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ON AN INTERIOR METRIC SPACE

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Abstract. In this paper, we present the proof of the property for interior metric space and geodesic space.

1. Introduction

Alexandrov space is a locally compact and complete space with an interior metric and curvature bounded above or below κ and introduced by A. D. Alexandrov. The Busemann *G-space* are special Alexandrov spaces admitting geodesic completeness, where the notion of curvature bounded above or below are defined by a similar manner. The most important problem discussed by these pioneers was if the differentiability assumption in Riemannian results is really essential. Now, many geometers focuss on this viewpoint and study a metric space. Alexandrov space is determined by a given curvature κ. Then the curvature depends completely on the metric. Therefore the geometric objects as length, area, angle, and volume etc. are determined by a given metric. Specially, the all metrics are *interior* metric. An *interior* metric space is one in which the distance between any two points is the infimum of the lenghs of curves joining them, where curvelengh is defined as usual ; the terms *inner* and *tight* have also been used. *Geodesic space* were first considered by Alexandrov[2], who defined upper curvature bounds for such spaces and gave a development method for transforming local curvature bounds into global ones.

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In this paper, we prove the theorem for interior metric space and survey the property of a main example called *bipartite graph*. These tell us the property of interior metric space.

2. Interior metric

DEFINITION 1. *A metric on a set X is said to be* interior *if for every $x, y \in X$ and for each $\epsilon > 0$ there exists an $\epsilon -$ midpoint$z$ between $x$and y, that is*

$$|xz|, |zy| \leq \frac{1}{2}|xy| + \epsilon.$$

In other words, $B_x(\frac{1}{2} + \epsilon) \cap B_y(\frac{1}{2} + \epsilon) \neq \emptyset$.

DEFINITION 2. *The* dilatation *of a map $f : X \rightarrow Y$ of metric spaces is defined to be*

$$dil(f) = sup_{x \neq y}|\frac{|f(x)f(y)|}{|xy|}.$$

The dilatation at $x \in X$ is defined to be

$$dil_x(f) = \lim_{\epsilon \longrightarrow 0} dil(f|_{B_\epsilon(x)}).$$

LEMMA 1. *For every x, y in a space X with an interior metric and for each $\delta > 0$, there exists a map*

$$z : \{ \text{ dyadic rationals in } [0, 1]\} \rightarrow X$$

with properties
(1) $z(0) = x, z(1) = y$
(2) $|z(\frac{k}{2^n})z(\frac{k+1}{2^n})| \leq \frac{1}{2^n}(|xy| + \delta)$, for all $n \geq 1$
and for all $k = 0, \cdots, 2^n - 1$.

Proof. We use an induction method. We put (1) and assume that z is already defined on rationals $\frac{k}{2^{n-1}}, k = 0, \cdots, 2^{n-1}$ with a condition stronger than (2).

Then we have

$$|z(\frac{k}{2^{n-1}})z(\frac{k+1}{2^{n-1}})| \leq \frac{1}{2^{n-1}}\{(|xy| + \delta(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}})\}.$$

For $k = 0, \cdots, 2^{n-1} - 1$, we can find $z(\frac{2k+1}{2^n})$ such that

$$|z(\frac{k}{2^{n-1}})z(\frac{2k+1}{2^n})| \leq \frac{1}{2}|z(\frac{k}{2^{n-1}})z(\frac{k+1}{2^{n-1}})| + \frac{\delta}{2^{2n}},$$

$$|z(\frac{2k+1}{2^n})z(\frac{k+1}{2^{n-1}})| \leq \frac{1}{2}|z(\frac{k}{2^{n-1}})z(\frac{k+1}{2^{n-1}})| + \frac{\delta}{2^{2n}}.$$

But, $\frac{1}{2}|z(\frac{k}{2^{n-1}})z(\frac{2k+1}{2^{n-1}})| + \frac{\delta}{2^{2n}} \leq \frac{1}{2^n}(|xy| + \delta(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n})$.
Hence,

$$|z(\frac{k}{2^{n-1}})z(\frac{2k+1}{2^n})| \leq \frac{1}{2^n}(|xy| + \delta'),$$

$$|z(\frac{2k+1}{2^n})z(\frac{k+1}{2^{n-1}})| \leq \frac{1}{2^n}(|xy| + \delta').$$

□

REMARK 1. *Above lemma is very useful in a construction method when we deal with interior metric space. This lemma tells us as follows ; Although we cannot take a midpoint exactly, an interior metric is sufficiently complementary.*
Also, this reflects the property of an interior metric to be local.

THEOREM 1. *For a interior metric on X we have*

$$dil(f) = sup_{x \in X} dil_x(f).$$

Proof. Assume that $x \neq y$ and $\delta > 0$. Since X has an interior metric, there is a function z satisfying (1),(2) in lemma.
For $k = 0$, $|xz(\frac{1}{2^n})| \leq \frac{1}{2^n}(|xy| + \delta)$, for all $n \geq 1$.
Thus for some $\epsilon > 0$, $z(\frac{1}{2^n}) \in B_{\frac{1}{2^n}(|xy|+\delta+\epsilon)}(x)$ and

$$(1)\ \frac{|f(x)f(y)|}{|xy|} \leq \frac{|f(x)f(y)|}{2^n|xz(\frac{1}{2^n})| - \delta}, \text{ for fixed } n \geq 1 \text{ and any } \delta > 0.$$

