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GEOMETRY OF HALF LIGHTLIKE SUBMANIFOLDS OF A SEMI-RIEMANNIAN SPACE FORM WITH A SEMI-SYMMETRIC METRIC CONNECTION

Dae Ho Jin*

Abstract. We study the geometry of half lightlike sbmanifolds M of a semi-Riemannian space form $\widetilde{M}(c)$ admitting a semi-symmetric metric connection subject to the conditions: (1) The screen distribution $S(TM)$ is totally umbilical (geodesic) and (2) the co-screen distribution $S(TM^{\perp})$ of M is a conformal Killing one.

1. Introduction

H. A. Hayden [3] introduced the notion of a semi-symmetric metric connection on a Riemannian manifold. K. Yano [8] studied some properties of a Riemannian manifold endowed with a semi-symmetric metric connection. T. Imai [4] found some properties of a hypersurface of a Riemannian manifold with a semi-symmetric metric connection. Z. Nakao [7] studied submanifolds of a Riemannian manifold with semi-symmetric metric connections.

The objective of this paper is the study of half lightlike version of above classical results. We focus on the geometry of half lightlike submanifolds M of a semi-Riemannian space form $\widetilde{M}(c)$ admitting a semi-symmetric metric connection subject to the conditions: (1) The screen distribution $S(TM)$ is totally umbilical and (2) the co-screen distribution $S(TM^{\perp})$ is a conformal Killing one. The reason for this geometric condition on M is due to the fact that such a class admits an integrable screen distribution and the induced Ricci tensor of M to be symmetric. In Section 2, we prove a classification theorem for such a class. This theorem shows that if the torsion vector field of $\widetilde{M}$ is tangent to M, then the local second fundamental forms B and C of M and $S(TM)$

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respectively satisfy either $B = 0$ or $C = 0$. In Section 3, we study the geometry of half lightlike submanifolds M of a semi-Riemannian space form $\widetilde{M}(c)$ admitting a semi-symmetric metric connection such that $S(TM)$ is totally geodesic and $S(TM^{\perp})$ is conformal Killing.

2. Semi-symmetric metric connection

Let $(\widetilde{M}, \widetilde{g})$ be a semi-Riemannian manifold. A connection $\widetilde{\nabla}$ on $\widetilde{M}$ is called a *semi-symmetric metric connection* [3] if it is metric, i.e., $\widetilde{\nabla}\widetilde{g} = 0$ and its torsion tensor $\widetilde{T}$ satisfies

$$\widetilde{T}(X, Y) = \pi(Y)X - \pi(X)Y, \tag{1.1}$$

for any vector fields X and Y of $\widetilde{M}$, where π is a 1-form defined by

$$\pi(X) = \widetilde{g}(X, \zeta),$$

and ζ is a vector field on $\widetilde{M}$, which called the *torsion vector field*.

It is well known [2] that the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ of half lightlike submanifolds M of a semi-Rimannian manifold of codimension 2 is a subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$. Thus there exist complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$ respectively, which called the *screen* and *co-screen distribution* on M ;

$$\begin{aligned} TM &= Rad(TM) \oplus_{orth} S(TM), \\ TM^{\perp} &= Rad(TM) \oplus_{orth} S(TM^{\perp}), \end{aligned} \tag{1.2}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a half lightlike submanifold by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of any vector bundle E over M. Consider the orthogonal complementary distribution $S(TM)^{\perp}$ to $S(TM)$ in $T\widetilde{M}$. Certainly $TM^{\perp}$ is a subbundle of $S(TM)^{\perp}$. As $S(TM^{\perp})$ is a non-degenerate subbundle of $S(TM)^{\perp}$, the orthogonal complementary distribution $S(TM^{\perp})^{\perp}$ to $S(TM^{\perp})$ in $S(TM)^{\perp}$ is also a non-degenerate distribution. Clearly $Rad(TM)$ is a subbundle of $S(TM^{\perp})^{\perp}$. Choose $L \in \Gamma(S(TM^{\perp}))$ as a unit vector field with $\widetilde{g}(L, L) = \epsilon = \pm 1$. For any null section ξ of $Rad(TM)$, there exists a uniquely defined null vector field $N \in \Gamma(S(TM^{\perp})^{\perp})$ satisfying

$$\widetilde{g}(\xi, N) = 1, \ \widetilde{g}(N, N) = \widetilde{g}(N, X) = \widetilde{g}(N, L) = 0, \ \forall\, X \in \Gamma(S(TM)).$$

Denote by $ltr(TM)$ the vector subbundle of $S(TM^\perp)^\perp$ locally spanned by N. Then we show that $S(TM^\perp)^\perp = Rad(TM) \oplus ltr(TM)$. Let $tr(TM) = S(TM^\perp) \oplus_{orth} ltr(TM)$. We call N, $ltr(TM)$ and $tr(TM)$ the *lightlike transversal vector field, lightlike transversal vector bundle* and *transversal vector bundle* of M with respect to the screen distribution $S(TM)$ respectively. Then $T\widetilde{M}$ is decomposed as follow:

$$(1.3)\quad \begin{aligned} T\widetilde{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^\perp). \end{aligned}$$

