

STABILITY OF CYCLIC FUNCTIONAL EQUATIONS IN BANACH MODULES OVER A C^* -ALGEBRA

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ABSTRACT. We prove the Cauchy-Rassias stability of cyclic functional equations in Banach modules over a unital C^* -algebra.

1. Introduction

Recently, T. Trif [6, Theorem 2.1] proved that, for vector spaces V and W , a mapping $f : V \rightarrow W$ with $f(0) = 0$ satisfies the functional equation

$$\begin{aligned} n \cdot {}_{n-2}C_{k-2} f\left(\frac{x_1 + \cdots + x_n}{n}\right) + {}_{n-2}C_{k-1} \sum_{i=1}^n f(x_i) \\ = k \sum_{1 \leq i_1 < \cdots < i_k \leq n} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right) \end{aligned}$$

for all $x_1, \dots, x_n \in V$ if and only if the mapping $f : V \rightarrow W$ satisfies the additive Cauchy equation $f(x + y) = f(x) + f(y)$ for all $x, y \in V$.

In [2], the author conjectured the following, and gave a partial answer for the conjecture.

Conjecture. Let p be an integer greater than 1. A mapping $f : V \rightarrow W$ with $f(0) = 0$ satisfies the functional equation

$$\begin{aligned} p^n f\left(\frac{x_1 + \cdots + x_{p^n}}{p^n}\right) + (pk - p) \sum_{i=1}^{p^{n-1}} f\left(\frac{x_{pi-p+1} + \cdots + x_{pi}}{p}\right) \\ (C) \quad = k \sum_{i=1}^{p^n} f\left(\frac{x_i + \cdots + x_{i+k-1}}{k}\right) \end{aligned}$$

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for all $x_1 = x_{p^n+1}, \dots, x_{k-1} = x_{p^n+k-1}, x_k, \dots, x_{p^n} \in V$ if and only if the mapping $f : V \rightarrow W$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in V$.

Throughout this paper, let A be a unital C^* -algebra with norm $|\cdot|$ and $\mathcal{U}(A)$ the unitary group of A . Let ${}_A\mathcal{B}$ and ${}_A\mathcal{C}$ be left Banach A -modules with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Let d, r and p be positive integers and b an integer greater than 1.

The main purpose of this paper is to prove the Cauchy-Rassias stability of the functional equation (C) in Banach modules over a unital C^* -algebra for a special case.

2. Cyclic functional equations

In this section, we are going to solve the conjecture for a special case.

THEOREM 2.1. *A mapping $f : V \rightarrow W$ with $f(0) = 0$ satisfies the functional equation*

$$(2.i) \quad \begin{aligned} (bp)^n f\left(\frac{x_1 + \dots + x_{(bp)^n}}{(bp)^n}\right) + bp(bp-2) \sum_{i=1}^{(bp)^{n-1}} f\left(\frac{x_{bpi-bp+1} + \dots + x_{bpi}}{bp}\right) \\ = (bp-1) \sum_{i=1}^{(bp)^n} f\left(\frac{x_i + \dots + x_{i+bp-2}}{bp-1}\right) \end{aligned}$$

for all $x_1 = x_{(bp)^n+1}, \dots, x_{bp-2} = x_{(bp)^n+bp-2}, x_{bp-1}, \dots, x_{(bp)^n} \in V$ if and only if the mapping $f : V \rightarrow W$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in V$.

Proof. Assume a mapping $f : V \rightarrow W$ satisfies (2.i). Put $x_{bi-b+1} = x_1, x_{bi-b+2}, \dots, x_{bi} = x_b$ in (2.i) for all $i = 1, \dots, p(bp)^{n-1}$. Then

$$(2.1) \quad \begin{aligned} (bp)^n f\left(\frac{x_1 + \dots + x_b}{b}\right) + (bp-2)(bp)^n f\left(\frac{x_1 + \dots + x_b}{b}\right) \\ = (bp-1)(bp)^{n-1} \sum_{i=1}^b pf\left(\frac{\sum_{m=1}^b px_m - x_i}{bp-1}\right) \end{aligned}$$

for all $x_1, \dots, x_b \in V$. Put $x_1 = x$ and $x_2 = \dots = x_b = y$ in (2.1).

Then we get

$$\begin{aligned}
 (2.2) \quad & (bp-1)(bp)^n f\left(\frac{x+(b-1)y}{b}\right) \\
 & = (bp-1)(bp)^{n-1} \left(pf\left(\frac{(p-1)x+(b-1)py}{bp-1}\right) \right. \\
 & \quad \left. + (b-1)p f\left(\frac{px+(bp-p-1)y}{bp-1}\right) \right)
 \end{aligned}$$

for all $x, y \in V$. Put $x = 0$ in (2.2). Replacing x and y by $-(bp-p-1)x$ and px in (2.2), respectively, we get

$$(bp-1)(bp)^n f\left(\frac{x}{b}\right) = (bp-1)(bp)^{n-1} pf(x)$$

for all $x \in V$. So we get

$$(2.3) \quad f\left(\frac{1}{b}x\right) = \frac{1}{b}f(x)$$

for all $x \in V$.

