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ON HS-ALGEBRAS

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Abstract. In this paper, we considered the congruence relation, isomorphism and obtained some properties of HS-algebras.

1. Introduction

The concept of Hilbert algebra was introduced in early 50-ties by L. Henkin and T. Skolem for some investigations of implication in intuitionistic and other classical logics. In 60-ties, these algebras were studied especially, by A. Horn and A. Diego [3] from algebraic point of view. Recently, the Hilbert algebras were treated by D. Buseneag [1, 2]. The present author introduced the notion of HS-algebra [4]. In this paper, we considered the congruence relation, isomorphism and obtained some properties of HS-algebras.

2. Preliminaries

A *Hilbert algebra* is a triple $(X, *, 1)$, where X is a nonempty set, " $*$ " is a binary operation on X, $1 \in X$ is an element such that the following three axioms are satisfied for every $x, y, z \in X$:
(H1) $x * (y * x) = 1$,
(H2) $(x * (y * z)) * ((x * y) * (x * z)) = 1$,
(H3) if $x * y = y * x = 1$ then $x = y$.

If X is a Hilbert algebra, then the relation $x \leq y$ if and only if $x*y = 1$ is a partial order on X, which will be called the *natural ordering* on X. With respect to this ordering, 1 is the largest element of X.

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In a Hilbert algebra X, the following properties hold([3]).

(H4) $x * x = 1$ for all $x \in X$,
(H5) $x * 1 = 1$ for all $x \in X$,
(H6) $x * (y * z) = (x * y) * (x * z)$ for all $x, yz \in X$,
(H7) $1 * x = x$ for all $x \in X$,
(H8) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.
(H9) $x * ((x * y) * y)) = 1$
(H10) $x \leq y$ implies $z * x \leq z * y$ and $y * z \leq x * z$ for all $x, y, z \in X$.

3. HS-algebras

Definition 3.1. By an *HS-algebra* $(X, \cdot, *)$ with two binary operations "$\cdot$" and "$*$" that satisfies the following axioms:

(HS1) $S(X) = (X, \cdot)$ is a semigroup,
(HS2) $H(X) = (X, *, 1)$ is a Hilbert algebra,
(HS3) $x \cdot (y * z) = x \cdot y * x \cdot z$ and $(x * y) \cdot z = x \cdot z * y \cdot z$ for any $x, y, z \in X$.

For convenience, we use the multiplication $x \cdot y$ by xy. X is a *multiplicatively abelian HS-algebra* if $S(X) = (X, \cdot)$ is abelian.

Example 3.2 [4]. Let $X = \{1, a, b, c\}$ in which "$*$" and "$\cdot$" are defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	a	b	1

$\cdot$	1	a	b	c
1	1	1	1	1
a	1	a	1	a
b	1	1	b	b
c	1	a	b	c

It is easy to check that $(X, \cdot, *)$ is an HS-algebra.

Example 3.3 [4]. Let $X = \{1, a, b, c\}$ in which "$*$" and "$\cdot$" are defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	1	1	c
c	1	1	1	1

$\cdot$	1	a	b	c
1	1	1	1	1
a	1	a	1	1
b	1	1	b	c
c	1	1	c	b

It is easy to check that $(X, \cdot, *)$ is an HS-algebra.

Example 3.4 [4]. Let $X = \{1, a, b, c\}$ in which "$*$" and "$\cdot$" are defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	a
c	1	1	1	1

$\cdot$	1	a	b	c
1	1	1	1	1
a	1	a	1	a
b	1	1	b	b
c	1	a	b	c

It is easy to check that $(X, \cdot, *)$ is an HS-algebra.

For any x, y in an HS-algebra X, we define $x \vee y$ as $(y * x) * x$. Note that $x \vee y$ is an upper bound of x and y.

Definition 3.5. An HS-algebra is said to be *commutative* if for all $x, y \in X$,

$$(y * x) * x = (x * y) * y, \text{ i.e., } x \vee y = y \vee x.$$

Lemma 3.6 [4]. Let X be an HS-algebra. Then the following identities hold.

