

## STABILITY OF A CUBIC FUNCTIONAL EQUATION IN $p$ -BANACH SPACES

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ABSTRACT. In this paper, we investigate the stability of a cubic functional equation

$$\begin{aligned} f(x+ny) + f(x-ny) + f(nx) \\ = n^2 f(x+y) + n^2 f(x-y) + (n^3 - 2n^2 + 2)f(x) \end{aligned}$$

in  $p$ -Banach spaces and in Banach modules, where  $n \geq 2$  is an integer.

### 1. Introduction

In 1940, S.M. Ulam [12] gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms:

*Let  $G_1$  be a group and let  $G_2$  be a metric group with the metric  $d(\cdot, \cdot)$ . Given  $\varepsilon > 0$ , does there exist a  $\delta > 0$  such that if a function  $h : G_1 \rightarrow G_2$  satisfies the inequality  $d(h(xy), h(x)h(y)) < \delta$  for all  $x, y \in G_1$  then there is a homomorphism  $H : G_1 \rightarrow G_2$  with  $d(h(x), H(x)) < \varepsilon$  for all  $x \in G_1$ ?*

The case of approximately additive mappings was solved by D. H. Hyers [5] under the assumption that  $G_1$  and  $G_2$  are Banach spaces. In 1978, Th. M. Rassias [9] gave a generalization of the Hyers' result. In 1994, P. Găvruta [4] also obtained a further generalization of the Rassias' result.

We first recall some basic facts concerning quasi-Banach spaces and some preliminary results.

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DEFINITION 1.1. ([1, 10]) Let  $X$  be a linear space. A *quasi-norm*  $\|\cdot\|$  is a real-valued function on  $X$  satisfying the following:

- (i)  $\|x\| \geq 0$  for all  $x \in X$  and  $\|x\| = 0$  if and only if  $x = 0$ .
- (ii)  $\|\lambda x\| = |\lambda| \cdot \|x\|$  for all scalar  $\lambda$  and all  $x \in X$ .
- (iii) There is a constant  $K \geq 1$  such that  $\|x + y\| \leq K(\|x\| + \|y\|)$  for all  $x, y \in X$ .

A quasi-normed space is a linear space together with a specified quasi-norm. A *quasi-Banach space* means a complete quasi-normed space. A quasi-norm  $\|\cdot\|$  is called a *p-norm* ( $0 < p \leq 1$ ) if the inequality

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

holds for all  $x, y \in X$ . In this case, a quasi-Banach space is called a *p-Banach space*. Clearly, *p*-norms are continuous, and in fact, if  $\|\cdot\|$  is a *p*-norm on  $X$ , then the formula  $d(x, y) := \|x - y\|^p$  defines a translation invariant metric for  $X$  and  $\|\cdot\|^p$  is a *p*-homogeneous *F*-norm. The Aoki-Rolewicz theorem [10] (see also [1, 8]) yields that each quasi-norm is equivalent to some *p*-norm for some  $0 < p \leq 1$ . Since it is much easier to work with *p*-norms than quasi-norms, henceforth we restrict our attention mainly to *p*-norms. In [11], J. Tabor has investigated a version of the Hyers-Rassias-Gajda theorem (see [3]) in quasi-Banach spaces.

The cubic function  $f(x) = cx^3$  ( $c \in \mathbb{R}$ ) satisfies the functional equation

$$(1.1) \quad f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x).$$

In this paper, we will prove the Hyers-Ulam-Rassias stability of the cubic functional equation

$$(1.2) \quad f(x + ny) + f(x - ny) + f(nx) = n^2 f(x + y) + n^2 f(x - y) + (n^3 - 2n^2 + 2)f(x)$$

in *p*-Banach spaces and in Banach modules, where  $n \geq 2$  is an integer.

## 2. Results in *p*-Banach spaces

From this section, let  $X$  be a linear space and  $Y$  a *p*-Banach space. For a mapping  $f : X \rightarrow Y$ , we define

$$Df(x, y) := f(x + ny) + f(x - ny) + f(nx) - n^2 f(x + y) - n^2 f(x - y) - (n^3 - 2n^2 + 2)f(x)$$

for all  $x, y \in X$ .

Now we have the stability of the equation (1.2).