By (2.3), it follows from (2.1) that

$$\begin{aligned}
 & (bp-1)p(bp)^{n-1} f(x_1 + \dots + x_b) \\
 & = (bp-1)(bp)^{n-1} \sum_{i=1}^b pf\left(\frac{\sum_{m=1}^b px_m - x_i}{bp-1}\right)
 \end{aligned}$$

for all $x_1, \dots, x_b \in V$. Let $v_i = \frac{\sum_{m=1}^b px_m - x_i}{bp-1}$ for all $i = 1, \dots, b$.

Then we get

$$\begin{aligned}
 (bp-1)p(bp)^{n-1} \sum_{i=1}^b f(v_i) & = (bp-1)p(bp)^{n-1} f(x_1 + \dots + x_b) \\
 & = (bp-1)p(bp)^{n-1} f(v_1 + \dots + v_b)
 \end{aligned}$$

for all $v_1, \dots, v_b \in V$. Let $v_3 = \dots = v_b = 0$. Then we get $f(v_1) + f(v_2) = f(v_1 + v_2)$ for all $v_1, v_2 \in V$. So the mapping $f : V \rightarrow W$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in V$.

The converse is obvious. $\square$

3. Stability of cyclic functional equations in Banach modules over a C^* -algebra associated with its unitary group

We are going to prove the Cauchy-Rassias stability of the functional equation (C) in Banach modules over a unital C^* -algebra associated with its unitary group for a special case.

THEOREM 3.1. *Let $f : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_A\mathcal{B}^{(bp)^n} \rightarrow [0, \infty)$ such that*

$$\begin{aligned}
 (3.i) \quad \tilde{\varphi}(x_1, \dots, x_{(bp)^n}) &:= \sum_{j=0}^{\infty} \left(\frac{d}{(bp)^n r}\right)^j \varphi\left(\left(\frac{(bp)^n r}{d}\right)^j x_1, \dots, \right. \\
 &\quad \left. \left(\frac{(bp)^n r}{d}\right)^j x_{(bp)^n}\right) < \infty \\
 (3.ii) \quad \|D_u f(x_1, \dots, x_{(bp)^n})\| &= \left\| \frac{d}{r} u f\left(\frac{rx_1 + \dots + rx_{(bp)^n}}{d}\right) \right. \\
 &\quad \left. + bp(bp-2) \sum_{i=1}^{(bp)^{n-1}} u f\left(\frac{x_{bpi-bp+1} + \dots + x_{bpi}}{bp}\right) \right. \\
 &\quad \left. - (bp-1) \sum_{i=1}^{(bp)^n} f\left(\frac{ux_i + \dots + ux_{i+bp-2}}{bp-1}\right) \right\| \\
 &\leq \varphi(x_1, \dots, x_{(bp)^n})
 \end{aligned}$$

for all $x_1 = x_{(bp)^{n+1}}, \dots, x_{bp-2} = x_{(bp)^n+bp-2}, x_{bp-1}, \dots, x_{(bp)^n} \in {}_A\mathcal{B}$ and all $u \in \mathcal{U}(A)$. Then there exists a unique A -linear mapping $T : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ such that

$$(3.iii) \quad \|f(x) - T(x)\| \leq \frac{1}{(bp)^n} \tilde{\varphi}(x, \dots, x)$$

for all $x \in {}_A\mathcal{B}$.

Proof. Put $u = 1 \in \mathcal{U}(A)$. Let $x_1 = \dots = x_{(bp)^n} = x$ in (3.ii). Then we get

$$\left\| \frac{d}{r} \cdot f\left(\frac{(bp)^n r}{d} x\right) + (bp-2)(bp)^n f(x) - (bp-1)(bp)^n f(x) \right\| \leq \varphi(x, \dots, x)$$

for all $x \in {}_A\mathcal{B}$. So one can obtain

$$\|f(x) - \frac{d}{(bp)^{n_r}} f\left(\frac{(bp)^{n_r}}{d} x\right)\| \leq \frac{1}{(bp)^n} \varphi(x, \dots, x),$$

hence

$$\begin{aligned} & \left\| \left(\frac{d}{(bp)^{n_r}}\right)^j f\left(\left(\frac{(bp)^{n_r}}{d}\right)^j x\right) - \left(\frac{d}{(bp)^{n_r}}\right)^{j+1} f\left(\left(\frac{(bp)^{n_r}}{d}\right)^{j+1} x\right) \right\| \\ & \leq \frac{1}{(bp)^n} \left(\frac{d}{(bp)^{n_r}}\right)^j \varphi\left(\left(\frac{(bp)^{n_r}}{d}\right)^j x, \dots, \left(\frac{(bp)^{n_r}}{d}\right)^j x\right) \end{aligned}$$

for all $x \in {}_A\mathcal{B}$. So we get

$$\begin{aligned} & \left\| f(x) - \left(\frac{d}{(bp)^{n_r}}\right)^j f\left(\left(\frac{(bp)^{n_r}}{d}\right)^j x\right) \right\| \\ (3.1) \quad & \leq \frac{1}{(bp)^n} \sum_{m=0}^{j-1} \left(\frac{d}{(bp)^{n_r}}\right)^m \varphi\left(\left(\frac{(bp)^{n_r}}{d}\right)^m x, \dots, \left(\frac{(bp)^{n_r}}{d}\right)^m x\right) \end{aligned}$$

for all $x \in {}_A\mathcal{B}$.

