

DIFFEOMORPHISMS WITH ROBUSTLY AVERAGE SHADOWING

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ABSTRACT. In this paper, we prove that for C^1 generically, if every hyperbolic periodic point in a chain component is uniformly far away from being nonhyperbolic, and it is C^1 -robustly average shadowable, then the chain component is hyperbolic.

1. Introduction

Let M be a closed manifold, and let $\text{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1 -topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM . Let $f \in \text{Diff}(M)$.

For $\delta > 0$ a sequence $\{x_i\}_{i=-\infty}^{\infty}$ in M is called a δ -average pseudo orbit of $f \in \text{Diff}(M)$ if there is a natural number $N = N(\delta) > 0$ such that for all $n \geq N$, and $k \in \mathbb{Z}$,

$$\frac{1}{n} \sum_{i=1}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$

We say that f has the *average-shadowing property* if for every $\epsilon > 0$ there is a $\delta > 0$ such that every δ -average pseudo orbit $\{x_i\}_{i=-\infty}^{\infty}$ is ϵ -shadowed in average by some $z \in M$, that is,

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$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n d(f^i(z), x_i) < \epsilon.$$

Let $\Lambda \subset M$ be a closed f -invariant set. We say that $f|_{\Lambda}$ has the average shadowing property if for every $\epsilon > 0$ there is $\delta > 0$ such that for any δ -average pseudo orbit $\{x_i\}_{i=a}^b \subset \Lambda$ ($-\infty \leq a < b \leq \infty$) of f , ϵ -shadowed in average by some $z \in \Lambda$.

For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^b$ ($-\infty \leq a < b \leq \infty$) in M is called a δ -pseudo orbit of f if $d(f(x_i), x_{i+1}) < \delta$ for all $a \leq i \leq b-1$. We say that f has the shadowing property if for any $\epsilon > 0$ there is $\delta > 0$ such that for any δ -pseudo orbit $\{x_i\}_{i \in \mathbb{Z}}$ of f there is $y \in M$ such that $d(f^i(y), x_i) < \epsilon$, for $i \in \mathbb{Z}$. For given $x, y \in M$, we write $x \rightsquigarrow y$ if for any $\delta > 0$, there is a δ -pseudo orbit $\{x_i\}_{i=a}^b$ ($a < b$) of f such that $x_a = x$ and $x_b = y$. The set of points $\{x \in M : x \rightsquigarrow x\}$ is called the *chain recurrent set* of f and is denoted by $\mathcal{R}(f)$. The relation $\rightsquigarrow$ induces on $\mathcal{R}(f)$ an equivalence relation, whose classes are called *chain components* of f . Denote by $C_f(p) = \{x \in M : x \rightsquigarrow p \text{ and } p \rightsquigarrow x\}$, where p is a hyperbolic saddle. Note that if p is not hyperbolic saddle then $C_f(p) = p$. Thus in this paper, we consider that p is a hyperbolic saddle.

It is well known that if p is a hyperbolic periodic point f with period k then the sets

$$W^s(p) = \{x \in M : f^{kn}(x) \rightarrow p \text{ as } n \rightarrow \infty\} \text{ and}$$

$$W^u(p) = \{x \in M : f^{-kn}(x) \rightarrow p \text{ as } n \rightarrow \infty\}$$

are C^1 -injectively immersed submanifolds of M . Every point $x \in W^s(p)$ $\overline{\cap} W^u(p)$ is called a *homoclinic point* of f . The closure of the homoclinic points of f associated to p is called the *homoclinic class* of f and it is denoted by $H_f(p)$.

Let $\Lambda \subset M$ be a f -invariant closed set. We say that Λ admits a *dominated splitting* if the tangent bundle $T_{\Lambda}M$ has a continuous Df -invariant splitting $E \oplus F$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E(x)}\| \cdot \|D_x f^{-n}|_{F(f^n(x))}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$.

We say that Λ is *hyperbolic* if the tangent bundle $T_{\Lambda}M$ has a Df -invariant splitting $E^s \oplus E^u$ and there exists constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E_x^s}\| \leq C\lambda^n \text{ and } \|D_x f^{-n}|_{E_x^u}\| \leq C\lambda^{-n}$$

for all $x \in \Lambda$ and $n \geq 0$.

We say that a subset $\mathcal{G} \subset \text{Diff}(M)$ is *residual* if $\mathcal{G}$ contains the intersection of a countable family of open and dense subsets of $\text{Diff}(M)$, and so $\mathcal{G}$ is dense in $\text{Diff}(M)$. A property $\mathcal{P}$ is said to be *generic* (C^1) if $\mathcal{P}$ holds for all diffeomorphisms which belong to some residual subset of $\text{Diff}(M)$.

In this paper, we introduce the notion of C^1 -robustly average shadowing property, and study the cases when the chain component $C_f(p)$ of f associated to a hyperbolic periodic point p has the C^1 -robustly average shadowing property.

Now, we introduce the C^1 -interior average shadowing property.

DEFINITION 1.1. Let $C_f(p)$ be the chain component having hyperbolic a hyperbolic periodic point p . We say that $C_f(p)$ has the *C^1 -robustly average shadowing property* if there exists a C^1 -neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, $g|_{C_g(p_g)}$ has the average shadowing property, where p_g is called the *continuation* of p .