THEOREM 2.1. Let  $f : X \rightarrow Y$  be a mapping for which there exists a function  $\varphi : X \times X \rightarrow [0, \infty)$  such that

$$\tilde{\varphi}(x, y) := \sum_{i=1}^{\infty} \frac{1}{n^{3pi}} \varphi(n^{i-1}x, n^{i-1}y)^p < \infty$$

and

$$(2.1) \quad \|Df(x, y)\| \leq \varphi(x, y)$$

for all  $x, y \in X$ . Then there exists a unique cubic mapping  $C : X \rightarrow Y$  satisfying (1.2) such that

$$(2.2) \quad \|f(x) - C(x)\| \leq \tilde{\varphi}(x, 0)^{\frac{1}{p}}$$

for all  $x \in X$ .

*Proof.* Letting  $y = 0$  in (2.1), we gain

$$\left\| f(x) - \frac{1}{n^3} f(nx) \right\|^p \leq \frac{1}{n^{3p}} \varphi(x, 0)^p$$

for all  $x \in X$ . Replacing  $x$  by  $nx$  in the above inequality and then dividing by  $n^{3p}$ , we get

$$\left\| \frac{1}{n^3} f(nx) - \frac{1}{n^6} f(n^2x) \right\|^p \leq \frac{1}{n^{6p}} \varphi(nx, 0)^p$$

for all  $x \in X$ . Adding the foregoing two inequalities, we have

$$\left\| f(x) - \frac{1}{n^6} f(n^2x) \right\|^p \leq \frac{1}{n^{3p}} \varphi(x, 0)^p + \frac{1}{n^{6p}} \varphi(nx, 0)^p$$

for all  $x \in X$ . Continuing in this way, one can obtain that

$$(2.3) \quad \left\| f(x) - \frac{1}{n^{3r}} f(n^r x) \right\|^p \leq \sum_{i=1}^r \frac{1}{n^{3pi}} \varphi(n^{i-1}x, 0)^p$$

for all  $x \in X$  and all  $r \in \mathbb{N}$ . For  $s = 1, 2, 3, \dots$ , dividing the preceding inequality by  $n^{3ps}$  and then substituting  $x$  by  $n^s x$ , we see that

$$\left\| \frac{1}{n^{3s}} f(n^s x) - \frac{1}{n^{3(s+r)}} f(n^{s+r} x) \right\|^p \leq \sum_{i=1}^r \frac{1}{n^{3pi}} \frac{1}{n^{3ps}} \varphi(n^{i+s-1}x, 0)^p$$

for all  $x \in X$  and all  $r \in \mathbb{N}$ . Taking  $s \rightarrow \infty$  in the previous inequality, we conclude that  $\left\{ \frac{1}{n^{3r}} f(n^r x) \right\}_{r=1}^{\infty}$  is a Cauchy sequence in the  $p$ -Banach space  $Y$  for all  $x \in X$ . This implies that the sequence  $\left\{ \frac{1}{n^{3r}} f(n^r x) \right\}_{r=1}^{\infty}$

converges in  $Y$  for all  $x \in X$ . Thus we can define a function  $C : X \rightarrow Y$  by

$$C(x) := \lim_{r \rightarrow \infty} \frac{1}{n^{3r}} f(n^r x)$$

for all  $x \in X$ . Then

$$\begin{aligned} \|DC(x, y)\|^p &= \lim_{r \rightarrow \infty} \frac{1}{n^{3pr}} \|Df(n^r x, n^r y)\|^p \\ &\leq \lim_{r \rightarrow \infty} \frac{1}{n^{3pr}} \varphi(n^r x, n^r y)^p = 0 \end{aligned}$$

for all  $x, y \in X$ . Hence the mapping  $C : X \rightarrow Y$  satisfies (1.2). By [6], the mapping  $C$  also satisfies 1.1). Taking  $r \rightarrow \infty$  in (2.3), we get the inequality (2.2).

