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COMPARISON THEOREMS FOR THE VOLUMES OF TUBES ABOUT METRIC BALLS IN $CAT(\kappa)$-SPACES

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Abstract. In this paper, we establish some comparison theorems about volumes of tubes in metric spaces with nonpositive curvature. First we compare the Hausdorff measure of tube about a metric ball contained in an $(n-1)$-dimensional totally geodesic subspace of an n-dimensional locally compact, geodesically complete Hadamard space with Lebesgue measure of its corresponding tube in Euclidean space $\mathbb{R}^n$, and then develop the result to the case of an m-dimensional totally geodesic subspace for $1 < m < n$ with an additional condition. Also, we estimate the Hausdorff measure of the tube about a shortest curve in a metric space of curvature bounded above and below.

1. Introduction

Comparison theorems for the volumes of regions in Riemannian manifolds with some curvature hypothesis play an important role in Riemannian geometry. From the Bishop-Günther inequalities, we obtain lower bounds, respectively upper bounds, for volumes of geodesic balls and tubes by imposing upper bounds, respectively lower bounds, on the sectional curvature. Assuming weaker conditions on the Ricci tensor or considering the ratio between the volumes of geodesic balls in the manifold and the model spaces, we can improve these inequalities. See [8, 10, 11] and the references therein.

From a practical point of view, it is interesting to improve the comparison theorems for tubes of subregions in Riemannian manifolds and to turn the attention from Riemannian manifolds to $CAT(\kappa)$-spaces. Motivated and inspired by the ongoing research with the comparison

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theorems, in this paper we are concerned with the comparison theorems for the volumes of tubes about metric balls in $CAT(\kappa)$-spaces.

The main purpose of this paper is to compare the volume of a tube in metric spaces with nonpositive curvature with the Lebesgue measure of the corresponding tube in Euclidean space $\mathbb{R}^n$. The Riemannian volume of a tube in a Riemannian manifold of curvature bounded above or below by 0 is closely related to the volume of the corresponding tube in Euclidean space [8]. As in Riemannian geometry, the volume of a region in a curved metric space with an intrinsic metric also depends on the curvature of the space [9, 12]. In [7], Lee et al. obtained the Bishop-Günther type inequality in the 2-dimensional singular Hadamard space with Nikolaev's area of surface by comparing the area of surface of a parallel set of a geodesic segment to the Lebesgue measure of the corresponding set in $\mathbb{R}^2$. In this paper, we improve this inequality and extend it to higher dimensional spaces. In this paper, we always assume the uniformity of the Hausdorff dimension. Namely, for each point in an n-dimensional Hausdorff space, any sufficiently small neighborhood has the Hausdorff dimension n.

First, we establish two comparison theorems about the volume of a tube in an n-dimensional Hadamard space with extendible geodesics. Let (X, d) be an n-dimensional geodesically complete Hadamard space, and let G be a totally geodesic subspace of X with codimension 1. Let $K_{o,\varepsilon,G}$ be a closed metric ball in G centered at $o \in G$ with radius ε. Then the tube $T(K_{o,\varepsilon,G}, r)$ of radius r about $K_{o,\varepsilon,G}$ is defined by the set

$$T(K_{o,\varepsilon,G}, r) \equiv \{x \in X : \pi_G(x) \in K_{o,\varepsilon,G} \text{ and } d(x, K_{o,\varepsilon,G}) \leq r\},$$

where $\pi_G(x)$ means the metric projection into G of $x \in X$. In what follows, $H^n(D)$ denotes the n-dimensional Hausdorff measure of a subset D of X and $L^n(E)$ denotes the n-dimensional Lebesgue measure of a subset E of $\mathbb{R}^n$. Then we obtain the following result:

THEOREM 1.1. *Let (X, d) be an n-dimensional locally compact, geodesically complete Hadamard space, and let G be a totally geodesic subspace of X with codimension* 1. *Then, for the tube $T(K_{o,\varepsilon,G}, r)$ of radius r about a closed metric ball $K_{o,\varepsilon,G}(\subset G)$, we have*

$$H^n(T(K_{o,\varepsilon,G}, r)) \geq L^n(T_0(\bar{K}, r)),$$

where $T_0(\bar{K}, r)$ is the corresponding tube in $\mathbb{R}^n$ of the tube $T(K_{o,\varepsilon,G}, r)$.

