JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **24**, No. 3, September 2011

# A NOTE ON WEAK QUASI GENERALIZED CONTINUITY ON BIGENERALIZED TOPOLOGICAL SPACES

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Abstract. We introduce the notion of weakly quasi generalized continuous functions, and investigate properties for the continuity.

## 1. Introduction

Császár [1] introduced the notions of generalized topologies and associated interior(closure) operators on generalized topological spaces. In [3], the concept of bitopology was introduced by Kelly. He called a set equipped with two topologies a bitopological space. Datta [2] has stated that a subset $S$ of bitopological space $(X, P, Q)$ is *quasiopen* if for every $x \in S$, there exists a $P$-open set $U$ such that $x \in U \subseteq S$, or a $Q$-open set $V$ such that $x \in V \subseteq S$. In [7], we introduced and investigated the notions of quasi generalized open sets and quasi generalized continuity. The purpose of this paper is to introduce and investigate the notion of weakly quasi generalized continuous functions, and investigate properties for the continuity.

## 2. Preliminaries

Let $X$ be a nonempty set and $\psi$ be a collection of subsets of $X$. Then $\psi$ is called a *generalized topology* (briefly GT) on $X$ iff $\emptyset \in \psi$ and $G_i \in \psi$ for $i \in I \neq \emptyset$ implies $G = \cup_{i \in I} G_i \in \psi$. We call the pair $(X, \psi)$ a

Received January 08, 20011; Revised September 01, 2011; Accepted September 02, 2011.
2010 Mathematics Subject Classification: Primary 54A05, 54C05.
Key words and phrases: generalized topological space, bigeneralized topological space, quasi generalized open sets, quasi generalized continuous function, weakly quasi generalized continuous function.
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*generalized topological space* (briefly GTS) on $X$. The elements of $\psi$ are called *$\psi$-open* sets and the complements are called *$\psi$-closed* sets. Denote $\mathcal{M}_\psi = \cup\{M \subseteq X : M \in \psi\}$.

If $\psi$ is a generalized topology on $X$ and $A \subseteq X$, the *interior* of $A$ (denoted by $i_\psi(A)$) is the union of all $G \subseteq A$, $G \in \psi$, and the *closure* of $A$ (denoted by $c_\psi(A)$) is the intersection of all $\psi$-closed sets containing $A$. Let $\psi$ and $\mu$ be generalized topologies on $X$ and $Y$, respectively. Then a function $f : (X, \psi) \to (Y, \mu)$ is said to be *$(\psi, \mu)$-continuous* [1] if $G \in \mu$ implies that $f^{-1}(G) \in \psi$.

Let $X$ be a nonempty set, and let $\psi_1, \psi_2$ be generalized topologies on $X$. A triple $(X, \psi_1, \psi_2)$ is called a *bigeneralized topological space* (briefly *biGTS*) [7]. Let $(X, \psi_1, \psi_2)$ be a biGTS. A subset $A$ of $X$ is said to be *quasi $(\psi_1, \psi_2)$-open* (briefly *quasi $q_\psi$-open*) [7] if for every $x \in A$, there exists a $\psi_1$-open set $U$ such that $x \in U \subseteq A$, or a $\psi_2$-open set $V$ such that $x \in V \subseteq A$. A subset $A$ of $X$ is said to be *quasi $q_\psi$-closed* if the complement of $A$ is quasi $q_\psi$-open.