Let P be the projection morphism of TM on $S(TM)$ with respect to the decomposition (1.2). The local Gauss and Weingarten formulas of M and $S(TM)$ are given by

$$(1.4)\qquad \widetilde{\nabla}_X Y = \nabla_X Y + B(X,Y)N + D(X,Y)L,$$

$$(1.5)\qquad \widetilde{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L,$$

$$(1.6)\qquad \widetilde{\nabla}_X L = -A_L X + \phi(X)N,$$

$$(1.7)\qquad \nabla_X PY = \nabla^*_X PY + C(X,PY)\xi,$$

$$(1.8)\qquad \nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \quad \forall\, X,\, Y \in \Gamma(TM)$$

respectively, where ∇ and ∇^* are induced connections on TM and $S(TM)$ respectively, B and D are called the *local second fundamental forms* of M, C is called the *local second fundamental form* on $S(TM)$. A_N, A^*_ξ and A_L are linear operators on TM and τ, ρ and ϕ are 1-forms on TM. We say that $h(X,Y) = B(X,Y)N + D(X,Y)L$ is the *second fundamental tensor* of M. The induced connection ∇ on M is not metric and satisfies

$$(1.9)\qquad (\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y),$$

for any X, Y, $Z \in \Gamma(TM)$, where η is a 1-form on TM such that

$$(1.10)\qquad \eta(X) = \widetilde{g}(X,N),\ \forall\, X \in \Gamma(TM).$$

But the connection ∇^* is metric. Using (1.1) and (1.4), we show that

$$(1.11)\qquad T(X,Y) = \pi(Y)X - \pi(X)Y, \quad \forall\, X,\, Y \in \Gamma(TM),$$

and B and D are symmetric, where T is the torsion tensor with respect to ∇. From (1.9) and (1.11), we show that the induced connection ∇ of M is a semi-symmetric non-metric connection of M. From the facts $B(X,Y) = \widetilde{g}(\widetilde{\nabla}_X Y, \xi)$ and $D(X,Y) = \epsilon\widetilde{g}(\widetilde{\nabla}_X Y, L)$, we know that B and D are independent of the choice of $S(TM)$ and satisfy

$$(1.12)\qquad B(X,\xi) = 0,\ D(X,\xi) = -\epsilon\phi(X),\ \forall\, X \in \Gamma(TM).$$

The above three local second fundamental forms are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \qquad \widetilde{g}(A^*_\xi X, N) = 0, \tag{1.13}$$

$$C(X,PY) = g(A_N X, PY), \qquad \widetilde{g}(A_N X, N) = 0, \tag{1.14}$$

$$\epsilon D(X,Y) = g(A_L X, Y) - \phi(X)\eta(Y), \quad \widetilde{g}(A_L X, N) = \epsilon\rho(X), \tag{1.15}$$

for all $X,\, Y \in \Gamma(TM)$. By (1.13) and (1.14), we show that A^*_ξ and A_N are $\Gamma(S(TM))$-valued shape operators related to B and C respectively and A^*_ξ is self-adjoint on TM and

$$A^*_\xi \xi = 0. \tag{1.16}$$

Denote by $\widetilde{R}$, R and R^* the curvature tensors of the semi-symmetric metric connection $\widetilde{\nabla}$ on $\widetilde{M}$, the induced connection ∇ on M and the induced connection ∇^* on $S(TM)$ respectively. Using the Gauss-Weingarten equations (1.4)$\sim$(1.8) for M and $S(TM)$, we obtain the Gauss-Codazzi equations for M and $S(TM)$:

$$\begin{aligned}\widetilde{g}(\widetilde{R}(X,Y)Z,\, PW) \;=\; & g(R(X,Y)Z,\, PW) \\ & + B(X,Z)C(Y,PW) - B(Y,Z)C(X,PW) \\ & + \epsilon\{D(X,Z)D(Y,PW) - D(Y,Z)D(X,PW)\},\end{aligned} \tag{1.17}$$

$$\begin{aligned}\widetilde{g}(\widetilde{R}(X,Y)Z,\, \xi) \;=\; & (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ & + [\tau(X) - \pi(X)]B(Y,Z) - [\tau(Y) - \pi(Y)]B(X,Z) \\ & + \phi(X)D(Y,Z) - \phi(Y)D(X,Z),\end{aligned} \tag{1.18}$$

$$\begin{aligned}\widetilde{g}(\widetilde{R}(X,Y)Z,\, N) \;=\; & \widetilde{g}(R(X,Y)Z,\, N) \\ & + \epsilon\{\rho(Y)D(X,Z) - \rho(X)D(Y,Z)\},\end{aligned} \tag{1.19}$$