(1) $x1 = 1$ and $1x = 1$ for all $x \in X$,
(2) $x \leq y$ implies $ax \leq ay$ and $xa \leq ya$ for all $x, y, a \in X$,
(3) $x(y \vee z) = xz \vee yz$ for all $x, y, z \in X$.

Definition 3.7 [4]. Let X and X' be HS-algebras. A mapping $f : X \to X'$ is called an *HS-algebra homomorphism* (briefly, *homomorphism*) if $f(x * y) = f(x) * f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in X$.

4. Congruence relation and isomorphism theorem

In what follows, let X denote an HS-algebra unless otherwise specified.

Definition 4.1. Let X be an HS-algebra and let ρ be a binary relation on X. Then

(1) ρ is said to be *right* (resp. *left*) *compatible* if $(x, y) \in \rho$ implies, $(x * z, y * z) \in \rho$ (resp. $(z * x, z * y) \in \rho$) and $(xz, yz) \in \rho$ (resp. $(zx, zy) \in \rho$) for all $x, y, z \in X$;
(2) ρ is said to be *compatible* if $(x, y) \in \rho$ and $(u, v) \in \rho$ imply $(x * u, y * v) \in \rho$ and $(xu, yv) \in \rho$ for all $x, y, u, v \in X$;
(3) A compatible equivalence relation is called a *congruence relation.*

Using the notion of left (resp. right) compatible relation, we give a characterization of a congruence relation.

Theorem 4.2. *Let X be an HS-algebra. Then an equivalence relation ρ on X is congruence if and only if it is both left and right compatible.*

Proof. Assume that ρ is a congruence relation on X. Let $x, y \in X$ be such that $(x, y) \in \rho$. Note that $(z, z) \in \rho$ for all $z \in X$ because ρ is reflexive. It follows from a congruence relation that $(x * z, y * z) \in \rho$ and $(xz, yz) \in \rho$. Hence ρ is right compatible. Similarly, ρ is left compatible.

Conversely, suppose that ρ is both left and right compatible. Let $x, y, u, v \in X$ be such that $(x, y) \in \rho$ and $(u, v) \in \rho$. Then $(x*u, y*u) \in \rho$ and $(xu, yu) \in \rho$. by the right compatibility. Using the left compatibility of ρ, we have $(y * u, y * v) \in \rho$ and $(yu, yv) \in \rho$. It follows from the transitivity of ρ that $(x * u, y * v) \in \rho$ and $(xu, yv) \in \rho$. Hence ρ is congruence. □

For an equivalence relation ρ on an HS-algebra X, we denote

$$x_\rho := \{y \in X \mid (x, y) \in \rho\} \text{ and } X/\rho := \{x_\rho \mid x \in X\}.$$

Theorem 4.3. *Let ρ be a congruence relation on a HS-algebra X. If X is commutative, X/ρ is a HS-algebra under the operations*

$$x_\rho * y_\rho = (x * y)_\rho \text{ and } (x_\rho)(y_\rho) = (xy)_\rho$$

for all $x_\rho, y_\rho \in X/\rho$.

Proof. Since ρ is a congruence relation, the operations are well-defined. Clearly, $(X/\rho, *)$ is a Hilbert-algebra and $(X/\rho, \cdot)$ is a semigroup. For every $x_\rho, y_\rho, z_\rho \in X/\rho$, we have

$$\begin{aligned} x_\rho(y_\rho * z_\rho) &= x_\rho(y * z)_\rho = (x(y * z))_\rho \\ &= (xy * xz)_\rho = (xy)_\rho * (xz)_\rho \\ &= x_\rho y_\rho * x_\rho z_\rho, \end{aligned}$$

and

$$\begin{aligned} (x_\rho * y_\rho)z_\rho &= (x * y)_\rho z_\rho = ((x * y)z)_\rho \\ &= (xz * yz)_\rho = (xz)_\rho * (yz)_\rho \\ &= x_\rho z_\rho * y_\rho z_\rho. \end{aligned}$$

Thus X/ρ is an HS-algebra. □

Theorem 4.4. *Let ρ be a congruence relation on an HS-algebra X. If X is commutative, the mapping $\rho^\star : X \to X/\rho$ defined by $\rho^\star(x) = x_\rho$ for all $x \in X$ is an HS-algebra homomorphism.*