It only remains to show the uniqueness of the cubic mapping  $C : X \rightarrow Y$ . Let  $C' : X \rightarrow Y$  be another cubic mapping satisfying (1.2) and (2.2). Then

$$\|C(x) - C'(x)\|^p = \frac{1}{n^{3pr}} \|C(n^r x) - C'(n^r x)\|^p \leq \frac{2}{n^{3pr}} \tilde{\varphi}(n^r x, 0)$$

for all  $x \in X$ . Taking  $r \rightarrow \infty$  in the above inequality, we conclude that  $C(x) = C'(x)$  for all  $x \in X$ .  $\square$

**COROLLARY 2.2.** *Let  $X$  be a quasi-normed space. Let  $\theta, p, q$  be real numbers with  $\theta \geq 0$ ,  $0 < p < 3$  and  $q \neq 0$  and let  $f : X \rightarrow Y$  be a mapping such that*

$$\|Df(x, y)\| \leq \theta(\|x\|^p + \|y\|^q)$$

*for all  $x, y \in X$ . Then there exists a unique cubic mapping  $C : X \rightarrow Y$  satisfying (1.2) such that*

$$\|f(x) - C(x)\| \leq \frac{\theta}{(n^{3p} - n^{p^2})^{\frac{1}{p}}} \|x\|^p$$

*for all  $x \in X$ .*

**THEOREM 2.3.** *Let  $f : X \rightarrow Y$  be a mapping for which there exists a function  $\varphi : X \times X \rightarrow [0, \infty)$  such that*

$$\tilde{\varphi}(x, y) := \sum_{i=1}^{\infty} n^{3p(i-1)} \varphi\left(\frac{x}{n^i}, \frac{y}{n^i}\right)^p < \infty$$

*and*

$$\|Df(x, y)\| \leq \varphi(x, y)$$

for all  $x, y \in X$ . Then there exists a unique cubic mapping  $C : X \rightarrow Y$  satisfying (1.2) such that

$$\|f(x) - C(x)\| \leq \tilde{\varphi}(x, 0)^{\frac{1}{p}}$$

for all  $x \in X$ .

*Proof.* The proof is similar to the proof of Theorem 2.1.  $\square$

**COROLLARY 2.4.** Let  $X$  be a quasi-normed space. Let  $\theta, p, q$  be real numbers with  $\theta \geq 0$ ,  $p > 3$  and  $q \neq 0$  and let  $f : X \rightarrow Y$  be a mapping such that

$$\|Df(x, y)\| \leq \theta(\|x\|^p + \|y\|^q)$$

for all  $x, y \in X$ . Then there exists a unique cubic mapping  $C : X \rightarrow Y$  satisfying (1.2) such that

$$\|f(x) - C(x)\| \leq \frac{\theta}{(n^{p^2} - n^{3p})^{\frac{1}{p}}} \|x\|^p$$

for all  $x \in X$ .

### 3. Results in Banach modules over a Banach algebra

In this section, let  $A$  be a unital Banach algebra with norm  $|\cdot|$ ,  $A_1 := \{a \in A : |a| = 1\}$ , and let  ${}_A\mathcal{M}$  and  ${}_A\mathcal{N}$  be left Banach  $A$ -modules with norms  $\|\cdot\|$  and  $\|\cdot\|$ , respectively.

For a mapping  $f : {}_A\mathcal{M} \rightarrow {}_A\mathcal{N}$ , we define

$$\begin{aligned} D_a f(x, y) &:= a^3 f(x + ny) + a^3 f(x - ny) + a^3 f(nx) \\ &\quad - n^2 f(ax + ay) - n^2 f(ax - ay) - (n^3 - 2n^2 + 2)f(ax) \end{aligned}$$

for all  $a \in A_1$  and all  $x, y \in {}_A\mathcal{M}$ .

**DEFINITION 3.1.** A cubic mapping  $C : {}_A\mathcal{M} \rightarrow {}_A\mathcal{N}$  is called  $A$ -cubic if  $C(ax) = a^3 C(x)$  for all  $a \in A$  and all  $x \in {}_A\mathcal{M}$ .

From now on, let  $A$  be a complex unital Banach  $*$ -algebra with norm  $|\cdot|$ .