By the corresponding tube $T_0(\bar{K}, r)$ we mean the set of all points m in $\mathbb{R}^n$ such that there exists a geodesic segment γ of length $\ell(\gamma) \leq r$ from m meeting the corresponding set $\bar{K}$ of $K_{o,\varepsilon,G}$ orthogonally.

We generalise Theorem 1.1 to the case that G is an m-dimensional totally geodesic subspace of X for $1 < m < n$ with an additional condition (cf. Theorem 2.3).

In order to measure the tube of a shortest curve, we shall consider spaces of curvature bounded above and below. Let I be a closed interval of $\mathbb{R}$ and $\gamma : I \to X$ a shortest curve parametrized by arc length, and consider the metric projection $\pi_\gamma : X \to \gamma(I)$. Now we assume that X is a geodesically complete space of curvature bounded above and below. Then the geodesic γ has an extension $\gamma_{ext} : \mathbb{R} \to X$. We define the tube $T(\gamma, r)$ of radius r about γ as the set

$$T(\gamma, r) = \{x \in X : \pi_\gamma(x) = \pi_{\gamma_{ext}}(x) \in \gamma(I), d_X(x, \gamma(I)) \leq r\}.$$

Then we have

Theorem 1.2. *Let (X, d_X) be an n-dimensional locally compact, geodesically complete simply connected space of curvature ≤ 0 and $\geq \kappa'(< 0)$, and let $\gamma : I \to X$ be a shortest curve parametrized by arc length. Then we have*

$$H^n(T(\gamma, r)) \geq L^n(B^{n-1}(o, r) \times I).$$

2. Main results

Throughout this paper, any metric space (X, d) is metrically connected and intrinsic, i.e. any two points of X can be joined by a curve with finite length and the metric $d(p, q)$ is the same as the infimum of lengths in the metric d of all curves joining $p, q \in X$.

A continuous curve $\gamma : I \to X$ in a metric space X is called a (unit speed) geodesic if each $t \in I$ has an open neighborhood $J \subset I$ such that $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in J$. The image $\gamma([a, b]) \subset X$ of $[a, b] \subset I$ is called a geodesic segment with endpoints $\gamma(a)$ and $\gamma(b)$. Usually we will denote the geodesic segment with endpoints p and q by pq and the geodesic passing through p and q by $\overline{pq}$, respectively. If $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in [a, b]$, then $\gamma : [a, b] \to X$ is called a shortest curve joining $\gamma(a)$ and $\gamma(b)$. A metric space X is said to be geodesic if for any two points in X there exists a shortest curve joining them.

A triangle in X with three shortest curves $\gamma_1, \gamma_2, \gamma_3$ as its sides is denoted by $\triangle(\gamma_1, \gamma_2, \gamma_3)$. When vertices of the triangle are p, q and r, we often denote the triangle by $\triangle pqr$. A triangle $\overline{\triangle}(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3)$ in a complete simply connected surface M_κ of constant curvature $\kappa \in \mathbb{R}$ is called a comparison triangle for $\triangle(\gamma_1, \gamma_2, \gamma_3)$ if $\ell(\bar{\gamma_i}) = \ell(\gamma_i)$ for $i = 1, 2, 3$.

