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EXPONENTIAL INEQUALITIES FOR ELNQD RANDOM VARIABLES WITH APPLICATIONS

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Abstract. In this paper we introduce the concept of extended linear negative quadrant dependence and obtain some exponential inequalities, complete convergence and almost sure convergence for extended linear negative quadrant dependent random variables.

1. Introduction

Lehman(1966) introduced a natural definition of negative dependence in the bivariate case. Two random variables X and Y are said to be negatively quadrant dependent(NQD) if for all real numbers x, y, $P(X \leq x, Y \leq Y) \leq P(X \leq x)P(Y \leq y)$ or $P(X > x, Y > Y) \leq P(X > x)P(Y > y)$. Joag-Dev and Proschan(1983) extended the concept of negative quadrant dependence to the multivariate case. A sequence $\{X_i, 1 \leq i \leq n\}$ of random variables is said to be negatively upper orthant dependent(NUOD) if for all real numbers $x_1, \cdots, x_n$,

$$P(X_1 > x_1, \cdots, X_n > x_n) \leq \prod_{i=1}^{n} P(X_i > x_i) \tag{1.1}$$

and it is said to be negatively lower orthant dependent(NLOD) if for all real numbers $x_1, \cdots, x_n$,

$$P(X_1 \leq x_1, \cdots, X_n \leq x_n) \leq \prod_{i=1}^{n} P(X_i \leq x_i). \tag{1.2}$$

A sequence $\{X_i, 1 \leq i \leq n\}$ of random variables is said to be negatively orthant dependent(NOD) if it is both NUOD and NLOD.

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Based on the concept of negative quadrant dependence, another notion of negative dependence was formulated by Newman(1984) as follows: A sequence $\{X_i, 1 \leq i \leq n\}$ of random variables is said to be linearly negative quadrant dependent(LNQD) if for any disjoint subsets A and B of $\{1, \cdots, n\}$ and positive r_j's, $\sum_{i\in A} r_iX_i$, $\sum_{j\in B} r_jX_j$ are NQD.

Recently, Liu(2009) introduced the concept of extended negative dependence in the multivariate case. A sequence $\{X_i, 1 \leq i \leq n\}$ of random variables is said to be extended negatively upper orthant dependent(ENUOD) if for all real numbers $x_1, \cdots, x_n$, there exists a constant $M > 0$ such that

$$P(X_1 > x_1, \cdots, X_n > x_n) \leq M \prod_{i=1}^{n} P(X_i > x_i) \tag{1.3}$$

and it is said to be extended negatively lower orthant dependent(ENLOD) if for all real numbers $x_1, \cdots, x_n$, there exists a constant $M > 0$ such that

$$P(X_1 \leq x_1, \cdots, X_n \leq x_n) \leq M \prod_{i=1}^{n} P(X_i \leq x_i). \tag{1.4}$$

A sequence $\{X_i, 1 \leq i \leq n\}$ of random variables is said to be extended negatively orthant dependent(ENOD) if it is both ENUOD and ENLOD.

From Definitions of NQD and ENOD we consider the concept of extended negative dependence in the bivariate case.

Definition 1.1. Two random variables X and Y are said to be extended negatively quadrant dependent(ENQD) if there exists a constant $M > 0$ such that for all real numbers x, y,

$$P(X \leq x, Y \leq Y) \leq MP(X \leq x)P(Y \leq y) \tag{1.5}$$

or

$$P(X > x, Y > Y) \leq MP(X > x)P(Y > y). \tag{1.6}$$

Remark 1.2. (1.5) and (1.6) are equivalent.

Example 1.3. (Farlie(1960) Let random variables X and Y have the Farlie-Gumber-Morgenstern bivariate distribution

$$H(x, y) = F(x)G(y)[1 + \rho(1 - F(x))(1 - G(y))], \ -1 \leq \rho \leq 1,$$

where $H(x, y)$ is the joint distribution of X and Y and $F(x), G(y)$ are marginal distributions, respectively. When $-1 \leq \rho \leq 0$ it is clear that

X and Y are ENQD as well as NQD, and when $0 \le \rho \le 1$ they are ENQD as well as PQD.

Recall that two random variables X and Y are called NQD if (1.5) or (1.6) holds when $M = 1$, they are called positively quadrant dependent(PQD) if the inequality (1.5) or (1.6) holds in the reverse direction when $M = 1$. Obviously, an NQD sequence must be an ENQD sequence. On the other hand, for some PQD sequences, it is possible to find a corresponding positive constant M such that (1.5) or (1.6) holds(see Example, when $0 \le \rho \le 1$). Therefore, the ENQD structure is substantially more comprehensive than the NQD structure in that it can reflect not only a negative dependence structure but also a positive one, to some extent.