Let $(X, \psi_1, \psi_2)$ be a biGTS and $A \subseteq X$. We define the *quasi $q_\psi$-closure* (briefly $c_{q_\psi}(A)$) and the *quasi $q_\psi$-interior* (briefly $i_{q_\psi}(A)$) as the following:

$$c_{q_\psi}(A) = \cap\{F : A \subseteq F \text{ for a quasi } q_\psi\text{-closed set } F\};$$

$$i_{q_\psi}(A) = \cup\{G : G \subseteq A \text{ for a quasi } q_\psi\text{-open set } G\}.$$

LEMMA 2.1 ([7]). *Let $(X, \psi_1, \psi_2)$ be a biGTS and $A \subseteq X$. Then*

*(1) $A$ is quasi $q_\psi$-open if and only if $A$ is a union of a $\psi_1$-open set and a $\psi_2$-open set.*

*(2) $A$ is quasi $q_\psi$-closed if and only if $A$ is a intersection of a $\psi_1$-closed set and a $\psi_2$-closed set.*

*(3) Any union of quasi $q_\psi$-open sets is quasi $q_\psi$-open.*

Let $(X, \psi_1, \psi_2)$ be a biGTS and $A \subseteq X$. The $\psi_i$-closure and $\psi_i$-interior of $A$ with respect to $\psi_i$ are denoted by $c_{\psi_i}(A)$ and $i_{\psi_i}(A)$, respectively, for $i = 1, 2$.

THEOREM 2.2 ([7]). *Let $(X, \psi_1, \psi_2)$ be a biGTS and $A \subseteq X$. Then*

*(1) $c_{q_\psi}(A) = c_{\psi_1}(A) \cap c_{\psi_2}(A)$.*

*(2) $i_{q_\psi}(A) = i_{\psi_1}(A) \cup i_{\psi_2}(A)$.*

*(3) $A$ is quasi $q_\psi$-closed iff $c_{q_\psi}(A) = A$.*

*(4) $A$ is quasi $q_\psi$-open iff $i_{q_\psi}(A) = A$.*

*(5) $x \in c_{q_\psi}(A)$ iff for every quasi $q_\psi$-open set $U$ containing $x$, $A \cap U \neq \emptyset$.*

*(6) $x \in i_{q_\psi}(A)$ iff there exists a quasi $q_\psi$-open set $U$ containing $x$ such that $x \in U \subseteq A$.*

*(7) $c_{q_\psi}(A) = X - i_{q_\psi}(X - A)$; $i_{q_\psi}(A) = X - c_{q_\psi}(X - A)$*

## 3. Weakly quasi $(q_\psi, q_\mu)$-continuous functions

Definition 3.1 ([7]). Let $(X, \psi_1, \psi_2)$ and $(Y, \mu_1, \mu_2)$ be two biGTS's. Then a function $f : X \to Y$ is said to be *quasi $(q_\psi, q_\mu)$-continuous* (or *quasi generalized continuous*) if for every quasi $q_\mu$-open set $U$ in $Y$, $f^{-1}(U)$ is quasi $q_\psi$-open in $X$.

Definition 3.2. Let $(X, \psi_1, \psi_2)$ and $(Y, \mu_1, \mu_2)$ be two biGTS's. Then a function $f : X \to Y$ is said to be *weakly quasi $(q_\psi, q_\mu)$-continuous* if for each $x \in X$ and each quasi $q_\mu$-open set $V$ containing $f(x)$, there exists a quasi $q_\psi$-open set $U$ containing $x$ such that $f(U) \subseteq c_{q_\mu}(V)$.

Let $\psi$ and $\psi'$ be generalized topologies on $X$ and $Y$, respectively. Then a function $f : X \to Y$ is said to be *weakly $(\psi, \psi')$-continuous* [5] if for each $x \in X$ and each $\psi'$-open set $V$ containing $f(x)$, there exists a $\psi$-open set $U$ containing $x$ such that $f(U) \subseteq c_{\psi'}(V)$.

Remark 3.3. Let $(X, \psi_1, \psi_2)$ and $(Y, \mu_1, \mu_2)$ be biGTS's. Then

(1) if $\psi_1 = \psi_2$ and $\mu_1 = \mu_2$ and if $f : (X, \psi_1, \psi_2) \to (Y, \mu_1, \mu_2)$ is a weakly quasi $(q_\psi, q_\mu)$-continuous (resp., quasi $(q_\psi, q_\mu)$-continuous) function, then it is weakly $(\psi_1, \mu_1)$-continuous (resp., $(\psi_1, \mu_1)$-continuous);

(2) if $f$ is a quasi $(q_\psi, q_\mu)$-continuous function, then it is weakly quasi $(q_\psi, q_\mu)$-continuous. But the converse may not be true in general as shown in the following example.