For any real number κ, a metric space X has Alexandrov curvature at most κ if, by definition, each point of M has an open neighborhood U, called a $CAT(\kappa)$ domain (or an R_κ domain), in which a minimizing geodesic exists joining any given pair of end points and for any minimizing geodesic triangle in U with perimeter less than $\frac{2\pi}{\sqrt{\kappa}}(\infty$ if $\kappa \leq 0)$, the distance between any two points on the triangle is no greater than the distance between corresponding points on the triangle in M_κ with the same side lengths.

A metric space Y is geodesically complete if for any geodesic $\gamma : I \to Y$ there exists an extension $\gamma_{ext} : \mathbb{R} \to Y$ of γ such that $\gamma_{ext}|_I = \gamma$.

From now on, let X be a locally compact, geodesically complete geodesic space of curvature bounded above by κ.

A mapping ϕ from a metric space (Y, d_Y) to a metric space (Z, d_Z) is said to be nonexpanding if $d_Z(\phi(p_1), \phi(p_2)) \leq d_Y(p_1, p_2)$ for any $p_1, p_2 \in Y$. If there exists a nonexpanding mapping from a convex domain D_1 onto another convex domain D_2, then $H^n(D_2) \leq H^n(D_1)$, which is generally known as Kolmogorov's Principle. For details we refer to [5].

A Hadamard space is a complete simply connected metric space of nonpositive curvature, which is a generalization of the Hadamard manifold. By the Hadamard-Cartan theorem, Hadamard spaces are geodesic.

A function $f : X \to \mathbb{R}$ defined on X is called (strictly) convex if for every nontrivial geodesic $\gamma : I \to X$, the real function $f \circ \gamma : I \to \mathbb{R}$ is (strictly) convex. It is known that for any given geodesics $\gamma_1, \gamma_2 : I \to Y$ in a Hadamard space (Y, d_Y), the function $f : I \to \mathbb{R}$ defined by $f(t) = d_Y(\gamma_1(t), \gamma_2(t))$ is convex in t. Also, for a convex subset C in Y, the function $d_C : Y \to \mathbb{R}$ defined by $d_C(z) = d_Y(z, C)$ is convex. If $f : C \to \mathbb{R}$ is a convex function on a (strongly) convex set C in Y, then for any $s \in \mathbb{R}$, the subset $f_{\leq s}$ defined by $f_{\leq s} \equiv \{p \in C : f(p) \leq s\}$ is (strongly) convex in C. Moreover, all metric balls and all ε-neighborhoods of a shortest curve in Y are always strongly convex in Y, respectively.

For a convex closed subset C of a Hadamard space Y, the metric projection $\pi_C : Y \to C$ defined by the closest point $\pi_C(p) \in C$ to the point p is well defined in Y, and the unique point $\pi_C(p) \in C$ is called the footpoint of p on C. Since a geodesic segment pq in Y is a closed convex subset of Y, each point $y \in Y$ has its footpoint on pq. It is known that the metric projection $\pi_C : Y \to C$ is a 1-Lipschitz retraction. Also, for the geodesic segment $y\,\pi_C(y)$ and a geodesic segment $\pi_C(y)\,z$, we have $\angle(\pi_C(y)y, \pi_C(y)z) \geq \frac{\pi}{2}$ for any $z \in C$ [6].

For a Riemannian manifold M and a topologically embedded submanifold P of M, a tube $T(P,r)$ of radius $r \geq 0$ about P is defined as the set of all points m in M such that there exists a geodesic segment γ of length $\ell(\gamma) \leq r$ from m meeting P orthogonally. It is known that for an analytic Riemannian manifold M and an analytically embedded submanifold P with compact closure, the volume of a tube of radius r about P in M is less (more) than that of the corresponding tube of P in $\mathbb{R}^n$ if $2 \leq \dim P \leq n-2$ and the sectional curvature of M is positive (negative) on P, respectively. If $\dim P = 1$ or $n-1$ and the Ricci curvature of M is positive (negative) on P, then the same comparison inequalities hold [8]. Weyl's result for Euclidean space is the following. Denote by $V_P^n(r)$ the volume of a tube $T(P,r)$ about a submanifold P of $\mathbb{R}^n$. Then,

