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SERIES RELATIONS FROM CERTAIN MODULAR TRANSFORMATION FORMULA

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Abstract. B. C. Berndt [4, 5] evaluated several classes of infinite series and established many relations between various infinite series. In this paper, continuing his work, we derive new relations between infinite series.

1. Introduction and preliminaries

In [5], B. C. Berndt proved a transformation formula for a large class of functions that includes the classical Dedekind eta function. From this formula, he [4, 5] evaluated several classes of infinite series and found a lot of relations between various infinite series. Some of the results have been stated in the Notebooks of Ramanujan [7]. He says [5] that the flavor of all his findings on series is much like that found in the Notebooks. One of them is Ramanujan's famous formula.([7])
For α, $\beta > 0$ with $\alpha\beta = \pi^2$,

$$\begin{aligned}
&\alpha^{-N}\left\{\frac{1}{2}\zeta(2N+1)+\sum_{n=1}^{\infty}\frac{n^{-2N-1}}{e^{2\alpha n}-1}\right\} \\
&= (-\beta)^{-N}\left\{\frac{1}{2}\zeta(2N+1)+\sum_{n=1}^{\infty}\frac{n^{-2N-1}}{e^{2\beta n}-1}\right\} \\
&\quad -2^{2N}\sum_{\ell=0}^{N+1}(-1)^{\ell}\frac{B_{2\ell}}{(2\ell)!}\frac{B_{2N+2-2\ell}}{(2N+2-2\ell)!}\alpha^{N+1-\ell}\beta^{\ell},
\end{aligned} \tag{1.1}$$

where N is any positive integer, B_ℓ is the ℓth Bernoulli number and $\zeta(s)$ is the Riemann zeta function.

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Recently he suggested that one could obtain analogous results of his work. Actually, the author derived a more generalized series relation than (1.1);

Theorem 1.1. ([6]). *Let α and β be positive real numbers with $\alpha\beta = \pi^2$. Let c denote a positive integer. Then, for any integer n,*

$$\alpha^{-n}\sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2k(\alpha-i\pi)/c}-1} = (-\beta)^{-n}\sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2k(\beta+i\pi)/c}-1}$$
$$-2^{2n}\sum_{j=1}^{c}\sum_{k=0}^{2n+2}\frac{B_k(j/c)\bar{B}_{2n+2-k}(j/c)}{k!(2n+2-k)!}\alpha^{n-k+1}(-i\pi)^k + I_0(n),$$

where

$$I_0(n) := \begin{cases} \frac{1}{2}\left((-\beta)^{-n} - \alpha^{-n}\right)\zeta(1+2n), & if\ n \neq 0, \\ -\frac{1}{4}(\log\beta - \log\alpha) + \frac{1}{4}i\pi, & if\ n = 0. \end{cases}$$

If $c = 1$ in Theorem 1.1, then Ramanujan's formula follows.

In this paper, we find several new series relations between infinite series, some of which are compared with series relations in [4, 5, 6]. For example,(see corollary 2.13 and 2.14)

$$\alpha^{1/2}\sum_{n=0}^{\infty}\frac{2(-1)^n}{e^{(2n+1)\alpha}-1} + \frac{1}{2}\alpha^{1/2} = \beta^{1/2}\sum_{n=1}^{\infty}\operatorname{sech}(n\beta) + \frac{1}{2}\beta^{1/2},$$

$$\sum_{n=1}^{\infty}\operatorname{sech}(n\pi) = \sum_{n=0}^{\infty}\frac{2(-1)^n}{e^{(2n+1)\pi}-1}.$$

The notation in this paper follows those in [5]. For a complex w, we choose the branch of the argument for a complex w defined by $-\pi \leq \arg w < \pi$. Let $e(w) = e^{2\pi i w}$ and $V\tau = V(\tau) = \frac{a\tau+b}{c\tau+d}$ always denote a modular transformation with $c > 0$ for every complex τ. Let $r = (r_1, r_2)$ and $h = (h_1, h_2)$ denote real vectors, and the associated vectors R and H are defined by

$$R = (R_1, R_2) = (ar_1 + cr_2, br_1 + dr_2)$$

and

$$H = (H_1, H_2) = (dh_1 - bh_2, -ch_1 + ah_2).$$

Let λ denote the characteristic function of the integers. For a real number x, $[x]$ denotes the greatest integer less than or equal to x and $\{x\} := x - [x]$. For real α, x and $\mathrm{Re}(s) > 1$, let

$$\psi(x, \alpha, s) := \sum_{n+\alpha>0}\frac{e(nx)}{(n+\alpha)^s}. \tag{1.2}$$

If x is an integer and α is not an integer, then $\psi(x,\alpha,s) = \zeta(s,\{\alpha\})$, where $\zeta(s,x)$ is the Hurwitz zeta-function. The function $\psi(x,\alpha,s)$ can be analytically continued to the entire s-plane [2] except for a possible simple pole at $s=1$ when x is an integer. Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$, the upper half-plane. For $\tau \in \mathbb{H}$ and an arbitrary complex numbers s, define

$$A(\tau,s;r,h) := \sum_{m+r_1>0}\sum_{n-h_2>0} \frac{e\left(mh_1 + ((m+r_1)\tau + r_2)(n-h_2)\right)}{(n-h_2)^{1-s}}.$$

Let

$$H(\tau,s;r,h) := A(\tau,s;r,h) + e\left(s/2\right) A(\tau,s;-r,-h).$$

We now state the theorem which is important for our results.

THEOREM 1.2. [4]. *Let* $Q = \{\tau \in \mathbb{C} \mid \mathrm{Re}(\tau) > -d/c\}$ *and* $\varrho = c\{R_2\} - d\{R_1\}$. *Then for* $\tau \in Q$ *and all* s,

$$\begin{aligned}
&(c\tau+d)^{-s} H(V\tau,s;r,h) = H(\tau,s;R,H)\\
&-\lambda(r_1)e(-r_1h_1)(c\tau+d)^{-s}\Gamma(s)(-2\pi i)^{-s}\left(\psi(h_2,r_2,s) + e\left(s/2\right)\psi(-h_2,-r_2,s)\right)\\
&+\lambda(R_1)e(-R_1H_1)\Gamma(s)(-2\pi i)^{-s}\left(\psi(H_2,R_2,s) + e\left(-s/2\right)\psi(-H_2,-R_2,s)\right)\\
&+(2\pi i)^{-s} L(\tau,s;R,H),
\end{aligned}$$

where

$$\begin{aligned}
&L(\tau,s;R,H)\\
&:= \sum_{j=1}^{c'} e(-H_1(j+[R_1]-c) - H_2([R_2]+1+[(jd+\varrho)/c]-d))\\
&\qquad \cdot \int_C u^{s-1} \frac{e^{-(c\tau+d)(j-\{R_1\})u/c}}{e^{-(c\tau+d)u} - e(cH_1+dH_2)} \frac{e^{\{(jd+\varrho)/c\}u}}{e^u - e(-H_2)} du,
\end{aligned}$$

where C is a loop beginning at $+\infty$, proceeding in the upper half-plane, encircling the origin in the positive direction so that $u=0$ is the only zero of

$$\left(e^{-(c\tau+d)u} - e(cH_1+dH_2)\right)\left(e^u - e(-H_2)\right)$$

lying "inside" the loop, and then returning to $+\infty$ in the lower half plane. Here, we choose the branch of u^s with $0 < \arg u < 2\pi$.

Remark 1.3. Theorem 1.2 is true for $\tau \in Q$. But, after the evaluation of $L(\tau,s;R,H)$ for an integer s, it will be valid for all $\tau \in \mathbb{H}$ by analytic continuation.

We shall use two kinds of polynomials. One is the Bernoulli polynomials $B_n(x)$, $n \geq 0$, defined by

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \ (|t| < 2\pi).$$

The n-th Bernoulli number B_n, $n \geq 0$, is defined by $B_n = B_n(0)$. Put $\bar{B}_n(x) = B_n(\{x\}),\ n \geq 0$. Recall that $B_{2n+1} = 0,\ n \geq 1$, and that $B_{2n+1}(1/2) = 0,\ n \geq 0$. The following formulas are helpful [1];

$$B_n(1-x) = (-1)^n B_n(x), \tag{1.3}$$

$$B_n\left(\frac{1}{2}\right) = -(1-2^{1-n})B_n,\ n \geq 0.$$

The other is the Euler polynomials $E_n(x)$, $n \geq 0$, defined by

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \ (|t| < \pi).$$

The Euler numbers E_n are defined by

$$E_n := 2^n E_n\left(\frac{1}{2}\right),\ n \geq 0.$$

Put $\bar{E}_n(x) = E_n(\{x\}),\ n \geq 0$. Recall also that $E_{2n+1}(1/2) = 0,\ n \geq 0$.