Let x be an element in ${}_A\mathcal{B}$. For positive integers l and m with $l > m$,

$$\begin{aligned} & \left\| \left(\frac{d}{(bp)^{n_r}}\right)^l f\left(\left(\frac{(bp)^{n_r}}{d}\right)^l x\right) - \left(\frac{d}{(bp)^{n_r}}\right)^m f\left(\left(\frac{(bp)^{n_r}}{d}\right)^m x\right) \right\| \\ & \leq \frac{1}{(bp)^n} \sum_{j=m}^{l-1} \left(\frac{d}{(bp)^{n_r}}\right)^j \varphi\left(\left(\frac{(bp)^{n_r}}{d}\right)^j x, \dots, \left(\frac{(bp)^{n_r}}{d}\right)^j x\right), \end{aligned}$$

which tends to zero as $m \rightarrow \infty$ by (3.i). So $\left\{\left(\frac{d}{(bp)^{n_r}}\right)^j f\left(\left(\frac{(bp)^{n_r}}{d}\right)^j x\right)\right\}$ is a Cauchy sequence for all $x \in {}_A\mathcal{B}$. Since ${}_A\mathcal{C}$ is complete, the sequence $\left\{\left(\frac{d}{(bp)^{n_r}}\right)^j f\left(\left(\frac{(bp)^{n_r}}{d}\right)^j x\right)\right\}$ converges for all $x \in {}_A\mathcal{B}$. We can define a mapping $T : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ by

$$(3.2) \quad T(x) = \lim_{j \rightarrow \infty} \left(\frac{d}{(bp)^{n_r}}\right)^j f\left(\left(\frac{(bp)^{n_r}}{d}\right)^j x\right)$$

for all $x \in {}_A\mathcal{B}$.

By (3.i) and (3.2), we get

$$\begin{aligned} & \|D_1T(x_1, \dots, x_{(bp)^n})\| \\ &= \lim_{j \rightarrow \infty} \left(\frac{d}{(bp)^n r} \right)^j \|D_1f\left(\left(\frac{(bp)^n r}{d}\right)^j x_1, \dots, \left(\frac{(bp)^n r}{d}\right)^j x_{(bp)^n}\right)\| \\ &\leq \lim_{j \rightarrow \infty} \left(\frac{d}{(bp)^n r} \right)^j \varphi\left(\left(\frac{(bp)^n r}{d}\right)^j x_1, \dots, \left(\frac{(bp)^n r}{d}\right)^j x_{(bp)^n}\right) = 0 \end{aligned}$$

for all $x_1, \dots, x_{(bp)^n} \in {}_A\mathcal{B}$. Hence $D_1T(x_1, \dots, x_{(bp)^n}) = 0$ for all $x_1, \dots, x_{(bp)^n} \in {}_A\mathcal{B}$. Put $x_1 = \dots = x_{(bp)^n} = x$ in $D_1T(x_1, \dots, x_{(bp)^n})$. Then

$$\frac{d}{r}T\left(\frac{(bp)^n r}{d}x\right) + (bp-2)(bp)^n T(x) - (bp-1)(bp)^n T(x) = 0$$

for all $x \in {}_A\mathcal{B}$. So

$$T\left(\frac{(bp)^n r}{d}x\right) = \frac{(bp)^n r}{d}T(x)$$

for all $x \in {}_A\mathcal{B}$. Since

$$\begin{aligned} \frac{d}{r}T\left(\frac{rx_1 + \dots + rx_{(bp)^n}}{d}\right) &= \frac{d}{r}T\left(\frac{(bp)^n r(x_1 + \dots + x_{(bp)^n})}{(bp)^n d}\right) \\ &= \frac{d}{r} \frac{(bp)^n r}{d} T\left(\frac{x_1 + \dots + x_{(bp)^n}}{(bp)^n}\right) \\ &= (bp)^n T\left(\frac{x_1 + \dots + x_{(bp)^n}}{(bp)^n}\right) \end{aligned}$$

for all $x_1, \dots, x_{(bp)^n} \in {}_A\mathcal{B}$,

$$\begin{aligned} (bp)^n T\left(\frac{x_1 + \dots + x_{(bp)^n}}{(bp)^n}\right) &+ bp(bp-2) \sum_{i=1}^{(bp)^{n-1}} T\left(\frac{x_{bpi-bp+1} + \dots + x_{bpi}}{bp}\right) \\ &= (bp-1) \sum_{i=1}^{(bp)^n} T\left(\frac{x_i + \dots + x_{i+bp-2}}{bp-1}\right) \end{aligned}$$

for all $x_1 = x_{(bp)^{n+1}}, \dots, x_{bp-2} = x_{(bp)^n+bp-2}, x_{bp-1}, \dots, x_{(bp)^n} \in {}_A\mathcal{B}$. By Theorem 2.1, T is additive. Moreover, by passing to the limit in (3.1) as $j \rightarrow \infty$, we get (3.iii).