$$V_P^n(r) = \frac{(\pi r^2)^{\frac{n-q}{2}}}{(\frac{n-q}{2})!} \sum_{i=0}^{[\frac{q}{2}]} \frac{k_{2i}(P) r^{2i}}{(n-q+2)(n-q+4)\cdots(n-q+2i)},$$

where $k_{2i}(P)$ is the $(2i)$-th integrated mean curvature of P. For details, we refer to [8] (p. 62). As in Riemannian geometry, volume of some domain in a nonregular metric space of curvature bounded above in the sense of Alexandrov is dependent on the curvature of the space [7, 9, 12]. In order to measure a tube volume in a curved metric space, we need to carefully choose a subspace P for a tube $T(P,r)$. It is because that, unlike a submanifold in Riemannian manifold, subspaces in a curved metric space may have nonconstant dimension, and hence it is unusual to calculate volume depending on dimension. In addition, orthogonality between subspace and geodesics is not natural in such a space.

Proposition 2.1. Let (X, d_X) be an n-dimensional locally compact, geodesically complete Hadamard space. Then, for any point $p \in X$, there exists an expanding mapping E_p from n-dimensional closed ball $B^n(o,\delta)$ in $\mathbb{R}^n$ to the closed metric ball $B(p,\delta)$ in X for some proper $\delta > 0$.

Proof. See Corollary 6.2 in [12].

Definition 2.1. Let (Z,d) be a metric space with an intrinsic metric d. A subspace $(Y, d|_Y)$ of (Z,d) is said to be totally geodesic if any geodesic in (Z,d) meeting Y in two points is totally contained in Y.

Theorem 2.2. *Let (X,d) be an n-dimensional locally compact, geodesically complete Hadamard space, and let G be a totally geodesic subspace* *of X with codimension 1. Then, for a tube $T(K_{p,\varepsilon,G}, r)$ of radius r about a closed metric ball $K_{p,\varepsilon,G}(\subset G)$, we have*

$$H^n(T(K_{p,\varepsilon,G}, r)) \geq L^n(T_0(\bar{K}, r)),$$

where $T_0(\bar{K}, r)$ is the tube in $\mathbb{R}^n$ corresponding to the $T(K_{p,\varepsilon,G}, r)$.

Proof. At first, we notice the closed metric ball $K_{p,\varepsilon,G}$ centered at p of radius ε in G of X is a convex Borel subset of X. Since $K_{p,\varepsilon,G}$ is convex in X, $K_{p,\varepsilon,G}$ is a Hadamard space as a metric subspace of X. Then from Proposition 2.1 there exists a nonexpanding mapping h from $K_{p,\varepsilon,G}$ onto the closed ball $B^{n-1}(o, \varepsilon)$ centered at the origin o of radius ε in an $(n-1)$-dimensional Euclidean space $\mathbb{R}^{n-1}$.

The corresponding tube $T_0(\bar{K}, r) \subset \mathbb{R}^n$ of a tube $T(K_{p,\varepsilon,G}, r)$ of radius r about $K_{p,\varepsilon,G}$ is a tube of radius r about the closed ball $B^{n-1}(o, \varepsilon) \subset \mathbb{R}^{n-1}$ centered at origin o of radius ε, and hence it is the same as the cylinder $B^{n-1}(o, \varepsilon) \times [-r, r]$ in $\mathbb{R}^n$. Devide the tube $T(K_{p,\varepsilon,G}, r)$ by three disjoint regions $K_{p,\varepsilon,G}$, $T^u_K(r)$ and $T^l_K(r)$, which are all Borel subsets of X. Here, $T^u_K(r)$ and $T^l_K(r)$ correspond to $B^{n-1}(o, \varepsilon) \times (0, r]$ and $B^{n-1}(o, \varepsilon) \times [-r, 0)$, respectively.