$$\begin{aligned}\epsilon\widetilde{g}(\widetilde{R}(X,Y)Z,\, L) \;=\; & (\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z) \\ & + \rho(X)B(Y,Z) - \rho(Y)B(X,Z) \\ & + \pi(Y)D(X,Z) - \pi(X)D(Y,Z),\end{aligned} \tag{1.20}$$

$$\begin{aligned}\widetilde{g}(\widetilde{R}(X,Y)\xi,\, N) \;=\; & g(A^*_\xi X,\, A_N Y) - g(A^*_\xi Y,\, A_N X) \\ & - 2d\tau(X,Y) + \rho(X)\phi(Y) - \rho(Y)\phi(X),\end{aligned} \tag{1.21}$$

$$\begin{aligned}g(R(X,Y)PZ,\, PW) \;=\; & g(R^*(X,Y)PZ,\, PW) \\ & + C(X,PZ)B(Y,PW) - C(Y,PZ)B(X,PW),\end{aligned} \tag{1.22}$$

$$\begin{aligned}\widetilde{g}(R(X,Y)PZ,\, N) \;=\; & (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\ & + [\tau(Y) + \pi(Y)]C(X,PZ) - [\tau(X) + \pi(X)]C(Y,PZ),\end{aligned} \tag{1.23}$$

for all $X,\, Y,\, Z,\, W \in \Gamma(TM)$.

The *Ricci curvature tensor*, denoted by $\widetilde{Ric}$, of $\widetilde{M}$ is defined by

$$\widetilde{Ric}(X,Y) = trace\{Z \to \widetilde{R}(Z,X)Y\},$$

for any X, $Y \in \Gamma(T\widetilde{M})$. Let $\dim \widetilde{M} = m+3$. Locally, $\widetilde{Ric}$ is given by

$$\widetilde{Ric}(X,Y) = \sum \epsilon_i \, \widetilde{g}(\widetilde{R}(E_i,X)Y,\, E_i), \tag{1.24}$$

where $\{E_1, \ldots, E_{m+3}\}$ is an orthonormal frame field of $T\widetilde{M}$ and ϵ_i ($= \pm 1$) denotes the causal character of respective vector field E_i. If the Ricci tensor $\widetilde{Ric}$ is of the form

$$\widetilde{Ric} = \widetilde{\kappa}\widetilde{g}, \quad \widetilde{\kappa} \text{ is a smooth function on } \widetilde{M},$$

then $\widetilde{M}$ is called to be an *Einstein manifold.* If $\widetilde{M}$ is connected Einstein manifold with $\dim(\widetilde{M}) = 2$, then $\widetilde{\kappa}$ is a constant. A semi-Riemannian manifold $\widetilde{M}$ of constant curvature c is called a *space form*, denote it by $\widetilde{M}(c)$. Then the curvature tensor $\widetilde{R}$ of $\widetilde{M}$ is given by

$$\widetilde{R}(X,Y)Z = c\{\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y\}, \quad \forall\, X,\, Y,\, Z \in \Gamma(T\widetilde{M}). \tag{1.25}$$

In general, $S(TM)$ is not necessarily integrable. The following result gives equivalent conditions for the integrability of $S(TM)$:

Theorem 2.1. *Let M be a half lightlike submanifold of a semi-Riemannian manifold $\widetilde{M}$ admitting a semi-symmetric metric connection. The following assertions are equivalent :*
(1) The screen distribution $S(TM)$ is an integrable distribution.
(2) C is symmetric, i.e., $C(X,Y) = C(Y,X)$ for all $X, Y \in \Gamma(S(TM))$.
(3) The shape operator A_N is self-adjoint with respect to g, i.e.,

$$g(A_N X, Y) = g(X, A_N Y), \quad \forall\, X,\, Y \in \Gamma(S(TM)).$$

Proof. First, note that a vector field X on M belongs to $S(TM)$ if and only if we have $\eta(X) = 0$. Next, by using (1.7) and (1.11), we have

$$C(X,Y) - C(Y,X) = \eta([X,Y]), \quad \forall\, X,\, Y \in \Gamma(S(TM)),$$

which implies the equivalence of (1) and (2). Finally, the equivalence of (2) and (3) follows from the first equation of (1.14) [denote $(1.14)_1$]. □

3. Totally umbilical screen distributions

Let $R^{(0,2)}$ denote the induced Ricci type tensor on M given by

$$R^{(0,2)}(X,Y) = trace\{Z \to R(Z,X)Y\}, \quad \forall\, X,\, Y \in \Gamma(TM). \tag{2.1}$$