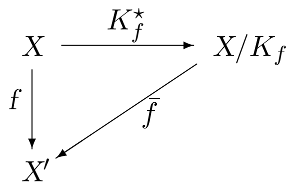
Proof. Let $x, y \in X$. Then $\rho^\star(x*y) = (x*y)_\rho = x_\rho * y_\rho = \rho^\star(x) * \rho^\star(y)$, and $\rho^\star(xy) = (xy)_\rho = (x_\rho)(y_\rho) = \rho^\star(x)\rho^\star(y)$. Hence $\rho^\star$ is an HS-algebra homomorphism. □

It is clear that $\rho^\star$ is clearly surjective.

THEOREM 4.5. *Let X and X' be commutative HS-algebras and let $f : X \to X'$ be an HS-algebra homomorphism. Then the set*

$$K_f := \{(x, y) \in X \times X \mid f(x) = f(y)\}$$

is a congruence relation on X and there exists a unique 1-1 HS-algebra homomorphism $\bar{f} : X/K_f \to X'$ such that $\bar{f} \circ K_f^\star = f$, where $K_f^\star : X \to X/K_f$. That is, the following diagram commute:



Proof. It is clear that K_f is an equivalence relation on X. Let $x, y, u, v \in X$ be such that $(x, y), (u, v) \in K_f$. Then $f(x) = f(y)$ and $f(u) = f(v)$, which imply that

$$f(x * u) = f(x) * f(u) = f(y) * f(v) = f(y * v)$$

and

$$f(xu) = f(x)f(u) = f(y)f(v) = f(yv).$$

It follows that $(x * u, y * v) \in K_f$ and $(xu, yv) \in K_f$. Hence K_f is a congruence relation on X. Let $\bar{f} : X/K_f \to X'$ be a map defined by $\bar{f}(xK_f) = f(x)$ for all $x \in X$. It is clear that $\bar{f}$ is well-defined. For any xK_f, $yK_f \in X/K_f$, we have

$$\begin{aligned}\bar{f}(xK_f * yK_f) &= \bar{f}((x * y)K_f) = f(x * y)\\ &= f(x) * f(y) = \bar{f}(xK_f) * \bar{f}(yK_f)\end{aligned}$$

and

$$\begin{aligned}\bar{f}((xK_f)(yK_f)) &= \bar{f}((xy)K_f) = f(xy)\\ &= f(x)f(y) = \bar{f}(xK_f)\bar{f}(yK_f).\end{aligned}$$

If $\bar{f}(xK_f) = \bar{f}(yK_f)$, then $f(x) = f(y)$ and so $(x, y) \in K_f$, that is, $xK_f = yK_f$. Thus $\bar{f}$ is a 1-1 HS-algebra homomorphism. Now let g be an HS-algebra homomorphism from X/K_f to X' such that $g \circ K_f^\star = f$. Then

$$g(xK_f) = g(K_f^\star(x)) = f(x) = \bar{f}(xK_f)$$

for all $xK_f \in X/K_f$. It follows that $g = \bar{f}$ so that $\bar{f}$ is unique. This completes the proof. $\square$

Corollary 4.6. *Let ρ and σ be congruence relations on an HS-algebra X such that $\rho \subseteq \sigma$. If X is commutative, the set*

$$\sigma/\rho := \{(x_\rho, y_\rho) \in X/\rho \times X/\rho \mid (x, y) \in \sigma\}$$

is a congruence relation on X/ρ and there exists a 1-1 *and onto HS-algebra homomorphism from $\frac{X/\rho}{\sigma/\rho}$ to X/σ.*

Proof. Let $g : X/\rho \to X/\sigma$ be a function defined by $g(x_\rho) = x_\sigma$ for all $x_\rho \in X/\rho$. Since $\rho \subseteq \sigma$, it follows that g is a well-defined onto HS-algebra homomorphism. According to Theorem 4.5, it is sufficient to show that $K_g = \sigma/\rho$. Let $(x_\rho, y_\rho) \in K_g$. Then $x_\sigma = g(x_\rho) = g(y_\rho) = y_\sigma$ and so $(x, y) \in \sigma$. Hence $(x_\rho, y_\rho) \in \sigma/\rho$, and thus $K_g \subseteq \sigma/\rho$.