Now let $L : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ be another additive mapping satisfying

$$\|f(x) - L(x)\| \leq \frac{1}{(bp)^n} \tilde{\varphi}(x, \dots, x)$$

for all $x \in {}_A\mathcal{B}$.

$$\begin{aligned} \|T(x) - L(x)\| &= \left(\frac{d}{(bp)^{nr}}\right)^j \|T\left(\left(\frac{(bp)^{nr}}{d}\right)^j x\right) - L\left(\left(\frac{(bp)^{nr}}{d}\right)^j x\right)\| \\ &\leq \left(\frac{d}{(bp)^{nr}}\right)^j \|T\left(\left(\frac{(bp)^{nr}}{d}\right)^j x\right) - f\left(\left(\frac{(bp)^{nr}}{d}\right)^j x\right)\| \\ &\quad + \left(\frac{d}{(bp)^{nr}}\right)^j \|f\left(\left(\frac{(bp)^{nr}}{d}\right)^j x\right) - L\left(\left(\frac{(bp)^{nr}}{d}\right)^j x\right)\| \\ &\leq \frac{2}{(bp)^n} \left(\frac{d}{(bp)^{nr}}\right)^j \tilde{\varphi}\left(\left(\frac{(bp)^{nr}}{d}\right)^j x, \dots, \left(\frac{(bp)^{nr}}{d}\right)^j x\right), \end{aligned}$$

which tends to zero as $j \rightarrow \infty$ by (3.i). Thus $T(x) = L(x)$ for all $x \in {}_A\mathcal{B}$. This proves the uniqueness of T .

By the assumption, for each $u \in \mathcal{U}(A)$,

$$\begin{aligned} \left(\frac{d}{(bp)^{nr}}\right)^j \|D_u f\left(\left(\frac{(bp)^{nr}}{d}\right)^j x, \dots, \left(\frac{(bp)^{nr}}{d}\right)^j x\right)\| \\ \leq \left(\frac{d}{(bp)^{nr}}\right)^j \varphi\left(\left(\frac{(bp)^{nr}}{d}\right)^j x, \dots, \left(\frac{(bp)^{nr}}{d}\right)^j x\right) \end{aligned}$$

for all $x \in {}_A\mathcal{B}$, and

$$\left(\frac{d}{(bp)^{nr}}\right)^j \|D_u f\left(\left(\frac{(bp)^{nr}}{d}\right)^j x, \dots, \left(\frac{(bp)^{nr}}{d}\right)^j x\right)\| \rightarrow 0$$

as $j \rightarrow \infty$ for all $x \in {}_A\mathcal{B}$. So

$$D_u T(x, \dots, x) = \lim_{j \rightarrow \infty} \left(\frac{d}{(bp)^{nr}}\right)^j D_u f\left(\left(\frac{(bp)^{nr}}{d}\right)^j x, \dots, \left(\frac{(bp)^{nr}}{d}\right)^j x\right) = 0$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. Hence

$$\begin{aligned} D_u T(x, \dots, x) &= \frac{d}{r} \cdot uT\left(\frac{(bp)^n r}{d} x\right) + (bp - 2)(bp)^n uT(x) \\ &\quad - (bp - 1)(bp)^n T(ux) = 0 \end{aligned}$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. So

$$uT(x) = T(ux)$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$.

Now let $a \in A$ ($a \neq 0$) and M an integer greater than $4|a|$. Then

$$\left|\frac{a}{M}\right| = \frac{1}{M}|a| < \frac{|a|}{4|a|} = \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}.$$

By [1, Theorem 1], there exist three elements $u_1, u_2, u_3 \in \mathcal{U}(A)$ such that $3\frac{a}{M} = u_1 + u_2 + u_3$. And $T(x) = T(3 \cdot \frac{1}{3}x) = 3T(\frac{1}{3}x)$ for all $x \in {}_A\mathcal{B}$. So $T(\frac{1}{3}x) = \frac{1}{3}T(x)$ for all $x \in {}_A\mathcal{B}$. Thus

$$\begin{aligned} T(ax) &= T\left(\frac{M}{3} \cdot 3\frac{a}{M}x\right) = M \cdot T\left(\frac{1}{3} \cdot 3\frac{a}{M}x\right) = \frac{M}{3}T\left(3\frac{a}{M}x\right) \\ &= \frac{M}{3}T(u_1x + u_2x + u_3x) = \frac{M}{3}(T(u_1x) + T(u_2x) + T(u_3x)) \\ &= \frac{M}{3}(u_1 + u_2 + u_3)T(x) = \frac{M}{3} \cdot 3\frac{a}{M}T(x) \\ &= aT(x) \end{aligned}$$

for all $x \in {}_A\mathcal{B}$. Obviously, $T(0x) = 0T(x)$ for all $x \in {}_A\mathcal{B}$. Hence

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in {}_A\mathcal{B}$. So the unique additive mapping $T : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ is an A -linear mapping, as desired. $\square$