**THEOREM 3.2.** Let  $f : {}_A\mathcal{M} \rightarrow {}_A\mathcal{N}$  be a mapping satisfying  $f(0) = 0$  for which there exists a function  $\phi : ({}_A\mathcal{M})^2 \rightarrow [0, \infty)$  such that

$$(3.1) \quad \tilde{\phi}(x, y) = \sum_{j=0}^{\infty} \frac{1}{n^{3j}} \phi(n^j x, n^j y) < \infty$$

and

$$(3.2) \quad \|D_a f(x, y)\| \leq \phi(x, y)$$

for all  $a \in A_1$  and all  $x, y \in {}_A\mathcal{M}$ . If  $f(tx)$  is continuous in  $t \in \mathbb{R}$  for each fixed  $x \in {}_A\mathcal{M}$ , then there exists a unique  $A$ -cubic mapping  $C : {}_A\mathcal{M} \rightarrow {}_A\mathcal{N}$  such that

$$(3.3) \quad \|f(x) - C(x)\| \leq \frac{1}{n^3} \tilde{\phi}(x, 0)$$

for all  $x \in {}_A\mathcal{M}$ .

*Proof.* Similar to the proof of [9, Theorem 3.1], there exists a unique cubic mapping  $C : {}_A\mathcal{M} \rightarrow {}_A\mathcal{N}$  satisfying (3.3). In fact, the mapping  $C$  is given by

$$C(x) = \lim_{r \rightarrow \infty} \left( \frac{1}{n^3} \right)^r f(n^r x)$$

for all  $x \in {}_A\mathcal{M}$ , where  $a = 1$ . Under the assumption that  $f(tx)$  is continuous in  $t \in \mathbb{R}$  for each fixed  $x \in {}_A\mathcal{M}$ , by the same reasoning as the proof of [2], the cubic mapping  $C : {}_A\mathcal{M} \rightarrow {}_A\mathcal{N}$  satisfies  $C(tx) = t^3 C(x)$  for all  $t \in \mathbb{R}$  and all  $x \in {}_A\mathcal{M}$ . That is,  $C$  is  $\mathbb{R}$ -cubic. By letting  $x = n^r x$  and  $y = 0$  in (3.2), we have

$$\left\| f(n^r ax) - \frac{1}{n^3} a^3 f(n^{r+1} x) \right\| \leq \frac{1}{n^3} \phi(n^r x, 0)$$

for all  $a \in A_1$  and all  $x \in {}_A\mathcal{M}$ . Note that for each  $a \in A$  and each  $z \in {}_A\mathcal{M}$ , we have

$$\|az\| \leq K|a| \cdot \|z\|$$

for some  $K > 0$ . Then we get

$$\begin{aligned} & \|f(n^r ax) - a^3 f(n^r x)\| \\ & \leq \left\| f(n^r ax) - \frac{1}{n^3} a^3 f(n^{r+1} x) \right\| + \left\| \frac{1}{n^3} a^3 f(n^{r+1} x) - a^3 f(n^r x) \right\| \\ & \leq \frac{1}{n^3} \phi(n^r x, 0)(1 + K) \end{aligned}$$

for all  $a \in A_1$  and all  $x \in {}_A\mathcal{M}$ . Hence, for all  $a \in A_1$  and all  $x \in {}_A\mathcal{M}$ ,  $n^{-3r} \|f(n^r ax) - a^3 f(n^r x)\| \rightarrow 0$  as  $r \rightarrow \infty$ . We may conclude that

$$C(ax) = \lim_{r \rightarrow \infty} \frac{f(n^r ax)}{n^{3r}} = a^3 \lim_{r \rightarrow \infty} \frac{f(n^r x)}{n^{3r}}$$

for all  $a \in A_1$  and all  $x \in {}_A\mathcal{M}$ . Note that  $C$  is  $\mathbb{R}$ -cubic and  $C(ax) = a^3C(x)$  for all  $a \in A_1$ . Thus we have

$$C(ax) = C\left(|a| \cdot \frac{a}{|a|}x\right) = |a|^3C\left(\frac{a}{|a|}x\right) = a^3C(x)$$

for all nonzero  $a \in A$  and all  $x \in {}_A\mathcal{M}$ . Also,  $C(0x) = 0^3C(x)$  for all  $x \in {}_A\mathcal{M}$ . Thus the cubic mapping  $C : {}_A\mathcal{M} \rightarrow {}_A\mathcal{N}$  is the unique  $A$ -cubic mapping as desired.  $\square$