Example 3.4. Let $X = \{a, b, c\}$ and $Y = \{a, b, c, d\}$. Consider

$\psi_1 = \{\emptyset, \{a\}\}$ and $\psi_2 = \{\emptyset, \{b, c\}\}$ on $X$;

$\mu_1 = \{\emptyset, \{a\}\}$ and $\mu_2 = \{\emptyset, \{a, b\}\}$ on $Y$.

Let us define a function $f : (X, \psi_1, \psi_2) \to (Y, \mu_1, \mu_2)$ as follows $f(a) = a$, $f(b) = c$, $f(c) = b$. Then it is obvious that $f$ is weakly quasi $(q_\psi, q_\mu)$-continuous but it is not quasi $(q_\psi, q_\mu)$-continuous.

Theorem 3.5. *Let $f : (X, \psi_1, \psi_2) \to (Y, \mu_1, \mu_2)$ be a function on two biGTS's $(X, \psi_1, \psi_2)$ and $(Y, \mu_1, \mu_2)$. Then the following are equivalent:*

*(1) $f$ is weakly quasi $(q_\psi, q_\mu)$-continuous.*

*(2) $f^{-1}(V) \subseteq i_{q_\psi}(f^{-1}(c_{q_\mu}(V)))$ for every quasi $q_\mu$-open subset $V$ of $Y$.*

*(3) $c_{q_\psi}(f^{-1}(i_{q_\mu}(F))) \subseteq f^{-1}(F)$ for every quasi $q_\mu$-closed set $F$ of $Y$.*

*(4)* $c_{q_\psi}(f^{-1}(i_{q_\mu}(c_{q_\mu}(B)))) \subseteq f^{-1}(c_{q_\mu}(B))$ *for every set* $B$ *of* $Y$.

*(5)* $f^{-1}(i_{q_\mu}(B)) \subseteq i_{q_\psi}(f^{-1}(c_{q_\mu}(i_{q_\mu}(B)))$ *for every set* $B$ *of* $Y$.

*(6)* $c_{q_\psi}(f^{-1}(V)) \subseteq f^{-1}(c_{q_\mu}(V))$ *for every quasi* $q_\mu$*-open subset* $V$ *of* $Y$.

*Proof.* (1) $\Rightarrow$ (2) Let $V$ be a quasi $q_\mu$-open subset of $Y$ and $x \in f^{-1}(V)$. Then there exists a quasi $q_\psi$-open subset $U$ of $X$ containing $x$ such that $f(U) \subseteq c_{q_\mu}(V)$. Since $x \in U \subseteq f^{-1}(c_{q_\mu}(V))$, from Theorem 2.2, $x \in i_{q_\psi}(f^{-1}(c_{q_\mu}(V)))$. Hence $f^{-1}(V) \subseteq i_{q_\psi}(f^{-1}(c_{q_\mu}(V)))$.

(2) $\Rightarrow$ (3) Let $F$ be a quasi $q_\mu$-closed subset in $Y$. Then by (2), $f^{-1}(Y-F) \subseteq i_{q_\psi}(f^{-1}(c_{q_\mu}(Y-F))) = i_{q_\psi}(f^{-1}(Y-i_{q_\mu}(F))) \subseteq X - c_{q_\psi}(f^{-1}(i_{q_\mu}(F)))$.

Thus $c_{q_\psi}(f^{-1}(i_{q_\mu}(F))) \subseteq f^{-1}(F)$.