Consider the induced quasi-orthonormal frame $\{\xi; W_a\}$ on M, where $Rad(TM) = Span\{\xi\}$ and $S(TM) = Span\{W_a\}$ and let $\{\xi, W_a; L, N\}$ be the corresponding frame field on $\widetilde{M}$. Using (1.24) and (2.1), we get

$$\begin{aligned}(2.2)\qquad \widetilde{Ric}(X,Y) = \sum_{a=1}^{m} \epsilon_a\, \widetilde{g}(\widetilde{R}(W_a, X)Y,\, W_a) \;+\; \widetilde{g}(\widetilde{R}(\xi, X)Y,\, N)\\ +\, \epsilon\, \widetilde{g}(\widetilde{R}(L, X)Y,\, L) \;+\; \widetilde{g}(\widetilde{R}(N, X)Y,\, \xi).\end{aligned}$$

$$(2.3)\qquad R^{(0,\,2)}(X,Y) = \sum_{a=1}^{m} \epsilon_a\, g(R(W_a, X)Y,\, W_a) + \widetilde{g}(R(\xi, X)Y,\, N).$$

Substituting (1.17) and (1.19) in (2.2) and using (1.13)∼(1.15) and (2.3), for any $X,\, Y \in \Gamma(TM)$, we obtain

$$\begin{aligned}(2.4)\qquad R^{(0,\,2)}(X,Y) = \widetilde{Ric}(X,Y) + B(X,Y)trA_{_N} + D(X,Y)trA_{_L}\\ -\, g(A_{_N}X,\, A^{*}_{\xi}Y) - \epsilon\, g(A_{_L}X,\, A_{_L}Y) + \rho(X)\phi(Y)\\ -\, \widetilde{g}(\widetilde{R}(\xi, Y)X,\, N) - \epsilon\, \widetilde{g}(\widetilde{R}(L, Y)X,\, L).\end{aligned}$$

This shows that $R^{(0,\,2)}$ is not symmetric. $R^{(0,\,2)}$ is called the *induced Ricci tensor*, denoted by Ric, of M if it is symmetric.

Using (1.21), (2.4) and the first Bianchi's identity, we obtain

$$R^{(0,\,2)}(X,\, Y) - R^{(0,\,2)}(Y,\, X) = 2d\tau(X,\, Y).$$

THEOREM 3.1. [5]. *Let M be a half lightlike submanifold of a semi-Riemannian manifold $\widetilde{M}$ admitting a semi-symmetric metric connection. Then the Ricci type tensor $R^{(0,\,2)}$ is symmetric if and only if the 1-form τ is closed, i.e., $d\tau = 0$, on any $\mathcal{U} \subset M$.*

If $\widetilde{M}$ is a semi-Riemannian space form $\widetilde{M}(c)$, then we have

$$\widetilde{R}(\xi, Y)X = c\widetilde{g}(X, Y)\xi, \quad \widetilde{R}(L, X)Y = c\widetilde{g}(X, Y)L$$

and $\widetilde{Ric}(X,Y) = (m+2)c\,\widetilde{g}(X,Y)$. Thus we obtain

$$\begin{aligned}(2.5)\qquad R^{(0,\,2)}(X,Y) = \; mcg(X,Y) + B(X,Y)trA_{_N} + D(X,Y)trA_{_L}\\ -\, g(A_{_N}X,\, A^{*}_{\xi}Y) - \epsilon\, g(A_{_L}X,\, A_{_L}Y) + \rho(X)\phi(Y).\end{aligned}$$

A vector field X on $\widetilde{M}$ is said to be a *conformal Killing vector field* [5] if $\widetilde{\mathcal{L}}_X\widetilde{g} = 2\alpha\widetilde{g}$ for any smooth function α, where $\widetilde{\mathcal{L}}_X$ denotes the Lie derivative with respect to X, that is,

$$(\widetilde{\mathcal{L}}_X\widetilde{g})(Y,Z) = X(\widetilde{g}(Y,Z)) - \widetilde{g}([X,Y],Z) - \widetilde{g}(Y,[X,Z]),$$

for all $X, Y, Z \in \Gamma(T\widetilde{M})$. In particular, if $\alpha = 0$, then X is called a *Killing vector field*. A distribution $\mathcal{G}$ on $\widetilde{M}$ is called a *conformal Killing* (or *Killing*) *distribution* if each vector field belonging to $\mathcal{G}$ is a conformal Killing (or Killing) vector field.

THEOREM 3.2. [5]. *Let M be a half lightlike submanifold of a semi-Riemannian manifold $\widetilde{M}$ admitting a semi-symmetric metric connection. If $S(TM^{\perp})$ is a conformal Killing distribution, then there exists a smooth function $\delta = -\{\alpha + \pi(L)\}$ such that*

$$D(X, Y) = \epsilon\delta\, g(X, Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{2.6}$$

Moreover, if $S(TM^{\perp})$ is a Killing distribution, then the equation (2.6) *holds and the function δ is given by $\delta = -\pi(L)$.*

DEFINITION 3.3. We say that the screen distribution $S(TM)$ of M is *totally umbilical* [2] in M if, on any coordinate neighborhood $\mathcal{U} \subset M$, there is a smooth function γ such that

$$C(X, PY) = \gamma\, g(X, Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{2.7}$$

In case $\gamma = 0$ on $\mathcal{U}$, we say that $S(TM)$ is *totally geodesic* in M.