Since $K_{p,\varepsilon,G}$ is a closed convex subset in $T(K_{p,\varepsilon,G}, r)$, there exists the metric projection π_K from $T(K_{p,\varepsilon,G}, r)$ onto $K_{p,\varepsilon,G}$, and so each point $x \in T(K_{p,\varepsilon,G}, r)$ has its footpoint $\pi_K(x) \in K_{p,\varepsilon,G}$. For $x \in T^u_K(r)$, denote the distance $d(x, K_{p,\varepsilon,G})$ between x and $K_{p,\varepsilon,G}$ by z_x. Then we define a mapping $F^u : T^u_K(r) \to T^u_0(r)$ by

$$F^u(x) = (h(\pi_K(x)), z_x) \in T^u_0(r),$$

where $T^u_0(r)$ denotes the cylinder $B^{n-1}(o, \varepsilon) \times (0, r]$ in $\mathbb{R}^n$. In a similar way, we define a mapping $F^l : T^l_K(r) \to T^l_0(r)$ by

$$F^l(x) = (h(\pi_K(x)), -z_x) \in T^l_0(r),$$

where $T^l_0(r)$ denotes the cylinder $B^{n-1}(o, \varepsilon) \times [-r, 0)$ in $\mathbb{R}^n$.

We will show that the mappings F^u and F^l are nonexpanding. For $x, y \in T^u_K(r)$ with $F^u(x) = (h(\pi_K(x)), z_x)$ and $F^u(y) = (h(\pi_K(y)), z_y)$, we have

$$d_X(\pi_K(x), \pi_K(y)) \geq d_0(h(\pi_K(x)), h(\pi_K(y))),$$

since the mapping $h : K_{p,\varepsilon,G} \to B^{n-1}(o, \varepsilon)$ is nonexpanding, where d_0 denotes the usual Euclidean metric on $\mathbb{R}^n$. On the other hand, the function $f : I \to \mathbb{R}$ defined by $f(t) = d(\gamma_1(t), \gamma_2(t))$ is convex in t for geodesics $\gamma_1, \gamma_2 : I \to X$ such that γ_1 is a geodesic joining two points x and $\pi_K(x)$ and γ_2 is a geodesic joining two points y and $\pi_K(y)$. Therefore, we have $d(x, y) \geq d_0(F^u(x), F^u(y))$, and so the mapping F^u

is nonexpanding.

Hence, from Kolmogorov's Principle, we obtain

$$H^n(T_K^u(r)) \geq L^n(T_0^u(r)).$$

By the same argument, we also obtain that

$$H^n(T_K^l(r)) \geq L^n(T_0^l(r)).$$

Since $T(K_{p,\varepsilon,G}, r)$ is the disjoint union of three Borel sets $K_{p,\varepsilon,G}$, $T_K^u(r)$ and $T_K^l(r)$, we obtain the desired inequality. □

Now we generalize Theorem 2.6 to the case of tube about metric ball in an m-dimensional totally geodesic subspace G of X for $1 < m < n$ under some condition.

THEOREM 2.3. *Let (X, d_X) be an n-dimensional locally compact, geodesically complete Hadamard space, and let $K_{o,\varepsilon,G}$ be a closed metric ball centered at a point $o \in G$ of radius ε in an m-dimensional$(m > 1)$ totally geodesic subspace G of X. We assume that for some $r > 0$, a closed metric ball in $\pi_G^{-1}(o)$ with center o and radius r is a convex subset of X. Then for the tube $T(K_{o,\varepsilon,G}, r)$ about $K_{o,\varepsilon,G}$, we have*

$$H^n(T(K_{o,\varepsilon,G}, r)) \geq L^n(B^m(\bar{o}, \varepsilon) \times B^{n-m}(\bar{o}, r)).$$