We take a δ such that

$$\delta \leq \frac{2\{|f(x)f(z(\frac{1}{2^n}))| - |f(x)f(y)|\} \cdot |xz(\frac{1}{2^n})|}{|f(x)f(z(\frac{1}{2^n}))|} \tag{2}$$

We can assume that
$|f(x)f(z(\frac{1}{2^n}))| = \max\{|f(z(\frac{k}{2^n}))f(z(\frac{k+1}{2^n}))|, k = 1, \cdots, 2^n - 1\}$ and

$$\begin{aligned} |f(x)f(y)| &\leq |f(x)f(z(\frac{1}{2^n}))| + \cdots + |f(z(\frac{2^n-1}{2^n}))y| \\ &\leq n \cdot \max\{|f(z(\frac{k}{2^n}))f(z(\frac{k+1}{2^n}))|, k = 0, \cdots, 2^n - 1\} \end{aligned}$$

In (1), substitution δ for δ in (2) provides

$$\frac{|f(x)f(y)|}{|xy|} \leq \frac{|f(x)f(z(\frac{1}{2^n}))|}{|xz(\frac{1}{2^n})|}.$$

If we continue above procedure, then we have

$$\frac{|f(x)f(y)|}{|xy|} \leq \lim_{n\to\infty} dil(f|_{B_{\frac{1}{2^n}(|xy|+\delta+\epsilon)}}(x)), \text{ for sufficiently small } \delta > 0.$$

Therefore,

$$\begin{aligned} dil(f) &= sup_{x\neq y}\frac{|f(x)f(y)|}{|xy|} \\ &\leq sup_{x\in X} \lim_{n\to\infty} dil(f|_{B_{\frac{1}{2^n}(|xy|+\delta+\epsilon)}}(x)) \\ &= sup_{x\in X} dil_x(f). \end{aligned}$$

$dil(f) \geq sup_{x\in X} dil_x(f)$ is trivial by the definition. □

3. Geodesic space

Definition 3. *A metric on X is said to be strictly interior if every $x, y \in X$ posses a midpoint z, that is,*

$$|xz| = |zy| = \frac{1}{2}|xy|.$$

Definition 4. *A geodesic in a metric space X is a locally-homothetic map $\gamma : D \to X$, that is, for some $v \geq 0$ every $t \in D$ possesses a neighborhood $U \subset D$ such that*

$$|\gamma(t')\gamma(t^{''})| = v|t' -^{t} {}''|, \text{ for all } t', t^{''} \in U.$$

If one can take $U = D$ then the geodesic γ is said to be minimizer.

Definition 5. *A metric space X is called geodesic if every two points in X can be connected by a minimizer.*

We can show that a geodesic X contains a shortest curve between any two points. A complete interior space is geodesic if it is compact[5], but might not be otherwise.

Example. Let X be a graph with two vertices and edges $e_n, n \geq 1$, between them such that the length of e_n is equal to $1 + \frac{1}{n}$. This space is called *bipartite graph.* Define the interior d on X by $d(a, b) = inf_{\gamma} L(\gamma)$, where $L(\gamma)$ is the length of γ and the infimum is taken over all graph γ connecting a and b. Then X is complete but not locally compact. Furthermore, X is not geodesic.

Proof. Let (X_n) be a Cauchy sequence on X as above metric. Then (X_n) is one of two cases. First, for a sufficiently large $n \geq N$, X_n are dense on an edge e_i, since the lengh of a path e_i crossing a vertex cannot be less than ϵ. Hence, Cauchy sequence converges. Secondly, (X_n) goes to each vertex. This Cauchy sequence converges to each vertex. Hence, X is complete. Since two vertices x, y cannot be covered by finite sets, X is not locally compact at each vertex. X contains a pair of points (two vertices) not joined by a shortest curve. Therfore, X is not geodesic. □

Theorem 2. *If X is complete and strictly interior, then it is geodesic.*

Proof. Since X is strictly interior, there is a map

$$\gamma : \{ \text{ dyadic rationals in } [0,1]\} \to X$$

such that for every pairs $x, y \in X$

$$(1) \qquad \gamma(0) = x, \gamma(1) = y$$

$$(2) \qquad |\gamma(\frac{k}{2^n})\gamma(\frac{k+1}{2^n})| = \frac{1}{2^n}|xy|, \text{ for all } n \geq 1 \text{ and for all } k = 0, \cdots, 2^n - 1.$$

Extend γ to a continuous map $\gamma' : [0,1] \to X$ by

$$\gamma'(t) = \begin{cases} \gamma(t), & \text{if } t \text{ is a dyadic rational} \\ \lim \gamma(t_i), & \text{if } t \text{ is not a dyadic rational} \end{cases}$$

where $\{t_i\}$ is a sequence of dyadic rationals converging to y.

Then γ' is a minimizer connecting x and y. □

References

1. S. B. Alexander and R. L. Bishop, The Hadamard-Cartan theorem in locally convex metric spaces, *Enseign. Math.*, 36(1990), 309-320.
2. A. D. Alexandrov A theorem on triangles in a metric space and some of its applications, *Trudy Math. Inst. Steks.*, 38(1951), 5-23.
3. W. Ballmann, *Lectures on spaces of nonpositive curvature*, DMV Seminar. Band 25, Basel ,Birkhäuser ,1995.
4. W. Ballmann, M. Gromov, and V. Schroeder, *Manifolds of nonpositive curvature*, Birkhäuser , Boston, Basel, 1990.
5. H. Busemann, *The Geometry of Geodesics,* Academic Press, New York, San Francisco, London, 1955.
6. S. Buyalo, Lectures on spaces of curvature bounded above, preprint

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