(3) $\Rightarrow$ (4) For a subset $B$ of $Y$, since $c_{q_\mu}(B)$ is quasi $q_\mu$-closed in $Y$, from (3), it follows $c_{q_\psi}(f^{-1}(i_{q_\mu}(c_{q_\mu}(B)))) \subseteq f^{-1}(c_{q_\mu}(B))$.

(4) $\Rightarrow$ (5) Let $B$ be a subset of $Y$. From (4) and Theorem 2.2, it follows $f^{-1}(i_{q_\mu}(B)) = X - f^{-1}(c_{q_\mu}(Y-B)) \subseteq X - c_{q_\psi}(f^{-1}(i_{q_\mu}(c_{q_\mu}(Y-B)))) = i_{q_\psi}(f^{-1}(c_{q_\mu}(i_{q_\mu}(B))))$. Thus we get the result.

(5) $\Rightarrow$ (6) Let $V$ be a quasi $q_\mu$-open subset of $Y$. Suppose $x \notin f^{-1}(c_{q_\mu}(V))$. Then by Theorem 2.2, there exists a quasi $q_\mu$-open set $U$ containing $f(x)$ such that $U \cap V = \emptyset$, and it implies $c_{q_\mu}(U) \cap V = \emptyset$. For the quasi $q_\mu$-open set $U$, by (5), $x \in f^{-1}(U) \subseteq i_{q_\psi}(f^{-1}(c_{q_\mu}(U)))$. Thus there exists a quasi $q_\psi$-open set $G$ containing $x$ such that $x \in G \subseteq f^{-1}(c_{q_\mu}(U))$. Since $c_{q_\mu}(U) \cap V = \emptyset$ and $f(G) \subseteq c_{q_\mu}(U)$, we have $G \cap f^{-1}(V) = \emptyset$. By Theorem 2.2, $x \notin c_{q_\psi}(f^{-1}(V))$ and consequently, $c_{q_\psi}(f^{-1}(V)) \subseteq f^{-1}(c_{q_\mu}(V))$.

(6) $\Rightarrow$ (1) Let $x \in X$ and $V$ a quasi $q_\mu$-open set in $Y$ containing $f(x)$. Then $V = i_{q_\mu}(V) \subseteq i_{q_\mu}(c_{q_\mu}(V))$, from (6), it follows

$x \in f^{-1}(V) \subseteq f^{-1}(i_{q_\mu}(c_{q_\mu}(V))) = X - f^{-1}(c_{q_\mu}(Y - c_{q_\mu}(V))) \subseteq X - c_{q_\psi}(f^{-1}(Y - c_{q_\mu}(V))) = i_{q_\psi}(f^{-1}(c_{q_\mu}(V)))$. So there exists a quasi $q_\psi$-open subset $U$ containing $x$ in $X$ such that $U \subseteq f^{-1}(c_{q_\mu}(V))$. Hence $f$ is weakly quasi $(q_\psi, q_\mu)$-continuous. □

Definition 3.6. Let $(X, \psi_1, \psi_2)$ be a biGTS. $X$ is said to be *G-quasi* $q_\psi$*-regular* if for each $x \in \mathcal{M}_{\psi_1} \cup \mathcal{M}_{\psi_2}$ and a quasi $q_\psi$-closed set $F$ with $x \notin F$, there exist disjoint quasi $q_\psi$-open sets $U$ and $V$ such that $x \in U$ and $F \cap (\mathcal{M}_{\psi_1} \cup \mathcal{M}_{\psi_2}) \subseteq V$.

Remark 3.7. Let $(X, \mu)$ be a generalized topological space. Then $X$ is said to be *relative G-regular* (simply, *G-regular*) [6] on $\mathcal{M}_\mu$ if for $x \in \mathcal{M}_\mu$ and a $\mu$-closed set $F$ with $x \notin F$, there exist $U, V \in \mu$ such that $x \in U$, $F \cap \mathcal{M}_\mu \subseteq V$ and $U \cap V = \emptyset$. For a biGTS $(X, \psi_1, \psi_2)$, if $\psi_1 = \psi_2$, then $X$ is the $G$-regular space.