For the rest of this paper, by M *is screen totally umbilical* we shall mean *the screen distribution $S(TM)$ is totally umbilical in M*.

THEOREM 3.4. *Let M be a screen totally umbilical half lightlike submanifold of a semi-Riemannian space form $\widetilde{M}(c)$ admitting a semi-symmetric metric connection and a conformal Killing co-screen distribution. Then $R^{(0,2)}$ is a symmetric Ricci tensor of M.*

Proof. Assume that $S(TM^{\perp})$ is a conformal Killing distribution. From $(1.12)_2$ and (2.6), we show that $\phi = 0$. From this result, (2.5) and the facts $A_{N}X = \gamma PX$ and A^{*}_{ξ} is self-adjoint, we deduce that $R^{(0,2)}$ is a symmetric Ricci tensor of M. □

Assume that $S(TM)$ is totally umbilical in M and $S(TM^{\perp})$ is conformal Killing on $\widetilde{M}$. Then (1.18) and (1.20) reduce to

$$\begin{aligned}(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) &= B(X, Z)\{\tau(Y) - \pi(Y)\}\\ &\quad - B(Y, Z)\{\tau(X) - \pi(X)\},\end{aligned} \tag{2.8}$$

$$\begin{aligned}(\nabla_X D)(Y, Z) - (\nabla_Y D)(X, Z) &= \rho(Y)B(X, Z) - \rho(X)B(Y, Z)\\ &\quad - \pi(Y)D(X, Z) + \pi(X)D(Y, Z).\end{aligned} \tag{2.9}$$

Applying ∇_Z to (2.7) and using (1.9), we have

$$(\nabla_X C)(Y, PZ) = X[\gamma]g(Y, PZ) + \gamma B(X, PZ)\eta(Y).$$

Substituting this equation into (1.23) and using (2.7), we get

$$\begin{aligned}\widetilde{g}(R(X,Y)PZ,\, N) \;=\; &\gamma\{B(X,PZ)\eta(Y) - B(Y,PZ)\eta(X)\}\\ &+ \{X[\gamma] - \gamma\tau(X) - \gamma\pi(X)\}g(Y,PZ)\\ &- \{Y[\gamma] - \gamma\tau(Y) - \gamma\pi(Y)\}g(X,PZ).\end{aligned}$$

Substituting this result, (1.25) and (2.6) into (1.19), we obtain

$$\begin{aligned}&\{X[\gamma] - \gamma\tau(X) - \gamma\pi(X) - \delta\rho(X) - c\eta(X)\}g(Y,PZ)\\ &\quad - \{Y[\gamma] - \gamma\tau(Y) - \gamma\pi(Y) - \delta\rho(Y) - c\eta(Y)\}g(X,PZ)\\ &= \gamma\{B(Y,PZ)\eta(X) - B(X,PZ)\eta(Y)\}, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM).\end{aligned}$$

Replacing Y by ξ to this equation and using $(1.12)_1$, we have

$$\gamma B(X,Y) = \{\xi[\gamma] - \gamma\tau(\xi) - \gamma\pi(\xi) - \delta\rho(\xi) - c\}g(X,Y). \tag{2.10}$$

THEOREM 3.5. *Let M be a screen totally umbilical half lightlike submanifold of a semi-Riemannian space form $\widetilde{M}^{m+3}(c), m > 2$, admitting a semi-symmetric metric connection and a conformal Killing co-screen distribution. If the torsion vector field ζ of $\widetilde{M}$ is tangent to M, then the local second fundamental forms B and C of M and $S(TM)$ respectively satisfy either $B = 0$ or $C = 0$ on any $\mathcal{U} \subset M$. Moreover we show that*

1. *$C = 0$ on any $\mathcal{U} \subset M$ implies that $S(TM)$ is a totally geodesical distribution,*
2. *$B = 0$ on any $\mathcal{U} \subset M$ implies that M is totally umbilical immersed in $\widetilde{M}(c)$ and the induced connection ∇ on M is a semi-symmetric metric connection.*

Proof. Assume that $\gamma \neq 0$: As ζ is tangent to M, (2.10) reduce to

$$B(X,Y) = \beta g(X,Y),\ \forall\, X,\, Y \in \Gamma(TM), \tag{2.11}$$

where $\beta = \gamma^{-1}(\xi[\gamma] - \gamma\tau(\xi) - \delta\rho(\xi) - c)$. Since $S(TM)$ is totally umbilical in M, by Theorem 2.1 $S(TM)$ is integrable. Let M^* be a leaf of $S(TM)$ and Ric^* be the symmetric Ricci tensor of M^*. From (1.17), (1.22), (1.25), (2.6), (2.7) and (2.11), we have

$$\begin{aligned}R^*(X,Y)Z &= (c + 2\beta\gamma + \epsilon\delta^2)\{g(Y,Z)X - g(X,Z)Y\},\\ Ric^*(X,Y) &= (c + 2\beta\gamma + \epsilon\delta^2)(m-1)g(X,Y).\end{aligned}$$