Let p be a hyperbolic periodic point of f . In [1], Bonatti and Crovisier proved that C^1 generically, a chain component $C_f(p)$ containing a periodic point p is the homoclinic class $H_f(p)$. Thus we get the the following Theorem.

The following fact is the main Theorem in this paper.

Theorem A. For C^1 -generic f , assume the chain component $C_f(p)$ is locally maximal and satisfies the following properties:

- (a) for any $g \in \mathcal{U}_0(f)$, if $q \in C_g(p_g) \cap P(g)$ has minimum period $\pi(q) \geq m$, then

$$\prod_{i=0}^{k-1} \|D_{g^{im}(q)} g^m|_{E^s|_{g^{im}(q)}}\| < K\lambda^k \text{ and}$$

$$\prod_{i=0}^{k-1} \|D_{g^{-im}(q)} g^m|_{E^u|_{g^{-im}(q)}}\| < K\lambda^k,$$

where $k = [\pi(q)/m]$, and p_g is the continuation of p .

- (b) $C_f(p)$ admits a dominated splitting $T_{C_f(p)}M = E \oplus F$ with $\dim E = \text{index}(p)$, for any $x \in C_f(p)$,

$$\|Df|_{E(x)}\|/m(Df|_{F(x)}) < \lambda^2.$$

- (c) f has the average shadowing property on $C_f(p)$.

Then $C_f(p)$ is hyperbolic for f .

2. Proof of the Theorem A

Let M be as before, and let $f \in \text{Diff}(M)$.

PROPOSITION 2.1 ([5]). *Let $\lambda \in (0, 1)$ be given. Let Λ be a compact f -invariant set with a continuous Df -invariant spitting $T_\Lambda M = E \oplus F$ such that $\|Df|_{E(x)}\|/m(Df|_{F(x)}) < \lambda^2$, for any $x \in \Lambda$. Suppose there is a point $y \in \Lambda$ satisfying*

$$\log \lambda < \log \lambda_1 = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(\|Df|_{E(f^i(y))}\|) < 0,$$

and

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(\|Df|_{E(f^i(y))}\|) < \log \lambda_1.$$

Then for any λ_2 and λ_3 with $\lambda < \lambda_2 < \lambda_1 < \lambda_3 < 1$, and any neighborhood U of Λ , there exists a hyperbolic periodic point q of index $\dim(E)$ such that its orbit $\mathcal{O}(q)$ is entirely contained in U and the derivatives along $\mathcal{O}(q)$ satisfy

$$\begin{aligned} \prod_{i=0}^{k-1} \|Df|_{E^s(f^i(q))}\| &< \lambda_3^k, \\ \prod_{i=k-1}^{\pi(q)-1} \|Df|_{E^s(f^i(q))}\| &> \lambda_2^{\pi(q)-k+1} \end{aligned}$$

for all $k = 1, 2, \dots, \pi(q)$. Furthermore, q can be chosen such that $\pi(q)$ is arbitrarily large.

LEMMA 2.2 ([3]). *Let $\Lambda \subset M$ be a closed f -invariant set and $\varphi(x)$ be a continuous function defined on Λ . For $\epsilon > 0$, there is $\delta > 0$ such that for sequences $\{x_i\}_{i \in \mathbb{N}}, \{y_i\}_{i \in \mathbb{N}}$ if*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n-1} d(x_i, y_i) < \delta,$$

then

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n-1} \|\varphi(x_i) - \varphi(y_i)\| < \epsilon.$$

LEMMA 2.3 ([3]). Let $C_f(p)$ satisfy (a)-(c) of Theorem A. Suppose that E is not contracting. Then for any $\lambda < \gamma_1 < \gamma_2 < 1$, there exists $x \in C_f(p)$ with $\mathcal{O}(x)$ such that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(\|Df|_{E(f^i(x))}\|) &< \log \gamma_1 \\ &< \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(\|Df|_{E(f^i(x))}\|) < \log \gamma_2. \end{aligned}$$

The following lemmas are well known results [5].

LEMMA 2.4. Assume Λ has a dominated splitting $E \oplus F$, and let $x \in U$ be a point such that $\mathcal{O}_f(x) \subset U$, where U is an admissible neighborhood of Λ . There is $\epsilon_1 > 0$ such that the local center unstable manifold $W_{\epsilon_1}^{cu}(x)$ is defined and is transverse to F . Such manifolds are of class C^1 . Moreover, there is a $N > 0$ such that for any $0 < \epsilon_2 < \epsilon_1$, there exists $\delta > 0$ such that $f^N(W_\delta^{cs}(x)) \subset W_{\epsilon_2}^{cs}(f^N(x))$, and $f^{-N}(W_\delta^{cu}(x)) \subset W_{\epsilon_2}^{cu}(f^{-N}(x))$.