#### 4. Results in Banach modules over a $C^*$ -algebra

In this section, we will investigate the stability of the given cubic functional equation (1.2) over a  $C^*$ -algebra. Throughout this section, let  $A$  be a unital  $C^*$ -algebra with a norm  $|\cdot|$ , and let  ${}_A\mathcal{M}$  and  ${}_A\mathcal{N}$  be left Banach  $A$ -modules with norms  $\|\cdot\|$  and  $\|\cdot\|$ , respectively. Put  $A_1 := \{a \in A \mid |a| = 1\}$ ,  $A_{in} := \{a \in A \mid a \text{ is invertible in } A\}$ ,  $A_{sa} := \{a \in A \mid a^* = a\}$ ,  $\mathcal{U}(A) := \{a \in A \mid aa^* = a^*a = 1\}$ ,  $A^+ := \{a \in A_{sa} \mid Sp(a) \subset [0, \infty)\}$  and  $A_1^+ := A_1 \cap A^+$ .

For explicitly later use, we state the following lemma.

LEMMA 4.1. [7] *Let  $a \in A$  and  $|a| < 1 - \frac{2}{k}$  for some integer  $k$  greater than 2. Then there are unitary elements  $u_1, \dots, u_k \in A$  such that  $ka = u_1 + \dots + u_k$ .*

THEOREM 4.2. *Let  $A$  be of real rank 0 and let  $f : {}_A\mathcal{M} \rightarrow {}_A\mathcal{N}$  be a mapping satisfying  $f(0) = 0$  for which there exists a function  $\phi : {}_A\mathcal{M} \times {}_A\mathcal{M} \rightarrow [0, \infty)$  satisfying (3.1) and (3.2) for all  $a \in (A_1^+ \cap A_{in}) \cup \{i\}$  and all  $x, y \in {}_A\mathcal{M}$ . For each fixed  $x \in {}_A\mathcal{M}$ , let the sequence  $\left\{\frac{f(n^r x)}{n^{3r}}\right\}$  converge uniformly on  $A_1$ . If  $f(ax)$  is continuous in  $a \in A_1 \cup \mathbb{R}$  for each fixed  $x \in {}_A\mathcal{M}$  then there exists a unique  $A$ -cubic mapping  $C : {}_A\mathcal{M} \rightarrow {}_A\mathcal{N}$  such that*

$$(4.1) \quad \|f(x) - C(x)\| \leq \frac{1}{n^3} \tilde{\phi}(x, 0)$$

for all  $x \in {}_A\mathcal{M}$ .

*Proof.* Similar to the proof of Theorem 3.2, there exists a unique cubic mapping  $C : {}_A\mathcal{M} \rightarrow {}_A\mathcal{N}$  satisfying (4.1) and

$$(4.2) \quad C(ax) = a^3C(x)$$

for all  $a \in (A_1^+ \cap A_{in}) \cup \{i\}$  and all  $x \in {}_A\mathcal{M}$ . From the continuity and the uniform convergence, one can show that  $C(ax)$  is continuous in  $a \in A_1$  for each  $x \in {}_A\mathcal{M}$ .

Let  $b \in A_1^+ \setminus A_{in}$ . Since  $A_{in} \cap A_{sa}$  is dense in  $A_{sa}$ , there exists a sequence  $\{b_m\}$  in  $A_{in} \cap A_{sa}$  such that  $b_m \rightarrow b$  as  $m \rightarrow \infty$ . Put  $d_m := \frac{1}{|b_m|} b_m$ . Then  $d_m \rightarrow b$  as  $m \rightarrow \infty$ . Put  $a_m = \sqrt{d_m^* d_m}$ . Then  $a_m \rightarrow b$  as  $m \rightarrow \infty$  and  $a_m \in A_1^+ \cap A_{in}$ . By the continuity of  $C$ ,

$$(4.3) \quad \lim_{m \rightarrow \infty} C(d_m x) = C\left(\lim_{m \rightarrow \infty} d_m x\right) = C(bx)$$

for all  $x \in {}_A\mathcal{M}$ . By (4.2),

$$\|C(a_m x) - b^3 C(x)\| = \|a_m^3 C(x) - b^3 C(x)\| \rightarrow \|b^3 C(x) - b^3 C(x)\| = 0$$

as  $m \rightarrow \infty$  for all  $x \in {}_A\mathcal{M}$ . By (4.3) and the above equation,

$$\begin{aligned} \|C(bx) - b^3 C(x)\| &\leq \|C(bx) - C(a_m x)\| + \|C(a_m x) - b^3 C(x)\| \\ &\rightarrow 0 \quad \text{as } m \rightarrow \infty \end{aligned}$$

for all  $x \in {}_A\mathcal{M}$ . By (4.2) and the above equation,  $C(ax) = a^3 C(x)$  for all  $a \in A_1^+ \cup \{i\}$  and all  $x \in {}_A\mathcal{M}$ .