2. Infinite series identities

From now on, we let V a modular transformation corresponding to

$$\begin{pmatrix} 1 & -1 \\ c & 1-c \end{pmatrix}$$

for $c > 0$. Put $r = (r_1, r_2/c)$. Then

$$R_1 = r_1 + r_2,\ R_2 = -r_1 - r_2 + \frac{r_2}{c}.$$

Replacing $c\tau + 1 - c$ by z, we have

$$V\tau = \frac{1}{c} - \frac{1}{cz},\ \tau = 1 - \frac{1}{c} + \frac{1}{c}z.$$

If $\tau \in Q$, then Re $z > 0$ and $z \in \mathbb{H}$. In this section, we consider three cases of $h = (h_1, h_2)$, i.e., $h = (1/2, 1/2),\ (1/2, 0)$ and $(0, 1/2)$. We also suppose that r_1 is an integer and r_2 is not an integer. By Theorem 1.2, for any integer m and $z \in \mathbb{H}$ with Re $z > 0$,

$$\begin{aligned} z^m H(V\tau, -m; r, h) &= H(\tau, -m; R, H) + (2\pi i)^m L(\tau, -m; R, H) \\ &\quad - e(-r_1 h_1) \lim_{s \to -m} (-2\pi i)^{-s} z^{-s} \Phi_+(s, r, h), \end{aligned} \tag{2.1}$$

where

$$\Phi_+(s,r,h) := \Gamma(s)\left(\psi\left(h_2, \frac{r_2}{c}, s\right) + e\left(\frac{s}{2}\right)\psi\left(-h_2, -\frac{r_2}{c}, s\right)\right).$$

Let $\Psi_0(x)$ be the digamma function defined by

$$\Psi_0(x) = \frac{d}{dx}\Gamma(x).$$

For brevity, we let

$$\mathcal{Z}_\pm(s,x) := \zeta(s,x) \pm \zeta(s,1-x), \tag{2.2}$$

and let

$$\mathfrak{Z}_\pm(s,x) := \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+x)^s} \pm \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1-x)^s} \tag{2.3}$$

for $0 < x < 1$ and $\mathrm{Re}\, s > 0$. Then $\mathfrak{Z}_\pm(s,x)$ can be analytically continued to an entire function.

We need the following basic equations to compute (2.1). For r_1 an integer,

$$\begin{aligned} H(V\tau, s; r, h) &= e(-r_1h_1)\sum_{n-h_2>0}\frac{e(r_2(n-h_2)/c)}{(n-h_2)^{1-s}}\frac{e(h_1+V\tau(n-h_2))}{1-e(h_1+V\tau(n-h_2))} \\ &+ e^{\pi is}e(-r_1h_1)\sum_{n+h_2>0}\frac{e(-r_2(n+h_2)/c)}{(n+h_2)^{1-s}}\frac{e(-h_1+V\tau(n+h_2))}{1-e(-h_1+V\tau(n+h_2))}, \end{aligned} \tag{2.4}$$

and, for R_1 not an integer,

$$\begin{aligned} H(\tau, s; R, H) &= e(-[R_1]H_1)\sum_{n-H_2>0}\frac{e((\{R_1\}\tau+R_2)(n-H_2))}{(n-H_2)^{1-s}(1-e(H_1+\tau(n-H_2)))} \\ &+ e^{\pi is}e(-([R_1]+1)H_1)\sum_{n+H_2>0}\frac{e(((1-\{R_1\})\tau-R_2)(n+H_2))}{(n+H_2)^{1-s}(1-e(-H_1+\tau(n+H_2)))}. \end{aligned} \tag{2.5}$$

THEOREM 2.1. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. Let r_2 be a real number, not integer. Then, for any integer k and for any positive even integer c,*

$$\begin{aligned} &\alpha^{-k}\sum_{n=0}^{\infty}\frac{2\cos((2n+1)\pi r_2/c)}{(2n+1)^{2k+1}(e^{(\alpha-\pi i)(2n+1)/c}+1)} \\ &= (-1)^{[r_2]}(-\beta)^{-k}\sum_{n=0}^{\infty}\frac{\sinh(((2\{r_2\}-1)(\beta+\pi i)-2\pi i r_2)(2n+1)/(2c))}{(2n+1)^{2k+1}\cosh((\beta+\pi i)(2n+1)/(2c))} \\ &\quad + \frac{(-1)^{[r_2]}}{4}\sum_{j=1}^{c}(-1)^{j+[\frac{j+[r_2]}{c}]}\sum_{\ell=0}^{2k}\frac{E_\ell\left(\frac{j-\{r_2\}}{c}\right)\bar{E}_{2k-\ell}\left(\frac{j+[r_2]}{c}\right)}{\ell!(2k-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell} \\ &\quad + J_1(k), \end{aligned}$$

where

$$J_1(k) := \begin{cases} \frac{(-1)^{[r_2/c]}}{2}(-\beta)^k\Gamma(-2k)\mathfrak{Z}_-(-2k,\{\frac{r_2}{c}\}) & if\ k<0, \\ \frac{(-1)^{[r_2/c]}}{2}\left(\log\cot\left(\frac{\pi}{2}\left\{\frac{r_2}{c}\right\}\right)-\frac{1}{2}\pi i\right) & if\ k=0, \\ \alpha^{-k}\sum_{n=0}^{\infty}\frac{e^{-(2n+1)\pi i r_2/c}}{(2n+1)^{2k+1}} & if\ k>0. \end{cases}$$

Proof. Let $h=(1/2,1/2)$ and $m=2k$ in (2.1). We have from (2.4) that

$$\begin{aligned} H(V\tau,-2k;r,h) &= (-1)^{r_1}\sum_{n=0}^{\infty}\frac{e(r_2(2n+1)/(2c))}{2^{-2k-1}(2n+1)^{2k+1}}\cdot\frac{-e((1-1/z)(2n+1)/(2c))}{1+e((1-1/z)(2n+1)/(2c))} \\ &+(-1)^{r_1}\sum_{n=0}^{\infty}\frac{e(-r_2(2n+1)/(2c))}{2^{-2k-1}(2n+1)^{2k+1}}\cdot\frac{-e((1-1/z)(2n+1)/(2c))}{1+e((1-1/z)(2n+1)/(2c))} \\ &= (-1)^{r_1+1}2^{2k+2}\sum_{n=0}^{\infty}\frac{\cos(\pi r_2(2n+1)/c)}{(2n+1)^{2k+1}(1+e^{-\pi i(1-1/z)(2n+1)/c})}. \end{aligned} \tag{2.6}$$

Since c is even, $H_1\equiv 0 \pmod 1$ and $H_2\equiv 1/2 \pmod 1$. Thus it follows from (2.5) that

$$\begin{aligned} H(\tau,-2k;R,H) &= \sum_{n=0}^{\infty}\frac{(-1)^{[r_1+r_2]}e^{\pi i(2n+1)(\{r_2\}(z-1)+r_2)/c}}{2^{-2k-1}(2n+1)^{2k+1}(1+e^{\pi i(z-1)(2n+1)/c})} \\ &\quad -\sum_{n=0}^{\infty}\frac{(-1)^{[r_1+r_2]}e^{-\pi i(2n+1)(\{r_2\}(z-1)+r_2)/c}e^{\pi i(z-1)(2n+1)/c}}{2^{-2k-1}(2n+1)^{2k+1}(1+e^{\pi i(z-1)(2n+1)/c})} \\ &= (-1)^{[r_1+r_2]}2^{2k+1}\sum_{n=0}^{\infty}\frac{\sinh(\pi i(2n+1)((2\{r_2\}-1)(z-1)+2r_2)/(2c))}{(2n+1)^{2k+1}\cosh(\pi i(z-1)(2n+1)/(2c))}. \end{aligned} \tag{2.7}$$

We see that

$$\frac{e^{-zu(j-\{R_1\})/c}}{e^{-zu}+1}=\frac{1}{2}\sum_{n=0}^{\infty}E_n\left(\frac{j-\{R_1\}}{c}\right)\frac{(-zu)^n}{n!},$$
$$\frac{e^{\{(j(1-c)+\varrho)/c\}u}}{e^u+1}=\frac{1}{2}\sum_{n=0}^{\infty}\bar{E}_n\left(\frac{j+\varrho}{c}\right)\frac{u^n}{n!},$$

and

$$\left[\frac{j(1-c)+\varrho}{c}\right]=-j-[R_1]-[R_2]+\left[\frac{j+[R_2]}{c}\right].$$

Then, by the residue theorem,

$$\begin{aligned} L(\tau,-2k;R,H) &= \frac{1}{4}\sum_{j=1}^{c}e\left(-\frac{1}{2}\left([R_2]+c+\left[\frac{j(1-c)+\varrho}{c}\right]\right)\right) \\ &\quad\cdot\int_C u^{-2k-1}\sum_{n=0}^{\infty}E_n\left(\frac{j-\{R_1\}}{c}\right)\frac{(-zu)^n}{n!}\cdot\sum_{m=0}^{\infty}\bar{E}_m\left(\frac{j+\varrho}{c}\right)\frac{u^m}{m!}du \\ &= \frac{(-1)^{[r_1+r_2]}}{2}\pi i\sum_{j=1}^{c}(-1)^{j+[(j+[r_2])/c]} \end{aligned}$$

$$\cdot \sum_{\ell=0}^{2k} \frac{E_\ell((j-\{r_2\})/c)}{\ell!} \cdot \frac{\bar{E}_{2k-\ell}((j+[r_2])/c)}{(2k-\ell)!}(-z)^\ell. \tag{2.8}$$

Now we compute $\Phi_+(s,r,h)$. It is easy to see that, for $x \notin \mathbb{Z}$,

$$\psi(1/2,x,s) = (-1)^{[x]}(2^{1-s}\zeta(s,\{x\}/2) - \zeta(s,\{x\})),$$
$$\psi(-1/2,-x,s) = (-1)^{[x]+1}(2^{1-s}\zeta(s,(1-\{x\})/2) - \zeta(s,1-\{x\})).$$