Proof. Since G is a totally geodesic subspace of X, the metric projection $\pi_G : X \to G$ is well-defined, and hence, for each point $x \in T(K_{o,\varepsilon,G}, r)$, there exists a unique footpoint $\pi_G(x)$ in $K_{o,\varepsilon,G}$. We denote it by x_K, for notational convenience. From Proposition 2.1, there exists a nonexpanding mapping E_o from $K_{o,\varepsilon,G}$ to an m-dimensional closed ball $B^m(\bar{o}, \varepsilon)$ in $\mathbb{R}^m$, where $\bar{o}$ is the origin of $\mathbb{R}^m$.

Let Q be the closed metric ball centered at o of radius r contained in $\pi_G^{-1}(o)$. By assumption that Q is convex in X, there exists a unique footpoint $\pi_Q(x)$ in Q, and denote it by x_Q. Then there exists a nonexpanding mapping J from $Q \subset \pi_G^{-1}(o)$ to $B^{n-m}(\bar{o}, r) \subset \mathbb{R}^{n-m}$. Thus

$$d_X(x_Q, y_Q) \geq d_0(J(x_Q), J(y_Q)).$$

Let $\overline{T}$ be the tube about the ball $B^m(\bar{o}, \varepsilon)$ in $\mathbb{R}^n$. Then

$$\overline{T} = B^m(\bar{o}, \varepsilon) \times B^{n-m}(\bar{o}, r),$$

which is a corresponding tube of $T(K_{o,\varepsilon,G}, r)$.

Now we define a mapping F from $T(K_{o,\varepsilon,G}, r)$ to $\overline{T}$ by

$$F(x) = (E_o(x_K), J(x_Q)).$$

Let $x, y \in T(K_{o,\varepsilon,G}, r)$. Since E_o is nonexpanding, we have

$$d_X(x_K, y_K) \geq d_0(E_o(x_K), E_o(y_K)).$$

Since the mapping J is also nonexpanding and $\angle(\overline{ox_K}, \overline{ox_Q}) \geq \frac{\pi}{2}$ for any point $x \in X$, $d_X(x, y) \geq d_0(F(x), F(y))$, i.e., F is nonexpanding. Therefore, we obtain the volume comparison inequality from Kolmogorov's Principle. □

A convex domain U in a locally compact metric space (M, d) is called an $R_{\kappa',\kappa}$-domain if for any triangle $\triangle$ in U (the perimeter of triangle is less than $2\pi/\sqrt{\kappa}$ if $\kappa > 0$),

$$d(p, q) \leq d_\kappa(\bar{p}, \bar{q}) \ , \ d(p, q) \geq d_{\kappa'}(\bar{p'}, \bar{q'}),$$

where for points p, q on sides of $\triangle$, $\bar{p}, \bar{q}$ are the corresponding points on the sides of the comparison triangle $\overline{\triangle} \subset M_\kappa$ and $\bar{p'}, \bar{q'}$ are the corresponding points on the sides of the comparison triangle $\overline{\triangle'} \subset M_{\kappa'}$. d_κ and $d_{\kappa'}$ denote the metrics of model surfaces M_κ and $M_{\kappa'}$, respectively. M is called a space of curvature $\leq \kappa$ and $\geq \kappa'$ if each point has a neighborhood that is an $R_{\kappa',\kappa}$-domain.

For $\delta > 0$, let $\gamma_1, \gamma_2 : [0, \delta] \to X$ be a pair of unit speed geodesics emanating from a point p in X. For $s, t \in (0, \delta]$ let $\overline{\triangle}_{st} = (\bar{p}, \bar{\gamma}_1(s), \bar{\gamma}_2(t)) \subset M_\kappa$ be the comparison triangle for the triangle $\triangle_{st} = (p, \gamma_1(s), \gamma_2(t))$. Then the angle $\angle(\gamma_1, \gamma_2)$ at p between γ_1 and γ_2 in X is defined by $\angle(\gamma_1, \gamma_2) = \lim_{s,t \to 0} \alpha(s, t)$, where $\alpha(s, t)$ is the angle of $\overline{\triangle}_{st}$ at $\bar{p}$ in M_κ. Two geodesics are said to be equivalent if the angle between them is zero. The direction space D_pX at $p \in X$ defined as the set of the equivalence classes of geodesics in X emanating from p is a metric space with the angle metric [2]. The tangent cone T_pX at $p \in X$ is the Euclidean cone over the direction space D_pM.