THEOREM 3.2. Let $f : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_A\mathcal{B}^{(bp)^n} \rightarrow [0, \infty)$ such that

$$(3.4) \quad \begin{aligned} \tilde{\varphi}(x_1, \dots, x_{(bp)^n}) &:= \sum_{j=0}^{\infty} \left(\frac{d}{(bp-1)r} \right)^j \varphi \left(\left(\frac{(bp-1)r}{d} \right)^j x_1, \dots, \right. \\ &\quad \left. \left(\frac{(bp-1)r}{d} \right)^j x_{(bp)^n} \right) < \infty \\ (3.5) \quad \|D_u f(x_1, \dots, x_{(bp)^n})\| &= \|(bp)^n u f\left(\frac{x_1 + \dots + x_{(bp)^n}}{(bp)^n}\right) \\ &\quad + bp(bp-2) \sum_{i=1}^{(bp)^{n-1}} u f\left(\frac{x_{bpi-bp+1} + \dots + x_{bpi}}{bp}\right) \\ &\quad - \frac{d}{r} \sum_{i=1}^{(bp)^n} f\left(\frac{ru x_i + \dots + ru x_{i+bp-2}}{d}\right)\| \leq \varphi(x_1, \dots, x_{(bp)^n}) \end{aligned}$$

for all $x_1 = x_{(bp)^{n+1}}, \dots, x_{bp-2} = x_{(bp)^n+bp-2}, x_{bp-1}, \dots, x_{(bp)^n} \in {}_A\mathcal{B}$ and all $u \in \mathcal{U}(A)$. Then there exists a unique A -linear mapping $T : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ such that

$$(3.6) \quad \|f(x) - T(x)\| \leq \frac{1}{(bp-1)(bp)^n} \tilde{\varphi}(x, \dots, x)$$

for all $x \in {}_A\mathcal{B}$.

Proof. Put $u = 1 \in \mathcal{U}(A)$. Let $x_1 = \dots = x_{(bp)^n} = x$ in (3.5). Then we get

$$\|(bp)^n f(x) + (bp-2)(bp)^n f(x) - \frac{(bp)^n d}{r} f\left(\frac{(bp-1)r}{d} x\right)\| \leq \varphi(x, \dots, x)$$

for all $x \in {}_A\mathcal{B}$. So one can obtain

$$\|f(x) - \frac{d}{(bp-1)r} f\left(\frac{(bp-1)r}{d} x\right)\| \leq \frac{1}{(bp-1)(bp)^n} \varphi(x, \dots, x),$$

hence

$$\begin{aligned} &\left\| \left(\frac{d}{(bp-1)r} \right)^j f\left(\left(\frac{(bp-1)r}{d}\right)^j x\right) - \left(\frac{d}{(bp-1)r} \right)^{j+1} f\left(\left(\frac{(bp-1)r}{d}\right)^{j+1} x\right) \right\| \\ &\leq \frac{1}{(bp-1)(bp)^n} \left(\frac{d}{(bp-1)r} \right)^j \varphi\left(\left(\frac{(bp-1)r}{d}\right)^j x, \dots, \left(\frac{(bp-1)r}{d}\right)^j x\right) \end{aligned}$$

for all $x \in {}_A\mathcal{B}$. So we get

$$(3.3) \quad \begin{aligned} & \|f(x) - (\frac{d}{(bp-1)r})^j f((\frac{(bp-1)r}{d})^j x)\| \leq \frac{1}{(bp-1)(bp)^n} \\ & \sum_{m=0}^{j-1} (\frac{d}{(bp-1)r})^m \varphi((\frac{(bp-1)r}{d})^m x, \dots, (\frac{(bp-1)r}{d})^m x) \end{aligned}$$

for all $x \in {}_A\mathcal{B}$.

Let x be an element in ${}_A\mathcal{B}$. For positive integers l and m with $l > m$,

$$\begin{aligned} & \|(\frac{d}{(bp-1)r})^l f((\frac{(bp-1)r}{d})^l x) - (\frac{d}{(bp-1)r})^m f((\frac{(bp-1)r}{d})^m x)\| \\ & \leq \frac{1}{(bp-1)(bp)^n} \sum_{j=m}^{l-1} (\frac{d}{(bp-1)r})^j \varphi((\frac{(bp-1)r}{d})^j x, \dots, (\frac{(bp-1)r}{d})^j x), \end{aligned}$$

which tends to zero as $m \rightarrow \infty$ by (3.iv). So $\{(\frac{d}{(bp-1)r})^j f((\frac{(bp-1)r}{d})^j x)\}$ is a Cauchy sequence for all $x \in {}_A\mathcal{B}$. Since ${}_A\mathcal{C}$ is complete, the sequence $\{(\frac{d}{(bp-1)r})^j f((\frac{(bp-1)r}{d})^j x)\}$ converges for all $x \in {}_A\mathcal{B}$. We can define a mapping $T : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ by

$$(3.4) \quad T(x) = \lim_{j \rightarrow \infty} (\frac{d}{(bp-1)r})^j f((\frac{(bp-1)r}{d})^j x)$$

for all $x \in {}_A\mathcal{B}$.