For Re $s<0$ and $0<x\le 1$ [8],

$$\Gamma(s)\zeta(s,x) = \frac{(2\pi)^s}{\sin(\pi s)} \sum_{n=1}^{\infty} \frac{\sin(2\pi n x + \pi s/2)}{n^{1-s}}. \tag{2.9}$$

It follows that for Re $s<0$,

$$\begin{aligned}\Phi_+(s,r,h) &= \Gamma(s)\left(\psi\left(\frac{1}{2},\frac{r_2}{c},s\right) + e\left(\frac{s}{2}\right)\psi\left(-\frac{1}{2},-\frac{r_2}{c},s\right)\right)\\ &= 2\pi^s e^{\pi i s/2}\sum_{n=0}^{\infty}\frac{e^{-\pi i r_2(2n+1)/c}}{(2n+1)^{1-s}}.\end{aligned} \tag{2.10}$$

In case of $s=0$, using the expansions at $s=0$,

$$\begin{aligned}2^{1-s} &= 2 - 2\log 2s + \cdots,\\ \zeta(s,x) &= \frac{1}{2} - x + \left(\log\Gamma(x) - \frac{1}{2}\log 2\pi\right)s + \cdots,\\ e^{\pi i s} &= 1 + \pi i s + \cdots,\end{aligned}$$

we have

$$\begin{aligned}&\lim_{s\to 0}\Phi_+(s,r,h)\\ &= \lim_{s\to 0}\Gamma(s)\left(\psi\left(\frac{1}{2},\frac{r_2}{c},s\right) + e\left(\frac{s}{2}\right)\psi\left(-\frac{1}{2},-\frac{r_2}{c},s\right)\right)\\ &= (-1)^{[r_2/c]}\left(\log\cot\left(\frac{\pi}{2}\left\{\frac{r_2}{c}\right\}\right) - \frac{1}{2}\pi i\right).\end{aligned} \tag{2.11}$$

Finally combining (2.6), (2.7), (2.8), (2.10), (2.11) and putting $z=\pi i/\alpha$ in (2.1), we prove the theorem. □

COROLLARY 2.2. *For any integer k and for any positive even integer c,*

$$\begin{aligned}&\alpha^{-k}\sum_{n=0}^{\infty}\frac{2\cos((2n+1)\pi/(2c))}{(2n+1)^{2k+1}(e^{(\alpha-\pi i)(2n+1)/c}+1)}\\ &= -(-\beta)^{-k}\sum_{n=0}^{\infty}\frac{i\sin((2n+1)\pi/(2c))}{(2n+1)^{2k+1}\cosh((\beta+\pi i)(2n+1)/(2c))}\\ &\quad+\frac{1}{4}\sum_{j=1}^{c}(-1)^{j+[j/c]}\sum_{\ell=0}^{2k}\frac{E_\ell\left(\frac{j-1/2}{c}\right)\bar{E}_{2k-\ell}\left(\frac{j}{c}\right)}{\ell!(2k-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell} + \mathcal{J}_1(k),\end{aligned}$$

where

$$\mathcal{J}_1(k) := \begin{cases} \frac{1}{2}(-\beta)^k\Gamma(-2k)\mathfrak{Z}_-(-2k, \frac{1}{2c}) & \textit{if } k < 0, \\ \frac{1}{2}\log\cot\left(\frac{\pi}{4c}\right) - \frac{1}{4}\pi i & \textit{if } k = 0, \\ \alpha^{-k}\sum_{n=0}^{\infty}\frac{e^{-(2n+1)\pi i/2c}}{(2n+1)^{2k+1}} & \textit{if } k > 0. \end{cases}$$

Proof. Put $r_2 = 1/2$ into Theorem 2.1. □

THEOREM 2.3. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. Let r_2 be a real number, not integer. Then, for any integer k and for any positive odd integer c,*

$$\begin{aligned} &\alpha^{-k}\sum_{n=0}^{\infty}\frac{2\cos((2n+1)\pi r_2/c)}{(2n+1)^{2k+1}(e^{(\alpha-\pi i)(2n+1)/c}+1)} \\ &= (-1)^{[r_2]}2^{-2k-1}(-\beta)^{-k}\sum_{n=1}^{\infty}\frac{\sinh(((2\{r_2\}-1)(\beta+\pi i)-2\pi i r_2)n/c)}{n^{2k+1}\cosh((\beta+\pi i)n/c)} \\ &\quad+\frac{(-1)^{[r_2]}}{2}\sum_{j=1}^{c}(-1)^{j+1}\sum_{\ell=0}^{2k+1}\frac{E_\ell\left(\frac{j-\{r_2\}}{c}\right)\bar{B}_{2k+1-\ell}\left(\frac{j+[r_2]}{c}\right)}{\ell!(2k+1-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell} \\ &\quad+J_1(k). \end{aligned}$$

Proof. Let $h = (1/2, 1/2)$ and $m = 2k$ in (2.1). Since c is odd, $H_1 \equiv 1/2 \pmod 1$ and $H_2 \equiv 0 \pmod 1$. Thus it follows from (2.5) that

$$\begin{aligned} H(\tau,-2k;R,H) &= \sum_{n=1}^{\infty}\frac{(-1)^{[r_1+r_2]}e^{2\pi i n(\{r_2\}(z-1)+r_2)/c}}{n^{2k+1}(1+e^{2\pi i n(z-1)/c})} \\ &\quad-\sum_{n=1}^{\infty}\frac{(-1)^{[r_1+r_2]}e^{-2\pi i n(\{r_2\}(z-1)+r_2)/c}e^{2\pi i n(z-1)/c}}{n^{2k+1}(1+e^{2\pi i n(z-1)/c})} \\ &= (-1)^{[r_1+r_2]}\sum_{n=0}^{\infty}\frac{\sinh(\pi i n((2\{r_2\}-1)(z-1)+2r_2)/c)}{n^{2k+1}\cosh(\pi i n(z-1)/c)}. \end{aligned} \tag{2.12}$$

We see that

$$\frac{e^{\{(j(1-c)+\varrho)/c\}u}}{e^u-1} = u^{-1}\sum_{n=0}^{\infty}\bar{B}_n\left(\frac{j(1-c)+\varrho}{c}\right)\frac{u^n}{n!}.$$

Then, by the residue theorem,

$$\begin{aligned} L(\tau,-2k;R,H) &= \frac{1}{2}\sum_{j=1}^{c}e\left(-\frac{1}{2}(j+[R_1]-c)\right) \\ &\quad\cdot\int_C u^{-2k-2}\sum_{n=0}^{\infty}E_n\left(\frac{j-\{R_1\}}{c}\right)\frac{(-zu)^n}{n!}\cdot\sum_{m=0}^{\infty}\bar{B}_m\left(\frac{j(1-c)+\varrho}{c}\right)\frac{u^m}{m!}du \\ &= \frac{(-1)^{[r_1+r_2]+1}}{\pi}i\sum_{j=1}^{c}(-1)^j \\ &\quad\cdot\sum_{\ell=0}^{2k+1}\frac{E_\ell((j-\{r_2\})/c)}{\ell!}\cdot\frac{\bar{B}_{2k+1-\ell}((j+[r_2])/c)}{(2k+1-\ell)!}(-z)^\ell. \end{aligned} \tag{2.13}$$

Put (2.6), (2.10), (2.11), (2.12) and (2.13) in (2.1) and let $z = \pi i/\alpha$. Then the desired results follow. □

Corollary 2.4. *For any integer k and for any positive odd integer c,*

$$\begin{aligned}
&\alpha^{-k}\sum_{n=0}^{\infty}\frac{2\cos((2n+1)\pi/(2c))}{(2n+1)^{2k+1}(e^{(\alpha-\pi i)(2n+1)/c}+1)}\\
&=-2^{-2k-1}(-\beta)^{-k}\sum_{n=1}^{\infty}\frac{i\sin(n\pi/c)}{n^{2k+1}\cosh((\beta+\pi i)n/c)}\\
&\quad+\frac{1}{2}\sum_{j=1}^{c}(-1)^{j+1}\sum_{\ell=0}^{2k+1}\frac{E_\ell\left(\frac{j-1/2}{c}\right)\bar{B}_{2k+1-\ell}\left(\frac{j}{c}\right)}{\ell!(2k+1-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell}+\mathcal{J}_1(k).
\end{aligned}$$

Proof. Put $r_2 = 1/2$ into Theorem 2.3. □

Corollary 2.5. *For any positive integer k,*

$$E_{2k} = (-1)^k 2^{2k+2}\pi^{-2k-1}(2k)!\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2k+1}}.$$

Proof. Let $c = 1$ in Corollary 2.4 and use the facts;

$$E_{2n} = 2^{2n}E_{2n}\left(\frac{1}{2}\right),\ E_{2n+1}\left(\frac{1}{2}\right) = 0,\ n \geq 0.$$