LEMMA 2.5. Let $W_{\epsilon_1}^{cs}(x)$ and $W_{\epsilon_1}^{cu}(x)$ be as in Lemma 2.4. For any $0 < \lambda < 1$, there is $\epsilon > 0$ such that:

- (a) If $\prod_{i=0}^{n-1} \|Df|_{E(f^i(x))}\| \leq \lambda^n$ for any $n \geq 1$, then for any $y \in W_\epsilon^{cs}(x)$, $d(f^n(x), f^n(y)) \rightarrow 0$ as $n \rightarrow \infty$.
- (b) If $\prod_{i=0}^{n-1} \|Df^{-1}|_{E(f^{-i}(x))}\| \leq \lambda^n$ for any $n \geq 1$, then for any $y \in W_\epsilon^{cu}(x)$, $d(f^n(x), f^n(y)) \rightarrow 0$ as $n \rightarrow -\infty$.

Note that let Λ be a closed f -invariant set. Then if Λ is not hyperbolic then either E is not contracting or F is not expanding.

LEMMA 2.6. Let Λ be a closed f -invariant set and let $T_\Lambda M = E \oplus F$. Suppose that E is not contracting. Then there is $y \in H_f(p)$ such that

$$\prod_{i=0}^{n-1} \|Df|_{E(f^i y)}\| \geq 1$$

for all $n \geq 1$.

Proof. Assume there is no $y \in H_f(p)$ such that

$$\prod_{i=0}^{n-1} \|Df|_{E(f^i y)}\| \geq 1$$

for all $n \geq 1$. Then for any $y \in H_f(p)$ there is $n(y) \in \mathbb{N}$ such that

$$\prod_{i=0}^{n(y)-1} \|Df|_{E(f^i y)}\| < 1.$$

Denote

$$\lambda(y) = \prod_{i=0}^{n(y)-1} \|Df|_{E(f^i y)}\|.$$

We can get a neighborhood $U(y)$ of y such that

$$\prod_{i=0}^{n(y)-1} \|Df|_{E(f^i x)}\| < \frac{1 + \lambda(y)}{2} < 1.$$

for any $x \in U(y)$. By the compactness of $H_f(p)$, we can find $y_1, y_2, \dots, y_k$ contained in $H_f(p)$ such that

$$H_f(p) \subset \bigcup_{i=1}^k U(y_i).$$

Let $N = \max\{n(y_1), n(y_2), \dots, n(y_k)\}$, $\lambda = \max\{(\frac{1+\lambda(y_i)}{2})^{1/N} : i = 1, 2, \dots, k\}$, $C = \lambda^{-N} \max\{\|Df^i\| : i = 0, 1, 2, \dots, N-1\}$. Then one can check that

$$\|Df^n|_{E(x)}\| < C\lambda^n$$

for any $x \in H_f(p)$ and any $n \geq 1$. □

Proof of Proposition 2.3: Suppose that $C_f(p)$ is not hyperbolic. Then we may assume that E is not contracting. Choose a neighborhood $\{W_k\}_{k \geq 1}$ of $C_f(p)$ such that for $k \geq 1$, W_k is decreasing, $W_k \subset U$ and $\bigcap_{k \geq 1} W_k = C_f(p)$. By [1], C^1 generically $C_f(p) = H_f(p)$. Let $\varphi(x) = \log \|Df|_{E(f^i(x))}\|$, for $x \in M$. Then by Lemma 2.2 and Lemma 2.3, for any $\lambda < \gamma_1 < \gamma_2 < 1$, there is a point $x \in H_f(p)$ such that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) < \log \gamma_1 < \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) < \log \gamma_2,$$

and for any k , we can choose a hyperbolic periodic point $q_k \in W_k$. Then we use Proposition 2.1, for any $\lambda < \gamma_1 < \gamma_2 < 1$, there is a periodic

point q arbitrarily closed to $H_f(p)$ such that

$$(2.1) \quad \prod_{i=0}^{k-1} \|Df|_{E^s(f^i(q))}\| < \gamma_2^k,$$

$$(2.2) \quad \prod_{i=k-1}^{\pi(q)-1} \|Df|_{E^s(f^i(q))}\| > \gamma_1^{\pi(q)-k+1}.$$

Since a periodic point q is arbitrarily close to $H_f(p)$, we may assume q_k converge to a point $z \in H_f(p)$. Thus for every q_k , it satisfies the inequalities (2.1) and (2.2). By Lemma 2.4, there is $\epsilon > 0$ such that $W_\epsilon^{cs}(q_k) \subset W^s(q_k)$ and $W_\epsilon^{cu}(q_k) \subset W^u(q_k)$. This means that there exists $N \geq 1$ such that for any $k, m \geq N$, q_k and q_m are homoclinic related, i.e., $W^s(q_k) \cap W^u(q_m)$ and $W^u(q_k) \cap W^s(q_m)$. Therefore, $z \in H_f(\mathcal{O}(q_k))$ for all $k \geq N$. Thus $q_k \in H_f(p) = C_f(p)$ for all $k \geq N$. Thus, due to (2.2),

$$\prod_{i=k-1}^{\pi(q)-1} \|Df|_{E^s(f^i(q))}\| > \gamma_1^{\pi(q)-k+1}$$

of q which contradicts the hypothesis (a). This ends the proof of proposition. $\square$

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