Thus M^* is an Einstein semi-Riemannian manifold of constant curvature $(c + 2\beta\gamma + \epsilon\delta^2)$ as $m > 2$. Differentiating (2.6) and (2.11) and using (2.8) and (2.9), for any $X,\, Y,\, Z \in \Gamma(S(TM))$, we have

$$\begin{aligned}&\{X[\beta] + \beta\tau(X) - \beta\pi(X) - \beta^2\eta(X)\}g(Y,Z)\\ &\quad = \{Y[\beta] + \beta\tau(Y) - \beta\pi(Y) - \beta^2\eta(Y)\}g(X,Z),\end{aligned}$$

$$\{X[\delta] + \epsilon\beta\rho(X) - \delta\pi(X) - \beta\delta\eta(X)\}g(Y, Z) \\ = \{Y[\delta] + \epsilon\beta\rho(Y) - \delta\pi(Y) - \beta\delta\eta(Y)\}g(X, Z).$$

From (2.11) and the last two equations, we have

$$\xi[\beta] = \beta^2 - \beta\tau(\xi), \quad \xi[\delta] = \beta\delta - \epsilon\beta\rho(\xi), \quad \xi[\gamma] = \beta\gamma + \gamma\tau(\xi) + \delta\rho(\xi) + c,$$

due to $\pi(\xi) = 0$. Since $(c + 2\beta\gamma + \epsilon\delta^2)$ is a constant, we get

$$0 = \xi[c + 2\beta\gamma + \epsilon\delta^2] = 2\beta(c + 2\beta\gamma + \epsilon\delta^2).$$

As $(c + 2\beta\gamma + \epsilon\delta^2)$ is a constant, we have $\beta = 0$ or $c + 2\beta\gamma + \epsilon\delta^2 = 0$. If $c + 2\beta\gamma + \epsilon\delta^2 = 0$, then M^* is a semi-Euclidean space. As the second fundamental form C of the totally umbilical semi-Euclidean space M^* as a submanifold of the semi-Riemannian space form $\widetilde{M}(c)$ vanishes [1, Section 2.3], we get $\gamma = 0$. It is a contradiction to $\gamma \neq 0$. Thus $\beta = 0$, i.e., $B = 0$. In this case, from (2.6) and (2.11), the second fundamental tensor h of M is given by $h = \mathcal{H}\, g$, where $\mathcal{H} = \beta N + \epsilon\delta L = \epsilon\delta L$ is the curvature vector field on M. Thus M is totally umbilical. As $B = 0$, we have $\nabla_X g = 0$ by (1.9). From this result and (1.11), we see that the induced connection ∇ on M is a semi-symmetric metric connection. $\square$

4. Totally geodesic screen distributions

THEOREM 4.1. *Let M be a screen totally geodesic half lightlike submanifold of a semi-Riemannian space form $\widetilde{M}(c)$ admitting a semi-symmetric metric connection and a conformal Killing co-screen distribution. Then $c + \delta\rho(\xi) = 0$ and M is an Einstein manifold. Moreover if $m > 1$, then the function δ, given by* (2.6)*, is a constant.*

Proof. As $C = 0$, we have $\widetilde{g}(R(X, Y)PZ,\, N) = 0$ due to (1.23). Using this, (1.19) and (2.6), we have

$$\widetilde{g}(\widetilde{R}(X, Y)PZ,\, N) = \delta\{g(X, PZ)\rho(Y) - g(Y, PZ)\rho(X)\}.$$

By Theorem 3.1 and Theorem 3.3, we get $d\tau = 0$ on TM. Thus we have $\widetilde{g}(\widetilde{R}(X, Y)\xi,\, N) = 0$ due to (1.21). From the above results, we deduce the following equation

$$\widetilde{g}(\widetilde{R}(X, Y)Z,\, N) = \delta\{g(X, Z)\rho(Y) - g(Y, Z)\rho(X)\}. \tag{3.1}$$

Replacing X by ξ and Z by X to (3.1) and using (1.25), we have

$$\{c + \delta\rho(\xi)\}g(X, Y) = 0, \quad \forall\, X,\, Y \in \Gamma(TM).$$

Thus we have $c + \delta\rho(\xi) = 0$. Substituting the relations $\phi = A_N = 0$, $tr\, A_L = m\delta + \epsilon\rho(\xi)$ and $A_L X = \delta PX + \epsilon\rho(X)\xi$ into (2.5) and using $c + \delta\rho(\xi) = 0$, we obtain

$$R^{(0,2)}(X, Y) = (m-1)(c + \epsilon\delta^2)\, g(X, Y), \quad \forall\, X,\, Y \in \Gamma(TM).$$