Conversely, if $(x_\rho, y_\rho) \in \sigma/\rho$, then $(x, y) \in \sigma$ and so $x_\sigma = y_\sigma$. It follows that

$$g(x_\rho) = x\sigma = y\sigma = g(y_\rho)$$

so that $(x_\rho, y_\rho) \in K_g$. Hence $K_g = \sigma/\rho$, and the proof is complete. $\square$

Definition 4.7. Let X be an HS-algebra. A subalgebra I of $(X, *)$ is called a *left ideal* of X if $XI \subseteq I$, a *right ideal* if $IX \subseteq I$, and an (*two-sided*) *ideal* if it is both a left and right ideal.

Theorem 4.8. *Let I be an ideal of an HS-algebra X. Then $\rho_I := (I \times I) \cup \Delta_X$ is a congruence relation on X, where $\Delta_X := \{(x, x) \mid x \in X\}$.*

Proof. Clearly, ρ_I is reflexive and symmetric. Noticing that $(x, y) \in \rho_I$ if and only if $x, y \in I$ or $x = y$, we know that if $(x, y) \in \rho_I$ and $(y, z) \in \rho_I$ then $(x, z) \in \rho_I$. Hence ρ_I is an equivalence relation on X. Assume that $(x, y) \in \rho_I$ and $(u, v) \in \rho_I$. Then we have the following four cases: (i) $x, y \in I$ and $u, v \in I$; (ii) $x, y \in I$ and $u = v$; (iii) $x = y$ and $u, v \in I$; and (iv) $x = y$ and $u = v$. In either case, we get $x*u = y*v$ or $(x * u, y * v) \in I \times I$, and $xu = yv$ or $(xu, yv) \in I \times I$. Therefore ρ_I is a congruence relation on X. $\square$

Let X be a multiplicatively abelian HS-algebra and ρ_X be a binary relation on X defined by

$$(a, b) \in \rho_X \Longleftrightarrow \exists u \in X \text{ such that } au = bu. \qquad (\bullet)$$

Clearly, ρ_X is reflexive and symmetric. Let $(a, b), (b, c) \in \rho_X$. Then there exist $u, v \in X$ such that $au = bu$ and $bv = cv$. These imply $a(buv) = (au)(bv) = (bu)(cv) = c(buv)$, whence ρ_X is transitive. Thus ρ_X is an equivalence relation on X.

Theorem 4.9. *Let X be a multiplicatively abelian HS-algebra and ρ_X be a binary relation on X defined by ($\bullet$). If X is commutative, ρ_X is a congruence relation on X, and X/ρ_X is a multiplicatively abelian HS-algebra.*

Proof. Let $(a,b),(c,d) \in \rho_X$, Then there exist $u, v \in X$ such that $au = bu$ and $cv = dv$. These imply $(ac)(uv) = (au)(cv) = (bu)(dv) = (bd)(uv)$ and $(a * c)(uv) = auv * cuv = buv * duv = (b * d)uv$. Hence $(ac, bd) \in \rho_X$ and $(a * c, b * d) \in \rho_X$. Thus ρ_X is a congruence relation on X, and clearly X/ρ_X is a multiplicatively abelian HS-algebra. □

Let X be a multiplicatively abelian HS-algebra. If X is commutative, a map $(\rho_X)^\star : X \to X/\rho_X$ defined by

$$(\rho_X)^\star(a) = a\rho_X$$

is a surjective HS-algebra homomorphism.

