thus by [6, Theorem 4.1], R_P satisfies the ACC on ideals, i.e. R_P is Noetherian. □

Let P be a prime ideal of R containing an ideal A of R. Then we call P a *Nagata prime of A* if there exists a multiplicative system S of R such that $S \cap A = \emptyset$ and such that $PR_S \supseteq AR_S$ and PR_S is maximal with respect to being contained in the set of zero-divisors mod AR_S. It is clear that any prime ideal of R which is minimal with respect to the property of containing A is a Nagata prime of A and such prime ideals are called *minimal prime divisor of A*.

In order to characterize SM domains via t-locally Noetherian domains, we need a couple of lemmas.

LEMMA 2.4. *Let R be an integral domain which satisfies the ACC on prime w-ideals and suppose that each w-finite ideal of R has only finitely many minimal prime divisors. If P is any prime w-ideal of R, then P is the unique minimal prime divisor of some w-finite ideal A of R.*

Proof. This follows from a slight modification of the proof of [1, Lemma 1.1]. □

COROLLARY 2.5. *Let R be an integral domain which satisfies the ACC on prime w-ideals. Then R is an SM domain if and only if each w-finite ideal A of R has only finitely many minimal prime divisors and $\sqrt{A}$ is w-finite.*

Proof. ($\Rightarrow$) By [11, Theorem 4.9 and Proposition 1.6].

($\Leftarrow$) Let P be a prime w-ideal of R. Then by Lemma 2.4, P is the unique minimal prime of some w-finite ideal A of R. Thus $P = \sqrt{A}$ and P is w-finite. Thus by [11, Theorem 4.3], R is an SM domain. □

LEMMA 2.6. *If R is a t-locally Noetherian domain, then R satisfies the ACC on prime w-ideals (and hence on prime t-ideals).*

Proof. Let $P_1 \subseteq P_2 \subseteq \cdots$ be a chain of prime w-ideals of R and set $P := \bigcup_{i=1}^{\infty} P_i$. Then P is a prime w-ideal of R (since w is of finite character) and R_P is Noetherian. Since $P_1R_P \subseteq P_2R_P \subseteq \cdots \subseteq PR_P$ is a chain of prime ideals of R_P, there exists an integer n such that $PR_P = P_{n+i}R_P$ for each nonnegative integer i. Hence $P = P_{n+i}$ for each nonnegative integer i. Therefore, R satisfies the ACC on prime w-ideals. □

Let P be a prime ideal of R containing an ideal A of R. Then P is called a *Bourbaki prime* (resp., Zariski-Samuel prime) of A if $P = A :_R x$ (resp., $P = \sqrt{A :_R x}$) for some $x \in R$. Finally we call P a *weak-Bourbaki prime of A* if there exists $x \in R$ such that P is a minimal prime ideal of $A :_R x$.

THEOREM 2.7. *The following conditions are equivalent for a t-locally Noetherian domain R.*

(1) *R is an SM domain.*

(2) *Each w-finite ideal of R may be expressed as a finite intersection of primary w-ideals of R.*
(3) *Each w-finite ideal of R has only finitely many Bourbaki primes.*
(4) *Each w-finite ideal of R has only finitely many Zariski-Samuel primes.*
(5) *Each w-finite ideal of R has only finitely many weak-Bourbaki primes.*
(6) *Each w-finite ideal I of R has only finitely many minimal prime divisors and $\sqrt{I}$ is w-finite.*

Proof. (1) $\Rightarrow$ (2) [11, Theorem 4.11].

(2) $\Rightarrow$ (3) Let I be a w-ideal of R, and let $I = Q_1 \cap \cdots \cap Q_n$ be a primary decomposition. It is obvious that if P is a prime ideal of R minimal over I, then $P = \sqrt{Q_i}$ for some Q_i . Thus the number of such P's is at most finite. Moreover $P = I :_R x$ for some $x \in R$ ([11, Theorem 4.9])

(3) $\Rightarrow$ (4) $\Rightarrow$ (5) This follows from the fact that (weak-Bourbaki prime) $\Rightarrow$ (Zariski-Samuel prime) $\Rightarrow$ (weak–Bourbaki prime) ([5, p.279]).

(5) $\Rightarrow$ (1) This follows from a slight modification of the proof of (vi) $\Rightarrow$ (i) in [3, Theorem 4.5].

(1) $\Leftrightarrow$ (6) This follows from Corollary 2.5 and Lemma 2.6. □

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