Thus M is Einstein manifold and δ is constant as $\dim M = m + 1 > 2$. □

THEOREM 4.2. *Let M be a screen totally geodesic half lightlike submanifold of a semi-Riemannian space form $\widetilde{M}(c)$ admitting a semi-symmetric metric connection and a Killing co-screen distribution. If the torsion vector field ζ of $\widetilde{M}$ is tangent to M, then $c = 0$, and M is a space of constant curvature* 0*, i.e., M is a flat manifold.*

Proof. Assume that ζ is tangent to M. Then we have $\pi(L) = 0$. Thus the conformal factor α is equal to $-\delta$. As $S(TM^{\perp})$ is a Killing distribution, we have $\alpha = 0$. Thus $\delta = 0$ and $c = 0$ due to $c + \delta\rho(\xi) = 0$. From (1.17), (1.19) and the fact $C = D = 0$, we have

$$g(R(X, Y)Z,\, W) = 0, \quad \widetilde{g}(R(X, Y)Z,\, N) = 0.$$

The Riemannian curvature tensor R of M is given by

$$R(X, Y)Z = \sum_{a=1}^{m} \epsilon_a g(R(X, Y)Z,\, W_a)W_a + \widetilde{g}(R(X, Y)Z, N)\xi \;=\; 0.$$

Therefore M is a space of constant curvature 0, i.e., M is flat. □

THEOREM 4.3. *Let M be a screen totally geodesic half lightlike submanifold of a semi-Riemannian space form $\widetilde{M}^{m+3}(c)$, $m > 1$, admitting a semi-symmetric metric connection and a conformal Killing co-screen distribution. If $\delta \neq 0$, then the torsion vector field ζ belongs to $S(TM)^{\perp}$ and the 1-form π satisfies $\pi(X) = \pi(\xi)\eta(X)$ for all $X \in \Gamma(TM)$.*

Proof. As $m > 1$, δ is constant. Comparing (1.25) and (3.1), we get

$$\{c\eta(X) + \delta\rho(X)\}g(Y, Z) = \{c\eta(Y) + \delta\rho(Y)\}g(X, Z),$$

for all X, Y, $Z \in \Gamma(TM)$. Taking $X = PX$, $Y = PY$ and $Z = PZ$ in this equation and using the fact $S(TM)$ is non-degenerate, we have

$$\delta\rho(PX)\, PY = \delta\rho(PY)\, PX, \quad \forall\, X,\, Y \in \Gamma(TM).$$

Suppose there exists a vector field $X_o \in \Gamma(TM_{|\mathcal{U}})$ such that $\delta\rho(PX_o) \neq 0$ at a point $p \in M$. It follows that all vectors from the fibre $S(TM)_p$ are collinear with $(PX_o)_p$. It is a contradiction as $m > 1$. Thus we have $\delta\rho(PX) = 0$ for any $X \in \Gamma(TM)$ and

$$\delta\rho(X) = \delta\rho(PX + \eta(X)\xi) = \delta\rho(\xi)\eta(X) = -c. \tag{3.2}$$

Assume that δ does not vanishes. Then we have $\rho(PX) = 0$ and

$$\rho(X) = -(c/\delta)\eta(X), \quad \forall\, X \in \Gamma(TM).$$

Substituting (2.6) into (2.9) and using the fact δ is a constant, we have

$$\begin{aligned}&\{\delta\eta(Y) - \epsilon\rho(Y)\}B(X,Z) - \{\delta\eta(X) - \epsilon\rho(X)\}B(Y,Z)\\ &= \delta\{\pi(X)g(Y,Z) - \pi(Y)g(X,Z)\}, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM).\end{aligned} \tag{3.3}$$

Taking $X = PX$, $Y = PY$ and $Z = PZ$ in this equation and using the facts $\rho(PX) = 0$, $\delta \neq 0$ and $S(TM)$ is non-degenerate, we have

$$\pi(PX)\, PY = \pi(PY)\, PX, \quad \forall\, X,\, Y \in \Gamma(TM).$$

As $m > 1$, $\pi(PX) = 0$ and $\pi(X) = \pi(PX + \eta(X)\xi) = \pi(\xi)\eta(X)$ for any $X \in \Gamma(TM)$. From the decomposition (1.3), the torsion vector field ζ is decomposed by

$$\zeta = \omega + \pi(N)\xi + \pi(\xi)N + \epsilon\pi(L)L,$$

where ω is a smooth vector field on $S(TM)$ and $\pi(N)\xi + \pi(\xi)N + \epsilon\pi(L)L \in \Gamma(S(TM)^{\perp})$. Taking the scalar product with X to the last equation and using $\pi(X) = \pi(\xi)\eta(X)$, we get $g(\omega,\, X) = 0$ for all $X \in \Gamma(TM)$. As $S(TM)$ is non-degenerate, we have $\omega = 0$. This implies that ζ belongs to $S(TM)^{\perp}$. □