Based on the concepts of extended negative quadrant dependence and linear negative quadrant dependence, another notion of extended negative dependence can be formulated as follows:

Definition 1.4. A sequence $\{X_i, 1 \le i \le n\}$ of random variables is said to be extended linearly negative quadrant dependent(ELNQD) if for any disjoint subsets A and B of $\{1, 2, \cdots, n\}$ and positive r_j's, $\sum_{i \in A} r_i X_i$, $\sum_{j \in B} r_j X_j$ are ENQD.

Remark 1.5. (i) NQD(LNQD) implies ENQD(ELNQD) and ELNQD implies ENQD.

(ii) ELNQD and ENQD do not imply each other.

The main purpose of this paper is to establish some exponential inequalities and complete convergence for the ELNQD random variables and to extend and improve the results of Ko et al.(2007), Nooghabi and Azarnoosh(2009) and Wang et al.(2010).

2. Some lemmas

Lemma 2.1. (Liu(2009) *Let two random variables X and Y be ENQD, then*

(i) if f and g are both nondecreasing(or both nonincreasing) functions, then $f(X)$ and $g(Y)$ are ENQD,

(ii) if X and Y are nonnegative random variables, then there exists a constant $M > 0$ such that $E(XY) \le MEXEY$,

(iii) especially, there exists a constant $M > 0$ such that for any real number h, $E(e^{h(X+Y)}) \le ME(e^{hX})E(e^{hY})$.

Lemma 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of ELNQD random variables and $t > 0$. Then for each $n \geq 1$, there exists a constant $M > 0$ such that*

$$E(\prod_{i=1}^{n} e^{tX_i}) \leq M \prod_{i=1}^{n} (Ee^{tX_i}). \tag{2.1}$$

Proof. For $t > 0$, it is easy to see that tX_i and $t\sum_{j=i+1}^{n} X_j$ are ENQD by the definition of ELNQD, which implies that $\exp(tX_i)$ and $\exp(t\sum_{j=i+1}^{n} X_j)$ are also ENQD for $i = 1, 2, \cdots, n-1$, by Lemma 2.1 (i). It follows from Lemma 2.1 (iii) and Definition 1.2 that

$$\begin{aligned}
E(\prod_{i=1}^{n} e^{tX_i}) &= E(\exp(tX_1)\exp(t\sum_{j=2}^{n} X_j)) \\
&\leq M_1 E[\exp(tX_1)]E[\exp(t\sum_{j=2}^{n} X_j)] \\
&= M_1 E[\exp(tX_1)]E[\exp(tX_2)\exp(t\sum_{j=3}^{n} X_j)] \\
&\leq M_1 M_2 E[\exp(tX_1)]E[\exp(tX_2)]E[\exp(t\sum_{j=3}^{n} X_j)] \\
&\leq \prod_{i=1}^{n-1} M_i \prod_{i=1}^{n} (Ee^{tX_i}) = M\prod_{i=1}^{n}(Ee^{tX_i}),
\end{aligned}$$

where $M = \prod_{i=1}^{n-1} M_i$. □

The following Lemma is an extension of Theorem 2 in Hoeffding(1963) to the ELNQD case.

Lemma 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of ELNQD random variables with $EX_n = 0$ for each $n \geq 1$. If there exist two sequences $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ such that $a_i \leq X_i \leq b_i$ for each $i \geq 1$, then for any $\epsilon > 0$*

$$P(|S_n| \geq n\epsilon) \leq 2M\exp\{-\frac{2n^2\epsilon^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\}. \tag{2.2}$$

Proof. Note that $E[\exp(\lambda X_i)] \leq \exp[\lambda^2(b_i - a_i)^2/8]$ for $\lambda > 0$ (see Lemma 3.1 in Devroye(1991)). Hence, it follows from Markov's inequality and (2.1) that

$$
\begin{aligned}
(2.3)\ P(S_n > n\epsilon) &= P(e^{\lambda S_n} > e^{\lambda n\epsilon}) \\
&\leq e^{-\lambda n\epsilon} E(e^{\lambda S_n}) \\
&\leq M e^{-\lambda n\epsilon} \prod_{i=1}^{n} E[\exp(\lambda X_i)] \\
&\leq M \exp(\lambda \sum_{i=1}^{n} (b_i - a_i)^2/8 - \lambda n\epsilon).
\end{aligned}
$$