□

Corollary 2.6. *Let r_2 be a real number, not integer. Then, for any integer k,*

$$\begin{aligned}
&\alpha^{-k}\sum_{n=0}^{\infty}\frac{2\cos((2n+1)\pi r_2)}{(2n+1)^{2k+1}(e^{(2n+1)\alpha}-1)}\\
&=(-1)^{[r_2]+1}2^{-2k-1}(-\beta)^{-k}\sum_{n=1}^{\infty}\frac{\sinh((2\{r_2\}-1)n\beta)}{n^{2k+1}\cosh(n\beta)}\\
&\quad+\frac{(-1)^{[r_2]+1}}{2}\sum_{\ell=0}^{k}\frac{E_{2\ell+1}(1-\{r_2\})B_{2k-2\ell}}{(2\ell+1)!(2k-2\ell)!}\alpha^{k-\ell}(-\beta)^{\ell+1}+\mathsf{J}_1(k),
\end{aligned}$$

where

$$\mathsf{J}_1(k) := \begin{cases}\frac{(-1)^{[r_2]+1}}{2}(-\beta)^k\Gamma(-2k)\mathfrak{z}_-(-2k,\{r_2\}) & \textit{if } k<0,\\ \frac{(-1)^{[r_2]+1}}{2}\log\cot\left(\frac{1}{2}\pi\{r_2\}\right) & \textit{if } k=0,\\ \alpha^{-k}\sum_{n=0}^{\infty}\frac{\cos((2n+1)\pi r_2)}{(2n+1)^{2k+1}} & \textit{if } k>0.\end{cases}$$

Proof. Put $c = 1$ in Theorem 2.3 and equate the real parts. □

Corollary 2.7. *For any integer $k > 0$,*

$$E_{2k}(1-\{r_2\}) = \frac{(-1)^{[r_2]+k}4(2k)!}{\pi^{2k+1}}\sum_{n=0}^{\infty}\frac{\sin((2n+1)\pi r_2)}{(2n+1)^{2k+1}}.$$

Proof. Put $c = 1$ in Theorem 2.3 and equate the imaginary parts. □

Theorem 2.8. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. Let r_2 be a real number, not integer. Then, for any integer k and for any positive even integer c,*

$$\begin{aligned}
&\alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{2i\sin((2n+1)\pi r_2/c)}{(2n+1)^{2k+1}(e^{(\alpha-\pi i)(2n+1)/c}+1)}\\
&= (-1)^{[r_2]+1}(-\beta)^{-k-1/2}\sum_{n=0}^{\infty}\frac{\cosh(((2\{r_2\}-1)(\beta+\pi i)-2\pi i r_2)(2n+1)/(2c))}{(2n+1)^{2k+1}\cosh\left(\frac{2n+1}{2c}(\beta+\pi i)\right)}\\
&\quad+\frac{(-1)^{[r_2]}}{4}\sum_{j=1}^{c}(-1)^{j+[\frac{j+[r_2]}{c}]}\\
&\qquad\cdot\sum_{\ell=0}^{2k+1}\frac{E_\ell\left(\frac{j-\{r_2\}}{c}\right)\bar{E}_{2k+1-\ell}\left(\frac{j+[r_2]}{c}\right)}{\ell!(2k-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell+1/2}+J_2(k),
\end{aligned}$$

where

$$J_2(k) := \begin{cases}
\frac{(-1)^{[r_2/c]+1}}{2}(-\beta)^{k+1/2}\Gamma(-2k-1)\mathfrak{Z}_+(-2k-1,\{\frac{r_2}{c}\}) & \text{if } k<-1,\\
\frac{(-1)^{[r_2/c]+1}}{2}(-\beta)^{-1/2}(\Psi_0(\{\frac{r_2}{c}\})+\Psi_0(1-\{\frac{r_2}{c}\}) & \\
\qquad -\Psi_0(\frac{1}{2}\{\frac{r_2}{c}\})-\Psi_0(\frac{1}{2}-\frac{1}{2}\{\frac{r_2}{c}\})-2\log 2) & \text{if } k=-1,\\
-\alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{e^{-(2n+1)\pi i r_2/c}}{(2n+1)^{2k+2}} & \text{if } k\geq 0.
\end{cases}$$

Proof. Let $h = (1/2, 1/2)$ and $m = 2k+1$ in (2.1). By the same way as we derived equations (2.6), (2.7) and (2.8), we have

$$\begin{aligned}
&H(V\tau,-2k-1;r,h) = (-1)^{r_1+1}2^{2k+3}\\
&\qquad\cdot\sum_{n=0}^{\infty}\frac{i\sin(\pi r_2(2n+1)/c)}{(2n+1)^{2k+2}(1+e^{-\pi i(1-1/z)(2n+1)/c})},
\end{aligned}\tag{2.14}$$

$$\begin{aligned}
&H(\tau,-2k-1;R,H) = (-1)^{[r_1+r_2]}2^{2k+2}\\
&\qquad\cdot\sum_{n=0}^{\infty}\frac{\cosh(\pi i(2n+1)((2\{r_2\}-1)(z-1)+2r_2)/(2c))}{(2n+1)^{2k+2}\cosh(\pi i(z-1)(2n+1)/(2c))},
\end{aligned}\tag{2.15}$$

and

$$\begin{aligned}
&L(\tau,-2k-1;R,H) = \frac{(-1)^{[r_1+r_2]}}{2}\pi i\sum_{j=1}^{c}(-1)^{j+[(j+[r_2])/c]}\\
&\qquad\cdot\sum_{\ell=0}^{2k+1}\frac{E_\ell((j-\{r_2\})/c)}{\ell!}\cdot\frac{\bar{E}_{2k+1-\ell}((j+[r_2])/c)}{(2k+1-\ell)!}(-z)^\ell.
\end{aligned}\tag{2.16}$$

In cases that $k \geq 0$ or $k < -1$, $\lim_{s\to -2k-1} \Phi_+(s,r,h)$ can be computed by the same matter as in the proof of Theorem 2.1. If $k=-1$, then, employing the following expansions at $s=1$,

$$\begin{aligned} 2^{1-s} &= 1-(\log 2)(s-1)+\cdots, \\ \zeta(s,x) &= \frac{1}{s-1}-\Psi_0(x)+\cdots, \\ e^{\pi is} &= -1-\pi i(s-1)+\cdots, \end{aligned}$$

we obtain that

$$\begin{aligned} \lim_{s\to 1}\Phi_+(s,r,h) &= (-1)^{[r_2/c]}(\Psi_0(\{\frac{r_2}{c}\})+\Psi_0(1-\{\frac{r_2}{c}\}) \\ &\quad -\Psi_0(\frac{1}{2}\{\frac{r_2}{c}\})-\Psi_0(\frac{1}{2}-\frac{1}{2}\{\frac{r_2}{c}\})-2\log 2). \end{aligned} \tag{2.17}$$

Let $z=\pi i/\alpha$ and put (2.14), (2.15), (2.16) and (2.17) in (2.1). Then we obtain the desired results. □

COROLLARY 2.9. *For any integer k and for any positive even integer c,*

$$\begin{aligned} &\alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{2i\sin((2n+1)\pi/(2c))}{(2n+1)^{2k+2}(e^{(\alpha-\pi i)(2n+1)/c}+1)} \\ &= -(-\beta)^{-k-1/2}\sum_{n=0}^{\infty}\frac{\cos((2n+1)\pi/(2c))}{(2n+1)^{2k+2}\cosh((\beta+\pi i)(2n+1)/(2c))} \\ &\quad +\frac{1}{4}\sum_{j=1}^{c}(-1)^{j+[j/c]}\sum_{\ell=0}^{2k+1}\frac{E_\ell\left(\frac{j-1/2}{c}\right)\bar{E}_{2k+1-\ell}\left(\frac{j}{c}\right)}{\ell!(2k-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell+1/2}+\mathcal{J}_2(k), \end{aligned}$$

where

$$\mathcal{J}_2(k) := \begin{cases} -\frac{1}{2}(-\beta)^{k+1/2}\Gamma(-2k-1)\mathfrak{Z}_+(-2k-1,\frac{1}{2c}) & if\ k<-1, \\ -\frac{1}{2}(-\beta)^{-1/2}(\Psi_0(\frac{1}{2c})+\Psi_0(1-\frac{1}{2c}) & \\ \qquad -\Psi_0(\frac{1}{4c})-\Psi_0(\frac{1}{2}-\frac{1}{4c})-2\log 2) & if\ k=-1, \\ -\alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{e^{-(2n+1)\pi i/(2c)}}{(2n+1)^{2k+2}} & if\ k\geq 0. \end{cases}$$

Proof. Put $r_2=1/2$ into Theorem 2.8. □

THEOREM 2.10. *Let $\alpha,\ \beta>0$ with $\alpha\beta=\pi^2$. Let r_2 be a real number, not integer. Then, for any integer k and for any positive odd integer c,*

$$\begin{aligned} &\alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{2i\sin((2n+1)\pi r_2/c)}{(2n+1)^{2k+1}(e^{(\alpha-\pi i)(2n+1)/c}+1)} \\ &= (-1)^{[r_2]+1}2^{-2k-2}(-\beta)^{-k-1/2}\sum_{n=1}^{\infty}\frac{\cosh(((2\{r_2\}-1)(\beta+\pi i)-2\pi i r_2)n/c)}{n^{2k+1}\cosh((\beta+\pi i)n/c)} \\ &\quad +\frac{(-1)^{[r_2]}}{2}\sum_{j=1}^{c}(-1)^{j+1}\sum_{\ell=0}^{2k+2}\frac{E_\ell\left(\frac{j-\{r_2\}}{c}\right)\bar{B}_{2k+2-\ell}\left(\frac{j+[r_2]}{c}\right)}{\ell!(2k+2-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell+1/2} \\ &\quad +J_2(k). \end{aligned}$$