Theorem 4.10. *Let X and X' be multiplicatively abelian HS-algebras with X/ρ_X and X'/ρ'_X, respectively and $\phi : X \to X'$ be an HS-algebra homomorphism. If X and X' are commutative, there exists a unique homomorphism $\phi/\rho : X/\rho_X \to X'/\rho_{X'}$ such that $\phi/\rho \circ (\rho_X)^\star = (\rho_{X'})^\star \circ \phi$.*

Proof. Define $\phi/\rho : X/\rho_X \to X'/\rho_{X'}$ by $\phi/\rho(a\rho_X) = \phi(a)\rho_{X'}$. If $a\rho_X = b\rho_X$, then there exists $u \in X$ such that $au = bu$. Thus $\phi(a)\phi(u) = \phi(b)\phi(u)$ and $(\phi(a), \phi(b)) \in \rho_{X'}$, so $\phi(a)\rho_{X'} = \phi(b)\rho_{X'}$. Therefore ϕ/ρ is well-defined. Next, we prove that ϕ/ρ is a homomorphism. In fact, $\phi/\rho(a\rho_X * b\rho_X) = \phi/\rho((a * b)\rho_X) = \phi(a * b)\rho_{X'} = (\phi(a) * \phi(b))\rho_{X'} = \phi(a)\rho_{X'} * \phi(b)\rho_{X'} = \phi/\rho(a\rho_X) * \phi/\rho(b\rho_X)$ and $\phi/\rho(a\rho_X \cdot b\rho_X) = \phi/\rho((ab)\rho_X) = \phi(ab)\rho_{X'} = (\phi(a) \cdot \phi(b))\rho_{X'} = \phi(a)\rho_{X'} \cdot \phi(b)\rho_{X'} = \phi/\rho(a\rho_X) \cdot \phi/\rho(b\rho_X)$. For any $a \in X$, we have $(\phi/\rho \circ (\rho_X)^\star)(a) = \phi/\rho((\rho_X)^\star(a)) = \phi/\rho(a\rho_X) = \phi(a)\rho_{X'} = (\rho_{X'})^\star(\phi(a)) = ((\rho_{X'})^\star \circ \phi)(a)$. Thus $\phi/\rho \circ (\rho_X)^\star = (\rho_{X'})^\star \circ \phi$. Finally, if there exists a homomorphism $g : X/\rho_X \to X'/\rho_{X'}$ such that $g \circ (\rho_X)^\star = (\rho_{X'})^\star \circ \phi$, then $g(a\rho_X) = g((\rho_X)^\star(a)) = (g \circ (\rho_X)^\star)(a) = ((\rho_{X'})^\star \circ \phi)(a) = (\rho_{X'})^\star(\phi(a)) = \phi(a)\rho_{X'} = \phi/\rho(a\rho_X)$. Thus $g = \phi/\rho$ and ϕ/ρ is unique. □

It is clear that $Hom(X, X')$ is a semigroup under multiplication defined by $(\phi_1 \cdot \phi_2)(a) = \phi_1(a) \cdot \phi_2(a)$. Likewise $Hom(X/\rho_X, X'/\rho_{X'})$ is a semigroup by Theorem 4.10, we can define a mapping

$$\Phi : Hom(X, X') \to Hom(X/\rho_X, X'/\rho_{X'})$$

by $\Phi(\phi) = \phi/\rho$. Then we have the following theorem.

THEOREM 4.11. *Let X and X' be multiplicatively abelian HS-algebras with X/ρ_X and $X'/\rho_{X'}$, respectively. If X and X' are commutative, the above mapping Φ given by $\Phi(\phi) = \phi/\rho$ is a semigroup homomorphism.*

Proof. Let $\phi_1, \phi_2 \in Hom(X, X')$ and $a\rho_X \in X/\rho_X$. Then $((\phi_1 \cdot \phi_2)/\rho)(a\rho_X) = ((\phi_1 \cdot \phi_2)(a))\rho_{X'} = (\phi_1(a) \cdot \phi_2(a))\rho_{X'} = \phi_1(a)\rho_{X'} \cdot \phi_2(a)\rho_{X'} = \phi_1/\rho(a\rho_X) \cdot \phi_2/\rho(a\rho_X) = (\phi_1/\rho \cdot \phi_2/\rho)(a\rho_X)$. Consequently, $(\phi_1 \cdot \phi_2)/\rho = \phi_1/\rho \cdot \phi_2/\rho$. Thus the map

$$\Phi : Hom(X, X') \rightarrow Hom(X/\rho_X, X'/\rho_{X'})$$

given by $\Phi(\phi) = \phi/\rho$ is a semigroup homomorphism. □

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