By minimizing(with respect to λ) the right-hand side of (2.3) we obtain

$$
P(S_n > n\epsilon) \leq M \exp[-\frac{2n^2\epsilon^2}{\sum_{i=1}^{n}(b_i - a_i)^2}]. \tag{2.4}
$$

Since $-X_n$'s are still ELNQD by (2.4) we obtain

$$
P(S_n < -n\epsilon) = P(-S_n > n\epsilon) \leq M \exp[-\frac{2n^2\epsilon^2}{\sum_{i=1}^{n}(b_i - a_i)^2}]. \tag{2.5}
$$

By (2.4) and (2.5) the desired result (2.2) follows. □

3. Results

THEOREM 3.1. *Let $\{X_n, n \geq 1\}$ be a sequence of ELNQD random variables with mean zero and finite variances. If there exists a positive number c such that $|X_i| \leq cB_n$ for each $1 \leq i \leq n$, $n \geq 1$, where $B_n = \sum_{i=1}^{n} EX_i^2$, then for any $\epsilon > 0$ and $n \geq 1$, there exists a constant $M > 0$ such that*

$$
P(S_n/B_n \geq \epsilon) \leq \begin{cases} M \exp(-\frac{\epsilon^2}{2}(1 - \frac{\epsilon c}{2})), & \text{if } \epsilon c \leq 1 \\ M \exp(-\frac{\epsilon}{4c}), & \text{if } \epsilon c > 1. \end{cases} \tag{3.1}
$$

Proof. For fixed $n \geq 1$, take $t > 0$ such that $tcB_n \leq 1$. It is easy to see that

$$
|EX_i^m| \leq (cB_n)^{m-2} EX_i^2,\ m \geq 2.
$$

Therefore by Taylor expansion and $EX_i = 0$ and the fact that $1+x \leq e^x$ we obtain

$$(3.2)\ Ee^{tX_i} \begin{aligned}[t] &= 1 + \sum_{m=2}^{\infty} \frac{t^m}{m!} EX_i^m \\ &\leq 1 + \frac{t^2}{2} EX_i^2 (1 + \frac{t}{3} cB_n + \frac{t^2}{12} c^2 B_n^2 + \cdots) \\ &\leq 1 + \frac{t^2}{2} EX_i^2 (1 + \frac{t}{2} cB_n) \\ &\leq \exp(\frac{t^2}{2} EX_i^2 (1 + \frac{t}{2} cB_n)). \end{aligned}$$

By Lemma 2.2 and (3.2)

$$Ee^{tS_n} = E(\prod_{i=1}^{n} e^{tX_i}) \leq M \prod_{i=1}^{n} E(e^{tX_i}) \leq M \exp(\frac{t^2}{2} B_n^2 (1 + \frac{t}{2} cB_n))$$

which yields

$$(3.3) \qquad P(S_n/B_n \geq \epsilon) \leq M \exp(-t\epsilon B_n + \frac{t^2}{2} B_n^2 (1 + \frac{t}{2} cB_n)).$$

By taking $t = \frac{\epsilon}{B_n}$ when $\epsilon c \leq 1$ and $t = \frac{1}{cB_n}$ when $\epsilon c \geq 1$ we obtain(3.1) from (3.3). □

Theorem 3.2. *Let $\{X_n, n \geq 1\}$ be a sequence of ELNQD random variables with mean zero. If there exists a positive constant b such that $|X_n| \leq b$ for each $n \geq 1$, then for any $\epsilon > 0$, there exists a constant $M > 0$ such that*

$$(3.4) \qquad P(|S_n| \geq \epsilon) \leq 2M \exp(-\frac{\epsilon^2}{2(2B_n^2 + b\epsilon)}),$$

where $B_n^2 = \sum_{i=1}^{n} EX_i^2$.