For a point p in a topological manifold M, a geodesic starts out in every direction at p, and the direction space D_pM with the angle metric is compact for each $p \in M$. In particular, D_pM is a geodesic space at the distances less than π [1, 4].

Remark 2.4. In a locally compact, geodesically complete, intrinsic metric space (M, d) of curvature $\leq \kappa$ and $\geq \kappa'$ the tangent cone T_pM for $p \in M$ is isometric to Euclidean space of the same finite dimension. (M, d) is a topological manifold of this dimension [3].

Suppose that $U \subset R_{K',K}$ is a sufficiently small convex domain, and AA' is a geodesic segment in U. Let $B(O, r)$ be the ball in the tangent space to M at A and $\zeta \in B(O, r)$. Draw the geodesic AH of length $||\zeta||$

from A in the direction of the vector ζ, namely, $H = \exp_A \zeta$. Let H' be the point symmetric to H with respect to the midpoint of the geodesic AA'. Then $\prod_{AA'}(\zeta)$ is defined by $\prod_{AA'}(\zeta) \equiv \exp_{A'}^{-1}(H')$.

Now let AB be an arbitrary geodesic segment in U. Divide AB into 2^j equal segments by points $A = A_0, A_1, A_2, \cdots, A_{2^j} = B$. Put $h_j = \frac{d(A,B)}{2^j}$. Also, we denote $\prod_{A_i A_{i+1}}$ constructed as above by $\prod_{i,i+1}$.

Definition 2.5. Let ζ be an arbitrary tangent vector to M at A, and put $\zeta' = h_j \frac{\zeta}{||\zeta||}$. Consider the map

$$\prod_j{}'(\zeta) = \prod_{2^j-1,2^j} \circ \prod_{2^j-2,2^j-1} \circ \cdots \circ \prod_{1,2} \circ \prod_{0,1}(\zeta').$$

We define $\prod_j(\zeta)$ as the vector with length $||\zeta||$ in the same direction as $\prod_j'(\zeta)$. Then the map $\prod : T_A M \to T_B M$ defined by

$$\prod(\zeta) = \lim_{j\to\infty} \prod_j(\zeta), \zeta \in T_A M$$

is called the parallel translation along the geodesic segment AB.

Proposition 2.2. The parallel translation along a geodesic segment AB preserves both angles between vectors and lengths of vectors, that is,

$$\angle(\zeta,\eta) = \angle(\prod(\zeta), \prod(\eta)) \ , \ ||\zeta|| = ||\prod(\zeta)||.$$

Proof. See [3].

Let $I = [a,b]$ be a closed interval of $\mathbb{R}$ and $\gamma : I \to X$ a shortest curve parametrized by arc length, and let $\pi_\gamma : X \to \gamma(I)$ be the metric projection. The tube $T(\gamma, r)$ of radius r about γ is defined by the set

$$T(\gamma, r) = \{x \in X : \pi_\gamma(x) = \pi_{\gamma_{ext}}(x) \in \gamma(I), d_X(x, \gamma(I)) \le r\},$$

where $\gamma_{ext} : J \to X$ is an extension of γ. Put

$$HB(a) = \{x \in X : \pi_\gamma(x) = a, \pi_{\gamma_{ext}}(x) \ne a, d_X(x, \gamma(I)) \le r\}$$

and

$$HB(b) = \{x \in X : \pi_\gamma(x) = b, \pi_{\gamma_{ext}}(x) \ne b, d_X(x, \gamma(I)) \le r\}.$$

Then

$$\{x \in X : d_X(x, \gamma(I)) \le r\} = T(\gamma, r) \cup HB(a) \cup HB(b).$$

Proposition 2.1 implies that $H^n(HB(a)) \ge \frac{1}{2} L^n(B^n(o,r))$ and $H^n(HB(b)) \ge \frac{1}{2} L^n(B^n(o,r))$.