By (3.iv) and (3.4), we get

$$\begin{aligned} & \|D_1 T(x_1, \dots, x_{(bp)^n})\| \\ & = \lim_{j \rightarrow \infty} (\frac{d}{(bp-1)r})^j \|D_1 f((\frac{(bp-1)r}{d})^j x_1, \dots, (\frac{(bp-1)r}{d})^j x_{(bp)^n})\| \\ & \leq \lim_{j \rightarrow \infty} (\frac{d}{(bp-1)r})^j \varphi((\frac{(bp-1)r}{d})^j x_1, \dots, (\frac{(bp-1)r}{d})^j x_{(bp)^n}) = 0 \end{aligned}$$

for all $x_1, \dots, x_{(bp)^n} \in {}_A\mathcal{B}$. Hence $D_1T(x_1, \dots, x_{(bp)^n}) = 0$ for all $x_1, \dots, x_{(bp)^n} \in {}_A\mathcal{B}$. Put $x_1 = \dots = x_{(bp)^n} = x$ in $D_1T(x_1, \dots, x_{(bp)^n})$. Then

$$(bp)^n T(x) + (bp-2) (bp)^n T(x) - \frac{(bp)^n d}{r} T\left(\frac{(bp-1)r}{d} x\right) = 0$$

for all $x \in {}_A\mathcal{B}$. So

$$T\left(\frac{(bp-1)r}{d} x\right) = \frac{(bp-1)r}{d} T(x)$$

for all $x \in {}_A\mathcal{B}$. Since

$$\begin{aligned} \frac{d}{r} T\left(\frac{rx_i + \dots + rx_{i+bp-2}}{d}\right) &= \frac{d}{r} T\left(\frac{(bp-1)r(x_i + \dots + x_{i+bp-2})}{(bp-1)d}\right) \\ &= \frac{d}{r} \frac{(bp-1)r}{d} T\left(\frac{x_i + \dots + x_{i+bp-2}}{bp-1}\right) \\ &= (bp-1) T\left(\frac{x_i + \dots + x_{i+bp-2}}{bp-1}\right) \end{aligned}$$

for all $x_i, \dots, x_{i+bp-2} \in {}_A\mathcal{B}$,

$$\begin{aligned} (bp)^n T\left(\frac{x_1 + \dots + x_{(bp)^n}}{(bp)^n}\right) + bp(bp-2) \sum_{i=1}^{(bp)^{n-1}} T\left(\frac{x_{bpi-bp+1} + \dots + x_{bpi}}{bp}\right) \\ = (bp-1) \sum_{i=1}^{(bp)^n} T\left(\frac{x_i + \dots + x_{i+bp-2}}{bp-1}\right) \end{aligned}$$

for all $x_1 = x_{(bp)^{n+1}}, \dots, x_{bp-2} = x_{(bp)^n+bp-2}, x_{bp-1}, \dots, x_{(bp)^n} \in {}_A\mathcal{B}$. By Theorem 2.1, T is additive. Moreover, by passing to the limit in (3.3) as $j \rightarrow \infty$, we get (3.vi).

Now let $L : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ be another additive mapping satisfying

$$\|f(x) - L(x)\| \leq \frac{1}{(bp-1)(bp)^n} \tilde{\varphi}(x, \dots, x)$$

for all $x \in {}_A\mathcal{B}$.

$$\begin{aligned}
\|T(x) - L(x)\| &= \left(\frac{d}{(bp-1)r}\right)^j \|T\left(\left(\frac{(bp-1)r}{d}\right)^j x\right) - L\left(\left(\frac{(bp-1)r}{d}\right)^j x\right)\| \\
&\leq \left(\frac{d}{(bp-1)r}\right)^j \|T\left(\left(\frac{(bp-1)r}{d}\right)^j x\right) - f\left(\left(\frac{(bp-1)r}{d}\right)^j x\right)\| \\
&\quad + \left(\frac{d}{(bp-1)r}\right)^j \|f\left(\left(\frac{(bp-1)r}{d}\right)^j x\right) - L\left(\left(\frac{(bp-1)r}{d}\right)^j x\right)\| \\
&\leq \frac{2}{(bp-1)(bp)^n} \left(\frac{d}{(bp-1)r}\right)^j \widetilde{\varphi}\left(\left(\frac{(bp-1)r}{d}\right)^j x, \dots, \left(\frac{(bp-1)r}{d}\right)^j x\right),
\end{aligned}$$

which tends to zero as $j \rightarrow \infty$ by (3.iv). Thus $T(x) = L(x)$ for all $x \in {}_A\mathcal{B}$. This proves the uniqueness of T .

By the assumption, for each $u \in \mathcal{U}(A)$,

$$\begin{aligned}
&\left(\frac{d}{(bp-1)r}\right)^j \|D_u f\left(\left(\frac{(bp-1)r}{d}\right)^j x, \dots, \left(\frac{(bp-1)r}{d}\right)^j x\right)\| \\
&\leq \left(\frac{d}{(bp-1)r}\right)^j \varphi\left(\left(\frac{(bp-1)r}{d}\right)^j x, \dots, \left(\frac{(bp-1)r}{d}\right)^j x\right)
\end{aligned}$$

for all $x \in {}_A\mathcal{B}$, and

$$\left(\frac{d}{(bp-1)r}\right)^j \|D_u f\left(\left(\frac{(bp-1)r}{d}\right)^j x, \dots, \left(\frac{(bp-1)r}{d}\right)^j x\right)\| \rightarrow 0$$

as $j \rightarrow \infty$ for all $x \in {}_A\mathcal{B}$. So

$$\begin{aligned}
&D_u T(x, \dots, x) \\
&= \lim_{j \rightarrow \infty} \left(\frac{d}{(bp-1)r}\right)^j D_u f\left(\left(\frac{(bp-1)r}{d}\right)^j x, \dots, \left(\frac{(bp-1)r}{d}\right)^j x\right) = 0
\end{aligned}$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. Hence

$$\begin{aligned}
&D_u T(x, \dots, x) \\
&= (bp)^n uT(x) + (bp-2)(bp)^n uT(x) - \frac{(bp)^n d}{r} T\left(\frac{(bp-1)r}{d} ux\right) \\
&= 0
\end{aligned}$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. So

$$uT(x) = T(ux)$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$.