The rest of the proof is similar to the proof of Theorem 3.2, which completes the proof.  $\square$

**THEOREM 4.3.** *Let  $A$  be of real rank 0, commutative. Let  $D := \{a \in A \mid Sp(a) \subset \mathbb{C} \setminus [0, \infty)\}$ ,  $E := \{a \in A \mid Sp(a) \subset \mathbb{C} \setminus (-\infty, 0]\}$  and  $D \cup E$  is dense in  $A_{in}$ . And let  $f : {}_A\mathcal{M} \rightarrow {}_A\mathcal{N}$  be a mapping satisfying  $f(0) = 0$  for which there exists a function  $\phi : {}_A\mathcal{M} \times {}_A\mathcal{M} \rightarrow [0, \infty)$  satisfying (3.1) and (3.2) for all  $a \in \exp(\mathcal{U}(A)) \cup \{1\}$  and all  $x, y \in {}_A\mathcal{M} \setminus \{0\}$ . For each fixed  $x \in {}_A\mathcal{M}$ , let the sequence  $\left\{\frac{f(n^r x)}{n^{3r}}\right\}$  converge uniformly on  $A_1$ . If  $f(ax)$  is continuous in  $a \in A_1 \cup \mathbb{R}$  for each fixed  $x \in {}_A\mathcal{M}$ , then there exists a unique  $A$ -cubic mapping  $C : {}_A\mathcal{M} \rightarrow {}_A\mathcal{N}$  such that*

$$(4.4) \quad \|f(x) - C(x)\| \leq \frac{1}{n^3} \tilde{\phi}(x, 0)$$

for all  $x \in {}_A\mathcal{M} \setminus \{0\}$ .

*Proof.* By the same reasoning as in the proof of Theorem 3.2, there exists a unique cubic mapping  $C : {}_A\mathcal{M} \rightarrow {}_A\mathcal{N}$  satisfying the inequality (4.4) for all  $x \in {}_A\mathcal{M}$ . By a similar method to the proof of Theorem 3.2, the cubic mapping  $C$  satisfies  $C(ax) = a^3 C(x)$  for all  $a \in \exp(\mathcal{U}(A)) \cup \{1\}$  and all  $x \in {}_A\mathcal{M}$ .

For every element  $a \in A_1 \cap D$ , there is a positive integer  $m$  such that

$$\left| \frac{1 + \log a}{m} \right| < 1 - \frac{2}{m}.$$

By Lemma 4.1, there are unitary element  $v_1, \dots, v_m \in \mathcal{U}(A)$  such that  $1 + \log a = v_1 + \dots + v_m$ . Then we have

$$\begin{aligned} C(eax) &= C(e^{1+\log a}x) = C(e^{v_1+\dots+v_m}x) = C(e^{v_1} \dots e^{v_m}x) \\ &= e^{3v_1} \dots e^{3v_m}C(x) = (e^{v_1+\dots+v_m})^3C(x) \\ &= (e^{1+\log a})^3C(x) = e^3a^3C(x) \end{aligned}$$

for all  $a \in A_1 \cap D$  and all  $x \in {}_A\mathcal{M} \setminus \{0\}$ . Hence we obtain  $e^3C(ax) = C(eax) = e^3a^3C(x)$  for all  $a \in A_1 \cap D$  and all  $x \in {}_A\mathcal{M} \setminus \{0\}$ . Thus  $C(ax) = a^3C(x)$  for all  $a \in A_1 \cap D$  and all  $x \in {}_A\mathcal{M}$ .

By the same process as the above argument, one can obtain that  $C(ax) = a^3C(x)$  for all  $a \in A_1 \cap E$  and all  $x \in {}_A\mathcal{M}$ . Since  $D \cup E$  is dense in  $A_{in}$ ,  $C(ax) = a^3C(x)$  for all  $a \in A_1 \cap A_{in}$  and all  $x \in {}_A\mathcal{M}$ .

The rest of the proof is the same as in the proof of Theorem 4.2, which completes the proof.  $\square$

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