Proof. Let $h = (1/2, 1/2)$ and $m = 2k+1$ in (2.1). In similar to (2.12) and (2.13), we have

$$H(\tau, -2k-1; R, H) = (-1)^{[r_1+r_2]} \sum_{n=1}^{\infty} \frac{\cosh(\pi i n((2\{r_2\}-1)(z-1)+2r_2))}{n^{2k+2}\cosh(\pi i n(z-1))},$$

$$L(\tau, -2k-1; R, H) = (-1)^{[r_1+r_2]+1}\pi i \sum_{j=1}^{c} (-1)^j$$

$$\cdot \sum_{\ell=0}^{2k+2} \frac{E_\ell((j-\{r_2\})/c)}{\ell!} \cdot \frac{\bar{B}_{2k+2-\ell}((j+[r_2])/c)}{(2k+2-\ell)!}(-z)^\ell. \tag{2.18}$$

Now put (2.14), (2.17) and (2.18) in (2.1) and put $z = \pi i/\alpha$ to complete the proof. □

Corollary 2.11. *For any integer k and for any positive odd integer c,*

$$\alpha^{-k-1/2} \sum_{n=0}^{\infty} \frac{2i\sin((2n+1)\pi/(2c))}{(2n+1)^{2k+2}(e^{(\alpha-\pi i)(2n+1)/c}+1)}$$
$$= -2^{-2k-2}(-\beta)^{-k-1/2} \sum_{n=1}^{\infty} \frac{\cos(n\pi/c)}{n^{2k+2}\cosh((\beta+\pi i)n/c)}$$
$$+\frac{1}{2}\sum_{j=1}^{c}(-1)^{j+1}\sum_{\ell=0}^{2k+2}\frac{E_\ell\left(\frac{j-1/2}{c}\right)\bar{B}_{2k+2-\ell}\left(\frac{j}{c}\right)}{\ell!(2k+2-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell+1/2} + \mathcal{J}_2(k).$$

Proof. Put $r_2 = 1/2$ in Theorem 2.10. □

Corollary 2.12.

$$\alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{2(-1)^n}{(2n+1)^{2k+2}(e^{(2n+1)\alpha}-1)}$$
$$= (-1)^{k+1}2^{-2k-2}\beta^{-k-1/2}\sum_{n=1}^{\infty}\frac{\operatorname{sech}(n\beta)}{n^{2k+2}}$$
$$+\frac{\pi}{2}\sum_{\ell=0}^{k+1}\frac{E_{2\ell}\left(\frac{1}{2}\right)B_{2k+2-2\ell}}{(2\ell)!(2k+2-2\ell)!}\alpha^{k-\ell+1/2}(-\beta)^\ell + \mathfrak{J}_2(k).$$

where

$$\mathfrak{J}_2(k) := \begin{cases} (-1)^k 2^{-2k-1}\beta^{k+1/2}\Gamma(-2k-1)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{-2k-1}} & \text{if } k < -1, \\ -\frac{1}{2}\alpha^{1/2} & \text{if } k = -1, \\ -\alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2k+2}} & \text{if } k \geq 0. \end{cases}$$

Proof. Put $c = 1$ in Corollary 2.11. □

Corollary 2.12 should be compared with Proposition 4.27 in [6] or Corollary 4.19 in [5].

COROLLARY 2.13.

$$\alpha^{1/2}\sum_{n=0}^{\infty}\frac{2(-1)^n}{e^{(2n+1)\alpha}-1}+\frac{1}{2}\alpha^{1/2}=\beta^{1/2}\sum_{n=1}^{\infty}\operatorname{sech}(n\beta)+\frac{1}{2}\beta^{1/2}.$$

Proof. Put $k=-1$ in Corollary 2.12. □

COROLLARY 2.14.

$$\sum_{n=1}^{\infty}\operatorname{sech}(n\pi)=\sum_{n=0}^{\infty}\frac{2(-1)^n}{e^{(2n+1)\pi}-1}.$$

Proof. Put $\alpha=\beta=\pi$ in Corollary 2.13. □

COROLLARY 2.15. *Let r_2 be a real number, not integer. Then, for any integer k,*

$$\begin{aligned}
&\alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{2\sin((2n+1)\pi r_2)}{(2n+1)^{2k+2}(e^{(2n+1)\alpha}-1)}\\
&=(-1)^{[r_2]+k+1}2^{-2k-2}\beta^{-k-1/2}\sum_{n=1}^{\infty}\frac{\cosh((2\{r_2\}-1)n\beta)}{n^{2k+2}\cosh(n\beta)}\\
&\quad+\frac{(-1)^{[r_2]}\pi}{2}\sum_{\ell=0}^{k+1}\frac{E_{2\ell}(1-\{r_2\})B_{2k+2-2\ell}}{(2\ell)!(2k+2-2\ell)!}\alpha^{k-\ell+1/2}(-\beta)^{\ell+1}+\mathsf{J}_2(k),
\end{aligned}$$

where

$$\mathsf{J}_2(k):=\begin{cases}\frac{(-1)^{[r_2]+k}}{2}\beta^{k+1/2}\Gamma(-2k-1)\mathfrak{Z}_+(-2k-1,\{r_2\}) & \text{if } k<-1,\\ \frac{(-1)^{[r_2]+1}}{2}\beta^{-1/2}(\Psi_0(\{r_2\})+\Psi_0(1-\{r_2\}) & \\ \qquad -\Psi_0(\frac{1}{2}\{r_2\})-\Psi_0(\frac{1}{2}-\frac{1}{2}\{r_2\})-2\log 2) & \text{if } k=-1,\\ -\alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{\sin((2n+1)\pi r_2)}{(2n+1)^{2k+2}} & \text{if } k\geq 0.\end{cases}$$

Proof. Put $c=1$ in Theorem 2.10 and equate the imaginary parts. □

COROLLARY 2.16. *For any integer $k\geq 0$,*

$$E_{2k+1}(1-\{r_2\})=\frac{(-1)^{[r_2]+k}4(2k+1)!}{\pi^{2k+2}}\sum_{n=0}^{\infty}\frac{\cos((2n+1)\pi r_2)}{(2n+1)^{2k+2}}.$$

Proof. Put $c=1$ in Theorem 2.10 and equate the real parts. □

THEOREM 2.17. *Let α, $\beta>0$ with $\alpha\beta=\pi^2$. Let r_2 be a real number, not integer. Then, for any integer k and for any positive even integer c,*

$$\begin{aligned}
&\alpha^{-k}\sum_{n=1}^{\infty}\frac{2\cos(2\pi n r_2/c)}{n^{2k+1}(e^{(\alpha-\pi i)2n/c}+1)}\\
&=(-1)^{[r_2]}(-\beta)^{-k}\sum_{n=1}^{\infty}\frac{\sinh(((2\{r_2\}-1)(\beta+\pi i)-2\pi i r_2)n/c)}{n^{2k+1}\cosh((\beta+\pi i)n/c)}
\end{aligned}$$

$$-(-1)^{[r_2]}2^{2k+1}\sum_{j=1}^{c}(-1)^j\sum_{\ell=0}^{2k+2}\frac{B_\ell\left(\frac{j-\{r_2\}}{c}\right)\bar{B}_{2k+2-\ell}\left(\frac{j+[r_2]}{c}\right)}{\ell!(2k+2-\ell)!}(-\pi i)^\ell\alpha^{k-\ell+1}$$
$$+J_3(k),$$

where

$$J_3(k):=\begin{cases}2^{2k}(-\beta)^k\Gamma(-2k)\mathcal{Z}_+\left(-2k,\{\frac{r_2}{c}\}\right) & if\ k<0,\\ -\log\left(1-e^{-2\pi i r_2/c}\right) & if\ k=0,\\ \alpha^{-k}\sum_{n=1}^{\infty}\frac{e^{-2\pi i n r_2/c}}{n^{2k+1}} & if\ k>0.\end{cases}$$

Proof. Let $h=(1/2,0)$ and $m=2k$ in (2.1). We find from (2.4) and (2.5) that

$$H(V\tau,-2k;r,h)=(-1)^{r_1+1}\sum_{n=1}^{\infty}\frac{2\cos(2\pi r_2 n/c)}{n^{2k+1}(1+e^{-2\pi i n(1-1/z)/c})},$$
$$H(\tau,-2k;R,H)=(-1)^{[r_1+r_2]}\sum_{n=1}^{\infty}\frac{\sinh(\pi i n((2\{r_2\}-1)(z-1)+2r_2)/c)}{n^{2k+1}\cosh(\pi i n(z-1)/c)}. \tag{2.19}$$

By the residue theorem, it is easily deduced that

$$L(\tau,-2k;R,H)=(-1)^{[r_1+r_2]}2\pi i\sum_{j=1}^{c}(-1)^j$$
$$\cdot\sum_{\ell=0}^{2k+2}\frac{B_\ell((j-\{r_2\})/c)}{\ell!}\frac{\bar{B}_{2k+2-\ell}((j+[r_2])/c)}{(2k+2-\ell)!}(-z)^{\ell-1}. \tag{2.20}$$

Since $h_2=0$,

$$\Phi_+(s,r,h)=\Gamma(s)(\zeta(s,\{r_2/c\})+e(s/2)\zeta(s,1-\{r_2\})).$$

For Re $s<0$, apply (2.9) to have

$$\Phi_+(s,r,h)=(2\pi)^s e^{\pi i s/2}\sum_{n=1}^{\infty}\frac{e^{-2\pi i n\{r_2/c\}}}{n^{1-s}}. \tag{2.21}$$