The rest of the proof is the same as the proof of Theorem 3.1. $\square$

THEOREM 3.3. *Let $f : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_A\mathcal{B}^{(bp)^n} \rightarrow [0, \infty)$ such that*

$$(3.vii) \quad \begin{aligned} \tilde{\varphi}(x_1, \dots, x_{(bp)^n}) &= \sum_{j=0}^{\infty} (bp)^j \varphi\left(-\frac{bp-2}{(bp)^j}x_1, \frac{1}{(bp)^j}x_2, \dots, \right. \\ &\quad \left. -\frac{bp-2}{(bp)^j}x_{(bp)^n-bp+1}, \frac{1}{(bp)^j}x_{(bp)^n-bp+2}, \dots, \right. \\ &\quad \left. \frac{1}{(bp)^j}x_{(bp)^n}\right) < \infty \end{aligned}$$

$$(3.viii) \quad \begin{aligned} \|D_u f(x_1, \dots, x_{(bp)^n})\| &= \|(bp)^n u f\left(\frac{x_1 + \dots + x_{(bp)^n}}{(bp)^n}\right) \\ &\quad + bp(bp-2) \sum_{i=1}^{(bp)^{n-1}} u f\left(\frac{x_{bpi-bp+1} + \dots + x_{bpi}}{bp}\right) \\ &\quad - (bp-1) \sum_{i=1}^{(bp)^n} f\left(\frac{ux_i + \dots + ux_{i+bp-2}}{bp-1}\right)\| \\ &\leq \varphi(x_1, \dots, x_{(bp)^n}) \end{aligned}$$

for all $x_1 = x_{(bp)^{n+1}}, \dots, x_{bp-2} = x_{(bp)^n+bp-2}, x_{bp-1}, \dots, x_{(bp)^n} \in {}_A\mathcal{B}$ and all $u \in \mathcal{U}(A)$. Then there exists a unique A -linear mapping $T : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ such that

$$(3.ix) \quad \|f(x) - T(x)\| \leq \frac{1}{(bp-1)(bp)^{n-1}} \tilde{\varphi}(x, \dots, x)$$

for all $x \in {}_A\mathcal{B}$.

Proof. Put $u = 1 \in \mathcal{U}(A)$. Let $x_{bpi-bp+1} = x$ and $x_{bpi-bp+2} = \dots = x_{bpi} = y$ in (3.viii) for all $i = 1, \dots, (bp)^{n-1}$. Then we get

$$\begin{aligned}
 & \| (bp)^n f\left(\frac{x + (bp-1)y}{bp}\right) + (bp-2)(bp)^n f\left(\frac{x + (bp-1)y}{bp}\right) \\
 & \quad - (bp-1)(bp)^{n-1} f(y) \\
 (3.5) \quad & \quad - (bp-1)(bp-1)(bp)^{n-1} f\left(\frac{x + (bp-2)y}{bp-1}\right) \| \\
 & \leq \varphi(x, y, y, \dots, x, y, \dots, y)
 \end{aligned}$$

for all $x, y \in {}_A\mathcal{B}$. Replacing x and y by $-(bp-2)x$ and x in (3.5), respectively, we get

$$\begin{aligned}
 & \| (bp-1)(bp)^n f\left(\frac{x}{bp}\right) - (bp-1)(bp)^{n-1} f(x) \| \\
 & \leq \varphi(-(bp-2)x, x, x, \dots, -(bp-2)x, x, \dots, x)
 \end{aligned}$$

for all $x \in {}_A\mathcal{B}$. So one can obtain

$$\begin{aligned}
 & \| f(x) - bp f\left(\frac{x}{bp}\right) \| \\
 & \leq \frac{1}{(bp-1)(bp)^{n-1}} \varphi(-(bp-2)x, x, x, \dots, -(bp-2)x, x, \dots, x),
 \end{aligned}$$

hence

$$\begin{aligned}
 & \| (bp)^j f\left(\frac{1}{(bp)^j} x\right) - (bp)^{j+1} f\left(\frac{1}{(bp)^{j+1}} x\right) \| \\
 & \leq \frac{(bp)^j}{(bp-1)(bp)^{n-1}} \varphi\left(-\frac{bp-2}{(bp)^j} x, \frac{1}{(bp)^j} x, \right. \\
 & \quad \left. \frac{1}{(bp)^j} x, \dots, -\frac{bp-2}{(bp)^j} x, \frac{1}{(bp)^j} x, \dots, \frac{1}{(bp)^j} x\right)
 \end{aligned}$$

for all $x \in {}_A\mathcal{B}$. So we get

$$\begin{aligned}
 & \| f(x) - (bp)^j f\left(\frac{1}{(bp)^j} x\right) \| \\
 (3.6) \quad & \leq \frac{1}{(bp-1)(bp)^{n-1}} \sum_{m=0}^{j-1} (bp)^m \varphi\left(-\frac{bp-2}{(bp)^m} x, \frac{1}{(bp)^m} x, \right. \\
 & \quad \left. \frac{1}{(bp)^m} x, \dots, -\frac{bp-2}{(bp)^m} x, \frac{1}{(bp)^m} x, \dots, \frac{1}{(bp)^m} x\right)
 \end{aligned}$$

for all $x \in {}_A\mathcal{B}$.