Proof. For any $0 < t \leq 1/b$ clearly, $|tX_i| \leq 1$. Hence, by Taylor expansion, $EX_i = 0,\ i \geq 1$ and the fact that $1 + x \leq e^x$

$$(3.5) \qquad Ee^{tX_i} = 1 + \sum_{m=2}^{\infty} \frac{E(tX_i)^m}{m!} \leq 1 + t^2 EX_i^2 \leq \exp(t^2 EX_i^2).$$

By Markov's inequality, Lemma 2.2 and (3.5) we get

$$\begin{aligned}
(3.6)\ P(S_n \geq \epsilon) &\leq e^{-t\epsilon}E(e^{tS_n}) \\
&\leq Me^{-t\epsilon}\prod_{i=1}^{n}E(e^{tX_i}) \\
&\leq M\exp(-t\epsilon + t^2B_n^2) \\
&\leq M\exp(-\frac{\epsilon^2}{2(2B_n^2+b\epsilon)})
\end{aligned}$$

by taking $t = \epsilon/(2B_n^2 + b\epsilon)$. Since $\{-X_n, n \geq 1\}$ is also a sequence of ELNQD random variables it follows from (3.6) that

$$P(S_n \leq -\epsilon) = P(-S_n \geq \epsilon) \leq M\exp(-\frac{\epsilon^2}{2(2B_n^2+b\epsilon)}). \tag{3.7}$$

From (3.6) and (3.7) we obtain

$$P(|S_n| \geq \epsilon) = P(S_n \geq \epsilon) + P(S_n \leq -\epsilon) \leq 2M\exp(-\frac{\epsilon^2}{2(2B_n^2+b\epsilon)}).$$

□

THEOREM 3.3. *Let $\{X_n, n \geq 1\}$ be a sequence of ELNQD random variables with mean zero. If there exists a sequence $\{c_n, n \geq 1\}$ of positive numbers such that $|X_i| \leq c_i, i \geq 1$, then for any $t > 0$, there exists a constant $M > 0$ such that*

$$E\exp(tS_n) \leq \exp(\frac{t^2}{2}\sum_{i=1}^{n}e^{tc_i}EX_i^2). \tag{3.8}$$

Proof. From conditions $EX_i = 0$ and $|X_i| \leq c_i$ and the facts that $e^x \leq 1 + x + \frac{1}{2}x^2e^{|x|}$ and $1 + x \leq e^x$, we have

$$\begin{aligned}
(3.9)\ Ee^{tX_i} &\leq 1 + \frac{t^2}{2}E(X_i^2e^{t|X_i|}) \\
&\leq 1 + \frac{t^2}{2}e^{tc_i}EX_i^2 \\
&\leq \exp(\frac{1}{2}t^2e^{tc_i}EX_i^2)
\end{aligned}$$

for any $t > 0$. By Lemma 2.2 and (3.9)

$$E\exp(t\sum_{i=1}^{n}X_i) \leq M\prod_{i=1}^{n}E(\exp(tX_i)) \leq M\exp(\frac{t^2}{2}\sum_{i=1}^{n}e^{tc_i}EX_i^2).$$

□

THEOREM 3.4. *Let $\{X_n, n \geq 1\}$ be a sequence of ELNQD random variables such that $|X_i| \leq c_n$, $1 \leq i \leq n$, $n \geq 1$, where $\{c_n, n \geq 1\}$ is a sequence of positive numbers. Then for any $\epsilon > 0$ such that $\epsilon \leq eB_n^2/(2c_n)$, $n \geq 1$, there exists a constant $M > 0$ such that*

$$P(|S_n - ES_n| \geq \epsilon) \leq 2M \exp(-\frac{\epsilon^2}{2eB_n^2}). \tag{3.10}$$

Proof. By Markov's inequality and Theorem 3.3 we obtain that for any $t > 0$, there exists a constant $M > 0$ such that

$$\begin{aligned} P(S_n - ES_n \geq \epsilon) &\leq Me^{-t\epsilon}E[\exp(t\sum_{i=1}^{n}(X_i - EX_i))] \\ &\leq M\exp(-t\epsilon + \frac{t^2}{2}e^{2tc_n}B_n^2). \end{aligned} \tag{3.11}$$

Take $t = \epsilon/(eB_n^2)$ then $2tc_n \leq 1$. Hence it follows from (3.11) that

$$P(S_n - ES_n \geq \epsilon) \leq M\exp(-\frac{\epsilon^2}{2eB_n^2}). \tag{3.12}$$