By using the following expansions at $s=0$, for $0<x\le 1$,

$$\zeta(s,x)=\frac{1}{2}-x+\left(\log\Gamma(x)-\frac{1}{2}\right)s+\cdots,$$
$$\Gamma(s)=\frac{1}{s}+\gamma+\cdots,$$
$$e^{\pi i s}=1+\pi i s+\cdots,$$

where γ is the Euler's constant, we find that

$$\lim_{s\to 0}\Phi_+(s,r,h)=-\log(1-e^{-2\pi i\{r_2/c\}}). \tag{2.22}$$

Now put $z=\pi i/\alpha$ and plug (2.19), (2.20), (2.21) and (2.22) into (2.1) to complete the proof. □

COROLLARY 2.18. *For any integer k and for any positive even integer c,*

$$\alpha^{-k}\sum_{n=1}^{\infty}\frac{2\cos(n\pi/c)}{n^{2k+1}(e^{(\alpha-\pi i)2n/c}+1)}$$
$$=-(-\beta)^{-k}\sum_{n=1}^{\infty}\frac{i\sin(n\pi/c)}{n^{2k+1}\,\cosh((\beta+\pi i)n/c)}$$
$$-2^{2k+1}\sum_{j=1}^{c}(-1)^j\sum_{\ell=0}^{2k+2}\frac{B_\ell\left(\frac{j-1/2}{c}\right)\bar{B}_{2k+2-\ell}\left(\frac{j}{c}\right)}{\ell!(2k+2-\ell)!}(-\pi i)^{\ell}\alpha^{k-\ell+1}+\mathcal{J}_3(k),$$

where

$$\mathcal{J}_3(k):=\begin{cases}2^{2k}(-\beta)^k\Gamma(-2k)\mathcal{Z}_+\left(-2k,\frac{1}{2c}\right) & if\ k<0,\\ -\log\left(1-e^{-\pi i/c}\right) & if\ k=0,\\ \alpha^{-k}\displaystyle\sum_{n=1}^{\infty}\frac{e^{-\pi in/c}}{n^{2k+1}} & if\ k>0.\end{cases}$$

Proof. Put $r_2=1/2$ in Theorem 2.17. □

THEOREM 2.19. *Let $\alpha,\ \beta>0$ with $\alpha\beta=\pi^2$. Let r_2 be a real number, not integer. Then, for any integer k and for any positive odd integer c,*

$$\alpha^{-k}\sum_{n=1}^{\infty}\frac{2\cos(2\pi nr_2/c)}{n^{2k+1}(e^{2n(\alpha-\pi i)/c}+1)}$$
$$=(-1)^{[r_2]}2^{2k+1}(-\beta)^{-k}\sum_{n=0}^{\infty}\frac{\sinh(((2\{r_2\}-1)(\beta+\pi i)-2\pi ir_2)(2n+1)/(2c))}{(2n+1)^{2k+1}\,\cosh((\beta+\pi i)(2n+1)/(2c))}$$
$$+(-1)^{[r_2]}2^{2k}\sum_{j=1}^{c}(-1)^{j+[\frac{j+[r_2]}{c}]}\sum_{\ell=0}^{2k+1}\frac{B_\ell\left(\frac{j-\{r_2\}}{c}\right)\bar{E}_{2k+1-\ell}\left(\frac{j+[r_2]}{c}\right)}{\ell!(2k+1-\ell)!}(-\pi i)^{\ell}\alpha^{k-\ell+1}$$
$$+J_3(k).$$

Proof. Let $h=(1/2,0)$ and $m=2k$ in (2.1). Since c is odd, $H_1\equiv 0 \pmod 1$ and $H_1\equiv 1/2 \pmod 1$. Then we have

$$\begin{aligned}H(\tau,-2k;R,H)&=(-1)^{[r_1+r_2]}2^{2k+1}\\ &\cdot\sum_{n=0}^{\infty}\frac{\sinh(\pi i(2n+1)((2\{r_2\}-1)(z-1)+2r_2)/(2c))}{(2n+1)^{2k+1}\cosh(\pi i(2n+1)(z-1)/(2c))}\end{aligned}\tag{2.23}$$

and

$$\begin{aligned}L(\tau,-2k;R,H)&=(-1)^{[r_1+r_2]+1}\pi i\sum_{j=1}^{c}(-1)^{j+[(j+[r_2])/c]}\\ &\cdot\sum_{\ell=0}^{2k+1}\frac{B_\ell((j-\{r_2\})/c)}{\ell!}\frac{\bar{E}_{2k+1-\ell}((j+[r_2])/c)}{(2k+1-\ell)!}(-z)^{\ell-1}.\end{aligned}\tag{2.24}$$

Let $z=\pi i/\alpha$ in (2.1) and use (2.19), (2.21), (2.22), (2.23) and (2.24) to arrive at the desired result. □

If $c=1$ in Theorem 2.19, then we obtain Theorem 5.11 in [5].

COROLLARY 2.20. *For any integer k and for any positive odd integer c,*

$$\alpha^{-k}\sum_{n=1}^{\infty}\frac{2\cos(n\pi/c)}{n^{2k+1}(e^{(\alpha-\pi i)2n/c}+1)}$$
$$=-2^{2k+1}(-\beta)^{-k}\sum_{n=0}^{\infty}\frac{i\sin((2n+1)\pi/(2c))}{(2n+1)^{2k+1}\cosh((\beta+\pi i)(2n+1)/(2c))}$$
$$+2^{2k}\sum_{j=1}^{c}(-1)^{j+[j/c]}\sum_{\ell=0}^{2k+1}\frac{B_\ell\left(\frac{j-1/2}{c}\right)\bar{E}_{2k+1-\ell}\left(\frac{j}{c}\right)}{\ell!(2k+1-\ell)!}(-\pi i)^{\ell}\alpha^{k-\ell+1}+\mathcal{J}_3(k).$$

Proof. Put $r_2=1/2$ in Theorem 2.19. □

COROLLARY 2.21. *For any integer k,*

$$\alpha^{-k}\sum_{n=1}^{\infty}\frac{2(-1)^n}{n^{2k+1}(e^{2n\alpha}+1)}$$
$$=-2^{2k+1}(-\beta)^{-k}\sum_{n=0}^{\infty}\frac{\operatorname{csch}((2n+1)\beta/2)}{(2n+1)^{2k+1}}$$
$$+2^{2k}\sum_{\ell=0}^{k}\frac{B_{2\ell}\left(\frac{1}{2}\right)E_{2k+1-2\ell}(0)}{(2\ell)!(2k+1-2\ell)!}\alpha^{k-\ell+1}(-\beta)^{\ell}+\mathfrak{J}_3(k),$$

where

$$\mathfrak{J}_3(k):=\begin{cases}\alpha^{-k}(2^{-2k}-1)\zeta(2k+1) & if\ \ k\neq 0,\\ -\log 2 & if\ \ k=0.\end{cases}$$

Proof. Let $c=2$ in Corollary 2.18 or $c=1$ in Corollary 2.20 and use the facts;

$$B_{2n+1}\left(\frac{1}{2}\right)=0,\ \ n\geq 0.$$

□

THEOREM 2.22. *Let $\alpha,\ \beta>0$ with $\alpha\beta=\pi^2$. Let r_2 be a real number, not integer. Then, for any integer k and for any positive even integer c,*

$$\alpha^{-k-1/2}\sum_{n=1}^{\infty}\frac{2i\sin(2\pi nr_2/c)}{n^{2k+2}(e^{(\alpha-\pi i)2n/c}+1)}$$
$$=(-1)^{[r_2]+1}(-\beta)^{-k-1/2}\sum_{n=1}^{\infty}\frac{\cosh(((2\{r_2\}-1)(\beta+\pi i)-2\pi ir_2)n/c)}{n^{2k+2}\cosh((\beta+\pi i)n/c)}$$
$$+(-1)^{[r_2]}2^{2k+2}\sum_{j=1}^{c}(-1)^j\sum_{\ell=0}^{2k+3}\frac{B_\ell\left(\frac{j-\{r_2\}}{c}\right)\bar{B}_{2k+3-\ell}\left(\frac{j+[r_2]}{c}\right)}{\ell!(2k+3-\ell)!}(-\pi i)^{\ell}\alpha^{k-\ell+3/2}$$

$$+J_4(k),$$

where

$$J_4(k) := \begin{cases} -2^{2k+1}(-\beta)^{k+1/2}\Gamma(-2k-1)\mathcal{Z}_-\left(-2k-1,\{\frac{r_2}{c}\}\right) & if\ k<-1, \\ -\frac{1}{2}(-\beta)^{-1/2}(\pi\cot(\pi\{\frac{r_2}{c}\})-\pi i) & if\ k=-1, \\ -\alpha^{-k-1/2}\sum_{n=1}^{\infty}\frac{e^{-2\pi i n r_2/c}}{n^{2k+2}} & if\ k\geq 0. \end{cases}$$