Let x be an element in ${}_A\mathcal{B}$. For positive integers l and m with $l > m$,

$$\begin{aligned} & \| (bp)^l f\left(\frac{1}{(bp)^l}x\right) - (bp)^m f\left(\frac{1}{(bp)^m}x\right) \| \\ & \leq \frac{1}{(bp-1)(bp)^{n-1}} \sum_{j=m}^{l-1} (bp)^j \varphi\left(-\frac{bp-2}{(bp)^j}x, \right. \\ & \quad \left. \frac{1}{(bp)^j}x, \frac{1}{(bp)^j}x, \dots, -\frac{bp-2}{(bp)^j}x, \frac{1}{(bp)^j}x, \dots, \frac{1}{(bp)^j}x\right), \end{aligned}$$

which tends to zero as $m \rightarrow \infty$ by (3.vii). So $\{(bp)^j f(\frac{1}{(bp)^j}x)\}$ is a Cauchy sequence for all $x \in {}_A\mathcal{B}$. Since ${}_A\mathcal{C}$ is complete, the sequence $\{(bp)^j f(\frac{1}{(bp)^j}x)\}$ converges for all $x \in {}_A\mathcal{B}$. We can define a mapping $T : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ by

$$(3.7) \quad T(x) = \lim_{j \rightarrow \infty} (bp)^j f\left(\frac{1}{(bp)^j}x\right)$$

for all $x \in {}_A\mathcal{B}$.

By (3.vii) and (3.7), we get

$$\begin{aligned} \|D_1 T(x_1, \dots, x_{(bp)^n})\| &= \lim_{j \rightarrow \infty} (bp)^j \|D_1 f\left(\frac{1}{(bp)^j}x_1, \dots, \frac{1}{(bp)^j}x_{(bp)^n}\right)\| \\ &\leq \lim_{j \rightarrow \infty} (bp)^j \varphi\left(\frac{1}{(bp)^j}x_1, \dots, \frac{1}{(bp)^j}x_{(bp)^n}\right) = 0 \end{aligned}$$

for all $x_1, \dots, x_{(bp)^n} \in {}_A\mathcal{B}$. Hence $D_1 T(x_1, \dots, x_{(bp)^n}) = 0$ for all $x_1, \dots, x_{(bp)^n} \in {}_A\mathcal{B}$. By Theorem 2.1, T is additive. Moreover, by passing to the limit in (3.6) as $j \rightarrow \infty$, we get (3.ix).

Now let $L : {}_A\mathcal{B} \rightarrow {}_A\mathcal{C}$ be another additive mapping satisfying

$$\|f(x) - L(x)\| \leq \frac{1}{(bp-1)(bp)^{n-1}} \tilde{\varphi}(x, \dots, x)$$

for all $x \in {}_A\mathcal{B}$.

$$\begin{aligned}
\|T(x) - L(x)\| &= (bp)^j \|T(\frac{1}{(bp)^j}x) - L(\frac{1}{(bp)^j}x)\| \\
&\leq (bp)^j \|T(\frac{1}{(bp)^j}x) - f(\frac{1}{(bp)^j}x)\| \\
&\quad + (bp)^j \|f(\frac{1}{(bp)^j}x) - L(\frac{1}{(bp)^j}x)\| \\
&\leq \frac{2}{(bp-1)(bp)^{n-1}} (bp)^j \tilde{\varphi}(\frac{1}{(bp)^j}x, \dots, \frac{1}{(bp)^j}x),
\end{aligned}$$

which tends to zero as $j \rightarrow \infty$ by (3.vii). Thus $T(x) = L(x)$ for all $x \in {}_A\mathcal{B}$. This proves the uniqueness of T .

By the assumption, for each $u \in \mathcal{U}(A)$,

$$(bp)^j \|D_u f(\frac{1}{(bp)^j}x, \dots, \frac{1}{(bp)^j}x)\| \leq (bp)^j \varphi(\frac{1}{(bp)^j}x, \dots, \frac{1}{(bp)^j}x)$$

for all $x \in {}_A\mathcal{B}$, and

$$(bp)^j \|D_u f(\frac{1}{(bp)^j}x, \dots, \frac{1}{(bp)^j}x)\| \rightarrow 0$$

as $j \rightarrow \infty$ for all $x \in {}_A\mathcal{B}$. So

$$D_u T(x, \dots, x) = \lim_{j \rightarrow \infty} (bp)^j D_u f(\frac{1}{(bp)^j}x, \dots, \frac{1}{(bp)^j}x) = 0$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. Hence

$$\begin{aligned}
&D_u T(x, \dots, x) \\
&= (bp)^n uT(x) + (bp-2)(bp)^n uT(x) - (bp-1)(bp)^n T(ux) = 0
\end{aligned}$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. So

$$uT(x) = T(ux)$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$.

The rest of the proof is the same as the proof of Theorem 3.1. $\square$

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