THEOREM 2.6. *Let (X, d_X) be an n-dimensional locally compact, geodesically complete simply connected space of curvature ≤ 0 and $\geq \kappa'$, where $\kappa' < 0$. Then for a shortest curve $\gamma : I \to X$ parametrized by arc length, we have*

$$H^n(T(\gamma, r)) \geq L^n(B^{n-1}(o, r) \times I),$$

where $B^{n-1}(o, r)$ is the closed ball centered at the origin o of radius r in $\mathbb{R}^{n-1}$.

Proof. Without loss of generality, we can assume that the domain I of the curve γ is a straight line segment in $\mathbb{R}^n$ such that the origin $o \in \mathbb{R}^n$ is the midpoint of I. Let $m \in X$ be the middle point of γ, i.e., $\gamma(0) = m$. By Proposition 2.1, there exists a nonexpanding map G from the metric ball $D^n(m, r)$ onto the Euclidean ball $B^n(o, r)$. Moreover, for each point $p \in D^n(m, r)$ such that $\pi_\gamma(p) = m$, $G(p)$ is orthogonal to I in $\mathbb{R}^n$. Also, each point of $\gamma(I)$ corresponds to the unique point of I in $\mathbb{R}^n$ and in particular, m corresponds to o. For each point x in $T(\gamma, r)$, let $t_x \in I$ be the corresponding point of $\pi_\gamma x \in \gamma(I)$.

Let v_x be the tangent vector at $\pi_\gamma x$, which belongs to the equivalence class of the geodesic $x\pi_\gamma x$ emanating from $\pi_\gamma x$. Consider the parallel translation $\prod(v_x)$ of v_x from $\pi_\gamma x$ to m along γ, and denote the image by P_x. By Proposition 2.2, the angle between P_x and the geodesic segment $m\pi_\gamma x$ at m is not less than $\pi/2$. Also, we notice that for $P_x \in D^n(m, r)$, $G(P_x) \in B^{n-1}(o, r)$.

Now we define a mapping $F : T(\gamma, r) \to B^{n-1}(o, r) \times I$ by

$$F(x) = (G(P_x), t_x).$$

The corresponding tube $\bar{T}(I, r)$ of $T(\gamma, r)$ is isometric to $B^{n-1}(o, r) \times I \subset \mathbb{R}^n$.

Let x, y be any two points in $T(\gamma, r)$. Denote the Euclidean parallel translation in $\mathbb{R}^n$ of $G(P_x)$ along I from o to t_x by z_x. Then $F(x) = z_x$. Also, $d(\pi_\gamma x, \pi_\gamma y) = |t_x - t_y|$, $d(\pi_\gamma x, x) = |t_x - z_x|$ and $d(\pi_\gamma y, y) = |t_y - z_y|$. Since G is a nonexpanding map,

$$d_X(P_x, P_y) \geq d_E(G(P_x), G(P_y)),$$

where d_E denote the usual Euclidean metric on $\mathbb{R}^n$. Notice that

$$\angle(x\pi_\gamma x, \pi_\gamma xm) \geq \frac{\pi}{2}\,, \ \angle(y\pi_\gamma y, \pi_\gamma ym) \geq \frac{\pi}{2}.$$

Therefore, we obtain the inequality

$$d_X(x, y) \geq d_E(z_x, z_y),$$

and this implies the consequence. □

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