Proof. Let $h=(1/2,0)$ and $m=2k+1$ in (2.1). We find from (2.4) and (2.5) that

$$H(V\tau,-2k-1;r,h)=(-1)^{r_1+1}\sum_{n=1}^{\infty}\frac{2i\sin(2\pi r_2 n/c)}{n^{2k+2}(1+e^{-2\pi n(1-1/z)/c})} \tag{2.25}$$

and

$$\begin{aligned} &H(\tau,-2k-1;R,H) \\ &=(-1)^{[r_1+r_2]}\sum_{n=1}^{\infty}\frac{\cosh(\pi i n((2\{r_2\}-1)(z-1)+2r_2)/c)}{n^{2k+2}\cosh(\pi i n(z-1)/c)}. \end{aligned} \tag{2.26}$$

Similarly as in (2.20), it is deduced that

$$\begin{aligned} L(\tau,-2k-1;R,H)&=(-1)^{[r_1+r_2]}2\pi i\sum_{j=1}^{c}(-1)^j \\ &\cdot\sum_{\ell=0}^{2k+3}\frac{B_\ell((j-\{r_2\})/c)}{\ell!}\frac{\bar{B}_{2k+3-\ell}((j+[r_2])/c)}{(2k+3-\ell)!}(-z)^{\ell-1}. \end{aligned} \tag{2.27}$$

For $k\geq 0$, apply (2.21) to obtain

$$\begin{aligned} &\Phi_+(-2k-1,r,h) \\ &=(2\pi)^{-2k-1}(-1)^{k+1}i\sum_{n=1}\frac{e^{-2\pi i n\{r_2/c\}}}{n^{2k+2}}. \end{aligned} \tag{2.28}$$

Using the expansions at $s=1$,

$$\begin{aligned} \zeta(s,x)&=\frac{1}{s-1}-\Psi_0(x)+\cdots, \\ e^{\pi i s}&=-1-\pi i(s-1)+\cdots, \end{aligned}$$

we have

$$\lim_{s\to 1}\Phi_+(s,r,h)=\pi\cot(\pi\{r_2/c\})-\pi i. \tag{2.29}$$

Lastly put (2.25)–(2.29) into (2.1) and let $z=\pi i/\alpha$ to complete the proof. □

Corollary 2.23. *For any integer k and for any positive even integer c,*

$$\alpha^{-k-1/2}\sum_{n=1}^{\infty}\frac{2i\sin(n\pi/c)}{n^{2k+2}(e^{(\alpha-\pi i)2n/c}+1)}$$
$$=-(-\beta)^{-k-1/2}\sum_{n=1}^{\infty}\frac{\cos(n\pi/c)}{n^{2k+2}\,\cosh((\beta+\pi i)n/c)}$$
$$+2^{2k+2}\sum_{j=1}^{c}(-1)^j\sum_{\ell=0}^{2k+3}\frac{B_\ell\left(\frac{j-1/2}{c}\right)\bar{B}_{2k+3-\ell}\left(\frac{j}{c}\right)}{\ell!(2k+3-\ell)!}(-\pi i)^\ell\alpha^{k-\ell+3/2}+\mathcal{J}_4(k),$$

where

$$\mathcal{J}_4(k):=\begin{cases}-2^{2k+1}(-\beta)^{k+1/2}\Gamma(-2k-1)\mathcal{Z}_-\left(-2k-1,\frac{1}{2c}\right) & \text{if } k<-1,\\ -\frac{1}{2}(-\beta)^{-1/2}(\pi\cot(\frac{\pi}{2c})-\pi i) & \text{if } k=-1,\\ -\alpha^{-k-1/2}\sum_{n=1}^{\infty}\frac{e^{-\pi in/c}}{n^{2k+2}} & \text{if } k\ge 0.\end{cases}$$

Proof. Put $r_2=1/2$ in Theorem 2.22. □

Corollary 2.23 should be compared with Corollary 2.11. If $c=2$ in Corollary 2.23, then we obtain Corollary 2.12.

Theorem 2.24. *Let α, $\beta>0$ with $\alpha\beta=\pi^2$. Let r_2 be a real number, not integer. Then, for any integer k and for any positive odd integer c,*

$$\alpha^{-k-1/2}\sum_{n=1}^{\infty}\frac{2i\sin(2\pi nr_2/c)}{n^{2k+2}(e^{(\alpha-\pi i)2n/c}+1)}$$
$$=(-1)^{[r_2]+1}2^{2k+2}(-\beta)^{-k-1/2}\sum_{n=0}^{\infty}\frac{\cosh(((2\{r_2\}-1)(\beta+\pi i)-2\pi ir_2)(2n+1)/(2c))}{(2n+1)^{2k+2}\,\cosh((\beta+\pi i)(2n+1)/(2c))}$$
$$+(-1)^{[r_2]+1}2^{2k+1}\sum_{j=1}^{c}(-1)^{j+[\frac{j+[r_2]}{c}]}$$
$$\cdot\sum_{\ell=0}^{2k+2}\frac{B_\ell\left(\frac{j-\{r_2\}}{c}\right)\bar{E}_{2k+2-\ell}\left(\frac{j+[r_2]}{c}\right)}{\ell!(2k+2-\ell)!}(-\pi i)^\ell\alpha^{k-\ell+3/2}+J_4(k).$$

Proof. Let $h=(1/2,0)$ and $m=2k+1$ in (2.1). Use (2.5) to obtain that

$$H(\tau,-2k-1;R,H)=(-1)^{[r_1+r_2]}2^{2k+2}\cdot\sum_{n=0}^{\infty}\frac{\cosh(\pi i(2n+1)((2\{r_2\}-1)(z-1)+2r_2)/(2c))}{(2n+1)^{2k+2}\cosh(\pi i(2n+1)(z-1)/(2c))}. \tag{2.30}$$

Apply the same method as in (2.24) to deduce that

$$L(\tau,-2k-1;R,H)=(-1)^{[r_1+r_2]+1}\pi i\sum_{j=1}^{c}(-1)^{j+[(j+[r_2])/c]}\cdot\sum_{\ell=0}^{2k+2}\frac{B_\ell((j-\{r_2\})/c)}{\ell!}\frac{\bar{E}_{2k+2-\ell}((j+[r_2])/c)}{(2k+2-\ell)!}(-z)^{\ell-1}. \tag{2.31}$$

Now plugging (2.25) and (2.28)–(2.31) into (2.1) with $z = \pi i/\alpha$, we obtain the desired result. □

COROLLARY 2.25. *For any integer k and for any positive odd integer c,*

$$\alpha^{-k-1/2}\sum_{n=1}^{\infty}\frac{2i\sin(n\pi/c)}{n^{2k+2}(e^{(\alpha-\pi i)2n/c}+1)}$$
$$= -2^{2k+2}(-\beta)^{-k-1/2}\sum_{n=0}^{\infty}\frac{\cos((2n+1)\pi/(2c))}{(2n+1)^{2k+2}\cosh((\beta+\pi i)(2n+1)/(2c))}$$
$$-2^{2k+1}\sum_{j=1}^{c}(-1)^{j+[j/c]}\sum_{\ell=0}^{2k+2}\frac{B_\ell\left(\frac{j-1/2}{c}\right)\bar{E}_{2k+2-\ell}\left(\frac{j}{c}\right)}{\ell!(2k+2-\ell)!}(-\pi i)^\ell\alpha^{k-\ell+3/2}+\mathcal{J}_4(k).$$

Proof. Put $r_2 = 1/2$ in Theorem 2.24. □

If we put $c = 1$ in Corollary 2.25, then we obtain a well-known formula for the zeta function;

$$\zeta(2k+2) = \frac{(-1)^{k+1}2^{2k+2}B_{2k+2}\pi^{2k+2}}{(2k+2)!},\ k \geq 0.$$

If $c = 1$ in Theorem 2.24, then we obtain Theorem 5.12 in [5].

THEOREM 2.26. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. Let r_2 be a real number, not integer. Then, for any integer k and for any positive integer c,*

$$\alpha^{-k}\sum_{n=0}^{\infty}\frac{2\cos((2n+1)\pi r_2/c)}{(2n+1)^{2k+1}(e^{(\alpha-\pi i)(2n+1)/c}-1)}$$
$$= (-\beta)^{-k}\sum_{n=0}^{\infty}\frac{\cosh(((2\{r_2\}-1)(\beta+\pi i)-2\pi i r_2)(2n+1)/(2c))}{(2n+1)^{2k+1}\sinh((\beta+\pi i)(2n+1)/(2c))}$$
$$-\frac{1}{4}\sum_{j=1}^{c}(-1)^{[\frac{j+[r_2]}{c}]}\sum_{\ell=0}^{2k}\frac{E_\ell\left(\frac{j-\{r_2\}}{c}\right)\bar{E}_{2k-\ell}\left(\frac{j+[r_2]}{c}\right)}{\ell!(2k-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell}+J_5(k),$$

where

$$J_5(k) := \begin{cases} \frac{(-1)^{[r_2/c]+1}}{2}(-\beta)^k\Gamma(-2k)\mathfrak{Z}_-(-2k,\{\frac{r_2}{c}\}) & \text{if } k<0,\\ \frac{(-1)^{[r_2/c]+1}}{2}(\log\cot(\frac{\pi}{2}\{\frac{r_2}{c}\})-\frac{1}{2}\pi i) & \text{if } k=0,\\ -\alpha^{-k}\sum_{n=0}^{\infty}\frac{e^{-(2n+1)\pi i r_2/c}}{(2n+1)^{2k+1}} & \text{if } k>0.\end{cases}$$