Let $-S_n = T_n = \sum_{i=1}^{n}(-X_n)$. Since $\{-X_n, n \geq 1\}$ is also a sequence of ELNQD random variables we also have

$$P(S_n - ES_n \leq -\epsilon) = P(T_n - ET_n \geq \epsilon) \leq M\exp(-\frac{\epsilon^2}{2eB_n^2}) \tag{3.13}$$

by (3.12). Combining (3.12) and (3.13) we get (3.10). □

COROLLARY 3.5. *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed ELNQD random variables. Assume that there exists a positive integer n_0 such that $|X_i| \leq c_n$, for each $1 \leq i \leq n$, $n \geq n_0$, where $\{c_n, n \geq 1\}$ is a sequence of positive numbers. Then for any $\epsilon > 0$ such that $\epsilon \leq eEX_1^2/(2c_n)$ and $n \geq n_0$, there exists a constant $M > 0$ such that*

$$P(|S_n - ES_n| \geq n\epsilon) \leq 2M\exp(-\frac{n\epsilon^2}{2eEX_1^2}). \tag{3.14}$$

THEOREM 3.6. *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed ELNQD random variables. Assume that there exists a positive integer n_0 such that $|X_i| \leq c_n$, for each $1 \leq i \leq n$, $n \geq n_0$, where $\{c_n, n \geq 1\}$ is a sequence of positive numbers satisfying*

$$0 < c_n \leq (\frac{enEX_1^2}{8})^{\frac{1}{3}}.$$

If $\epsilon_n = (2eEX_1^2 c_n/n)^{\frac{1}{2}}$*, then for* $n \geq n_0$*, there exists a constant* $M > 0$ *such that*

$$P(\frac{1}{n}|S_n - ES_n| \geq \epsilon_n) \leq 2Me^{-c_n}. \tag{3.15}$$

Proof. Clearly, $2\epsilon c_n \leq eEX_1^2$ and $n\epsilon_n^2/(2eEX_1^2) = c_n$.

By Corollary 3.5 we have that for $n \geq n_0$, there exists a constant $M > 0$ such that

$$P(\frac{1}{n}|S_n - ES_n| \geq \epsilon_n) \leq 2M\exp(-\frac{n\epsilon_n^2}{2eEX_1^2}) = 2Me^{-c_n}.$$

Hence the proof is complete. □

Next, we consider the complete convergence and almost sure convergence for ELNQD sequences.

Theorem 3.7. *Let* $\{X_n, n \geq 1\}$ *be a sequence of ELNQD random variables with* $EX_i = 0$ *and* $|X_i| \leq b$*, for each* $i \geq 1$*, where* b *is a positive constant. Then, for any* $r > 0$

$$\sum_{n=1}^{\infty} P(|S_n| > n^r\epsilon) < \infty. \tag{3.16}$$

Proof. Let $B = \sum_{n=1}^{\infty} EX_n^2 < \infty$. For any $\epsilon > 0$, it follows from Theorem 3.2 that for a constant $M > 0$

$$\begin{aligned}\sum_{n=1}^{\infty} P(|S_n| > n^r\epsilon) &\leq 2M\sum_{n=1}^{\infty}\exp(-\frac{n^{2r}\epsilon^2}{2(2B + bn^r\epsilon)}) \\ &\leq 2M\sum_{n=1}^{\infty}[\exp(-c)]^{n^r} < \infty\end{aligned}$$

where c is a positive number not depending on n, which implies (3.16). □

Taking $c_n = \delta \ln n$ and $\delta > 1$ in Theorem 3.7 we get the following result.

Theorem 3.8. *Let* $\{X_n, n \geq 1\}$ *be a sequence of identically distributed ELNQD random variables. Assume that there exists a positive integer* n_0 *such that* $|X_i| \leq \delta \ln n$*, for each* $1 \leq i \leq n$*,* $n \geq n_0$ *and some* $\delta > 1$*. Denote* $\epsilon_n = (2\delta eEX_1 \ln n/n)^{\frac{1}{2}}$*. Then*

$$\sum_{n=1}^{\infty} P(\frac{1}{n}|S_n - ES_n| \geq \epsilon_n) < \infty. \tag{3.17}$$

Theorem 3.9. *Let $\{X_n, n \geq 1\}$ be a sequence of ELNQD random variables with $|X_i| \leq c < \infty$, for each $i \geq 1$, where c is a positive constant. Then, for any $r > \frac{1}{2}$*

$$\sum_{n=1}^{\infty} P(|S_n - ES_n| > n^r \epsilon) < \infty. \tag{3.18}$$

Proof. From Lemma 2.3, for any $\epsilon > 0$, there exists a constant $M > 0$ such that

$$\sum_{n=1}^{\infty} P(|S_n - ES_n| > n^r \epsilon) \leq 2M \sum_{n=1}^{\infty} [\exp(-\frac{\epsilon^2}{2c^2})]^{n^{2r-1}} < \infty,$$

which yields (3.18). Hence the proof is complete. □

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