Corollary 4.4. *Let M be a screen totally geodesic half lightlike submanifold of a semi-Riemannian space form $\widetilde{M}^{m+3}(c)$, $m > 1$, admitting a semi-symmetric metric connection and a conformal Killing co-screen distribution with non-vanishing conformal factor. If the torsion vector field ζ is tangent to M, then the 1-form π vanishes identically on TM.*

Proof. If ζ is tangent to M, then we have $\pi(\xi) = g(\zeta,\, \xi) = 0$. Thus $\pi(X) = 0$ for all $X \in \Gamma(TM)$ due to $\pi(X) = \pi(\xi)\eta(X)$. In case ζ is tangent to M, we know that $\alpha = -\delta$. Thus if the conformal factor α does not vanishes, then we have $\delta \neq 0$. □

The *type number* $t^*(x)$ of M at any point x is the rank of A^*_ξ.

Theorem 4.5. *Let M be a screen totally geodesic half lightlike submanifold of a semi-Riemannian space form $\widetilde{M}^{m+3}(c)$, $m > 1$, admitting a semi-symmetric metric connection and a conformal Killing co-screen* *distribution. If the torsion vector field ζ is tangent to M and the type number t^* satisfies $t^*(x) \geq 1$ for any $x \in M$, then M is a flat manifold.*

Proof. Under the assumption of this theorem, we have

$$g(R(X,Y)Z,\, PW) = (c + \epsilon\delta^2)\{g(Y,Z)g(X,PW) - g(X,Z)g(Y,PW)\}, \tag{3.4}$$

$$\widetilde{g}(R(X,Y)Z,\, N) = \{c\eta(X) + \delta\rho(X)\}g(Y,Z) - \{c\eta(Y) + \delta\rho(Y)\}g(X,Z). \tag{3.5}$$

for all $X, Y, Z, W \in \Gamma(TM)$. Due to (3.2), we have $\widetilde{g}(R(X,Y)Z,\, N) = 0$. Replacing Y by ξ to (3.3), we obtain

$$\{\delta - \epsilon\rho(\xi)\}B(X,Y) = 0, \quad \forall\, X,\, Y \in \Gamma(TM).$$

As δ is a constant and $c + \delta\rho(\xi) = 0$, $\rho(\xi)$ and $\delta - \epsilon\rho(\xi)$ are constants. Assume that $t^*(x) \geq 1$ for any $x \in M$. Then we have $\delta - \epsilon\rho(\xi) = 0$. This implies $c + \epsilon\delta^2 = 0$ due to $c + \delta\rho(\xi) = 0$. From this and (3.4). we get $g(R(X,Y)Z,\, PW) = 0$. Thus M is a flat manifold. □

Definition 4.6. M is said to be *irrotational*[6] if $\widetilde{\nabla}_X\xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$, i.e., $D(X,\xi) = 0 = \phi(X)$ for all $X \in \Gamma(TM)$.

Remark 4.7. Instead of the condition $S(TM^{\perp})$ *is conformal Killing distribution* of Theorem 4.1 $\sim$ 4.4, even though we use the condition M *is irrotational* of Definition 4.6 given above, it is easy to find that we can establish the same results Theorem 4.1 $\sim$ 4.4 except Theorem 4.2.

References

[1] B. Y. Chen, *Geometry of Submanifolds*, Marcel Dekker, New York, 1973.

[2] K. L. Duggal and D. H. Jin, *Half-lightlike submanifolds of codimension* 2, Math. J. Toyama Univ. **22** (1999), 121-161.

[3] H. A. Hayden, *Subspace of a space with torsion*, Proc. London Math. Soc. **34** (1932), 27-50.

[4] T. Imai, *Hypersurfaces of a Riemannian manifold with semi-symmetric metric connection*, Tensor, N. S., 23, 1972, 300-306.

[5] D. H. Jin and J. W. Lee, *Einstein half lightlike submanifolds of a Lorentzian space form with a semi-symmetric metric connection*, to appear Mediterranean J. of Mathematics.

[6] D. N. Kupeli, *Singular Semi-Riemannian Geometry*, Mathematics and Its Applications, Kluwer Acad. Publishers, Dordrecht, 1996.

[7] Z. Nakao, *Submanifolds of a Riemannian manifold with semi-symmetric metric connection*, Proc. Amer. Math. Soc. **54** (1976), 261-266.

[8] K. Yano, *On semi-symmetric metric connection*, Rev. Roumaine Math. Pures Appl. **15** (1970), 1579-1586.

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Department of Mathematics
Dongguk University
Gyeongju 780-714, Republic of Korea
E-mail: jindh@dongguk.ac.kr