Proof. Let $h = (0, 1/2)$ and $m = 2k$ in (2.1). In this case, $H_1 \equiv H_2 \equiv 1/2 \pmod 1$. We find from (2.4) and (2.5) that

$$H(V\tau,-2k;r,h) = 2^{2k+1}\sum_{n=0}^{\infty}\frac{2\cos(\pi r_2(2n+1)/c)}{(2n+1)^{2k+1}(e^{-\pi i(2n+1)(1-1/z)/c}-1)} \tag{2.32}$$

and

$$H(\tau,-2k;R,H) \\ = 2^{2k+1}\sum_{n=0}^{\infty}\frac{\cosh(\pi i(2n+1)((2\{r_2\}-1)(z-1)+2r_2)/(2c))}{(2n+1)^{2k+1}\sinh(-\pi i(2n+1)(z-1)/(2c))}. \tag{2.33}$$

By the same way as in the proof of Theorem 2.1, it is deduced that

$$L(\tau,-2k;R,H) = \frac{\pi i}{2}\sum_{j=1}^{c}(-1)^{[(j+[r_2])/c]} \\ \cdot\sum_{\ell=0}^{2k}\frac{E_\ell((j-\{r_2\})/c)}{\ell!}\cdot\frac{\bar{E}_{2k-\ell}((j+[r_2])/c))}{(2k-\ell)!}(-z)^{\ell}. \tag{2.34}$$

For $s=-2k$, the function $\Phi(s,r,h)$ is also computed by the same way as in the proof of Theorem 2.1, namely, for $k>0$,

$$\Phi(-2k,r,h) = 2(\pi i)^{-2k}\sum_{n=0}^{\infty}\frac{e^{\pi i r_2(2n+1)/c}}{(2n+1)^{2k+1}} \tag{2.35}$$

and

$$\lim_{k\to 0}\Phi(-2k,r,h) = (-1)^{[r_2/c]}\left(\log\cot\left(\frac{\pi}{2}\left\{\frac{r_2}{c}\right\}\right)-\frac{\pi i}{2}\right). \tag{2.36}$$

Finally, employing (2.32)–(2.36) and putting $z=\pi i/\alpha$, we readily obtain the desired result. □

If $c=1$ in Theorem 2.26, then we obtain Theorem 4.17 in [5].

Corollary 2.27. *For any integer k and for any positive integer c,*

$$\alpha^{-k}\sum_{n=0}^{\infty}\frac{2\cos((2n+1)\pi/(2c))}{(2n+1)^{2k+1}(e^{(\alpha-\pi i)(2n+1)/c}-1)} \\ = (-\beta)^{-k}\sum_{n=0}^{\infty}\frac{\cos((2n+1)\pi/(2c))}{(2n+1)^{2k+1}\sinh((\beta+\pi i)(2n+1)/(2c))} \\ -\frac{1}{4}\sum_{j=1}^{c}(-1)^{[j/c]}\sum_{\ell=0}^{2k}\frac{E_\ell\left(\frac{j-1/2}{c}\right)\bar{E}_{2k-\ell}\left(\frac{j}{c}\right)}{\ell!(2k-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell}+\mathcal{J}_5(k),$$

where

$$\mathcal{J}_5(k) := \begin{cases} -\frac{1}{2}(-\beta)^k\Gamma(-2k)\mathfrak{Z}_-(-2k,\frac{1}{2c}) & if\ k<0, \\ -\frac{1}{2}\log\cot\left(\frac{\pi}{4c}\right)+\frac{1}{4}\pi i & if\ k=0, \\ -\alpha^{-k}\sum_{n=0}^{\infty}\frac{e^{-(2n+1)\pi i/(2c)}}{(2n+1)^{2k+1}} & if\ k>0. \end{cases}$$

Proof. Put $r_2=1/2$ in Theorem 2.26. □

If $c=1$ in Corollary 2.27, then we also obtain Corollary 2.5 using $E_{2n}(0)=0$ for $n\geq 1$.

THEOREM 2.28. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. Let r_2 be a real number, not integer. Then, for any integer k and for any positive integer c,*

$$\begin{aligned}
&\alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{2i\sin((2n+1)\pi r_2/c)}{(2n+1)^{2k+2}(e^{(\alpha-\pi i)(2n+1)/c}-1)}\\
&=-(-\beta)^{-k-1/2}\sum_{n=0}^{\infty}\frac{\sinh(((2\{r_2\}-1)(\beta+\pi i)-2\pi i r_2)(2n+1)/(2c))}{(2n+1)^{2k+2}\sinh((\beta+\pi i)(2n+1)/(2c))}\\
&-\frac{1}{4}\sum_{j=1}^{c}(-1)^{[\frac{j+[r_2]}{c}]}\sum_{\ell=0}^{2k+1}\frac{E_\ell\left(\frac{j-\{r_2\}}{c}\right)\bar{E}_{2k+1-\ell}\left(\frac{j+[r_2]}{c}\right)}{\ell!(2k+1-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell+1/2}\\
&+J_6(k),
\end{aligned}$$

where

$$J_6(k):=\begin{cases}\frac{(-1)^{[r_2/c]}}{2}(-\beta)^{k+1/2}\Gamma(-2k-1)\mathfrak{Z}_+(-2k-1,\{\frac{r_2}{c}\}) & if\ k<-1,\\ \frac{(-1)^{[r_2/c]}}{2}(-\beta)^{-1/2}(\Psi_0(\{\frac{r_2}{c}\})+\Psi_0(1-\{\frac{r_2}{c}t\}) & \\ \qquad -\Psi_0(\frac{1}{2}\{\frac{r_2}{c}\})-\Psi_0(\frac{1}{2}-\frac{1}{2}\{\frac{r_2}{c}\})-2\log 2) & if\ k=-1,\\ \alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{e^{-(2n+1)\pi i r_2/c}}{(2n+1)^{2k+2}} & if\ k\geq 0.\end{cases}$$

Proof. Let $h=(0,1/2)$ and $m=2k+1$ in (2.1). It is easily deduced from (2.4) and (2.5) that

$$\begin{aligned}&H(V\tau,-2k-1,;r,h)\\ &=2^{2k+2}\sum_{n=0}^{\infty}\frac{2i\sin(\pi r_2(2n+1)/c)}{(2n+1)^{2k+2}(e^{-\pi i(2n+1)(1-1/z)/c}-1)}\end{aligned} \tag{2.37}$$

and

$$\begin{aligned}&H(\tau,-2k-1,;R,H)\\ &=2^{2k+2}\sum_{n=0}^{\infty}\frac{\sinh(\pi i(2n+1)((2\{r_2\}-1)(z-1)+2r_2)/(2c))}{(2n+1)^{2k+2}\sinh(-\pi i(2n+1)(z-1)/(2c))}.\end{aligned} \tag{2.38}$$

By the same way as in the proof of Theorem 2.8, we obtain that

$$\begin{aligned}L(\tau,-2k-1;R,H)&=\frac{\pi i}{2}\sum_{j=1}^{c}(-1)^{[(j+[r_2])/c]}\\ &\cdot\sum_{\ell=0}^{2k+1}\frac{E_\ell((j-\{r_2\})/c)}{\ell!}\cdot\frac{\bar{E}_{2k+1-\ell}((j+[r_2])/c))}{(2k+1-\ell)!}(-z)^\ell.\end{aligned} \tag{2.39}$$

The function $\Phi(s,r,h)$ has the same results in the proof of Theorem 2.8. Now combine these results with (2.37)–(2.39) and let $z=\pi i/\alpha$ to complete the proof. □

If $c=1$ in Theorem 2.28, then we obtain Theorem 4.18 in [5].

COROLLARY 2.29. *For any integer k and for any positive integer c,*

$$\alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{2i\sin((2n+1)\pi/(2c))}{(2n+1)^{2k+2}(e^{(\alpha-\pi i)(2n+1)/c}-1)}$$

$$= (-\beta)^{-k-1/2} \sum_{n=0}^{\infty} \frac{i \sin((2n+1)\pi/(2c))}{(2n+1)^{2k+2} \sinh((\beta+\pi i)(2n+1)/(2c))}$$

$$-\frac{1}{4} \sum_{j=1}^{c} (-1)^{[j/c]} \sum_{\ell=0}^{2k+1} \frac{E_\ell\left(\frac{j-1/2}{c}\right) \bar{E}_{2k+1-\ell}\left(\frac{j}{c}\right)}{\ell!(2k+1-\ell)!} (-\pi i)^{\ell+1} \alpha^{k-\ell+1/2} + \mathcal{J}_6(k),$$

where

$$\mathcal{J}_6(k) := \begin{cases} \frac{1}{2}(-\beta)^{k+1/2}\Gamma(-2k-1)\mathfrak{Z}_+(-2k-1, \frac{1}{2c}) & \text{if } k < -1, \\ \frac{1}{2}(-\beta)^{-1/2}(\Psi_0(\frac{1}{2c}) + \Psi_0(1-\frac{1}{2c}) \\ \qquad -\Psi_0(\frac{1}{4c}) - \Psi_0(\frac{1}{2}-\frac{1}{4c}) - 2\log 2) & \text{if } k = -1, \\ \alpha^{-k-1/2}\sum_{n=0}^{\infty} \frac{e^{-(2n+1)\pi i/(2c)}}{(2n+1)^{2k+2}} & \text{if } k \geq 0. \end{cases}$$

Proof. Put $r_2 = 1/2$ in Theorem 2.28. □

We obtain Corollary 4.19 in [5] by putting $c = 1$ in Corollary 2.29.

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