We recall: Let $(X, \psi_1, \psi_2)$ be a biGTS. Then $X$ is said to be *bi-strong* [7] if $X = \mathcal{M}_{\psi_1} \cup \mathcal{M}_{\psi_2}$.

Lemma 3.8. *Let $(X, \psi_1, \psi_2)$ be a biGTS. If $X$ is bi-strong and G-quasi $q_\psi$-regular, then for each $x \in X$ and a quasi $q_\psi$-closed set $F$ with $x \notin F$, there exist disjoint quasi $q_\psi$-open sets $U$ and $V$ such that $x \in U$ and $F \subseteq V$.*

Theorem 3.9. *Let $(X, \psi_1, \psi_2)$ be a biGTS. Then $X$ is G-quasi $q_\psi$-regular if and only if for $x \in \mathcal{M}_\psi$ and each quasi $q_\psi$-open set $U$ containing $x$, there is a quasi $q_\psi$-open set $V$ containing $x$ such that $x \in V \subseteq c_{q_\psi}(V) \cap \mathcal{M}_\psi \subseteq U$ where $\mathcal{M}_\psi = \mathcal{M}_{\psi_1} \cup \mathcal{M}_{\psi_2}$.*

*Proof.* Let $X$ be $G$-quasi $q_\psi$-regular. Then for $x \in \mathcal{M}_\psi$ and a quasi $q_\psi$-open set $U$ containing $x$, since the quasi $q_\psi$-closed set $U^c$ does not contain the point $x$, there are disjoint quasi $q_\psi$-open sets $V, W$ such that $x \in V$ and $U^c \cap \mathcal{M}_\psi \subseteq W$. Since $V \subseteq W^c$ and $W^c$ is quasi $q_\psi$-closed, $c_\psi V \subseteq W^c$ and $c_\psi(V) \cap (U^c \cap \mathcal{M}_\psi) \subseteq c_\psi V \cap W = \emptyset$. It implies $c_\psi V \cap \mathcal{M}_\psi \subseteq U$.

For the converse, let $F$ be a quasi $q_\psi$-closed set and $x \notin F$ for $x \in \mathcal{M}_\psi$. Then by hypothesis, there is a quasi $q_\psi$-open set $V$ containing $x$ such that $x \in V \subseteq c_\psi V \cap \mathcal{M}_\psi \subseteq F^c$, so $c_\psi V \cap \mathcal{M}_\psi \cap F = \emptyset$ and $\mathcal{M}_\psi \cap F \subseteq (c_\psi V)^c$. Hence $X$ is $G$-quasi $q_\psi$-regular. □

Corollary 3.10. Let $(X, \psi_1, \psi_2)$ be a biGTS, and let $X$ be bi-strong. Then $X$ is $G$-quasi $q_\psi$-regular if and only if for $x \in X$ and each quasi $q_\psi$-open set $U$ containing $x$, there is a quasi $q_\psi$-open set $V$ containing $x$ such that $x \in V \subseteq c_{q_\psi}(V) \subseteq U$.

Let $(X, \psi_1, \psi_2)$ and $(Y, \mu_1, \mu_2)$ be two biGTS's. In Theorem 4.3 (6) of [7], we showed that $f$ is quasi $(q_\psi, q_\mu)$-continuous if and only if for each $x \in X$ and each quasi $q_\mu$-open set $V$ containing $f(x)$, there exists a quasi $q_\psi$-open set $U$ containing $x$ such that $f(U) \subseteq V$.

From the above fact, we have the following theorem.

Theorem 3.11. *Let $f : (X, \psi_1, \psi_2) \to (Y, \mu_1, \mu_2)$ be a function on biGTS's. Let $Y$ be G-quasi $q_\mu$-regular and $f(\mathcal{M}_\psi) \subseteq \mathcal{M}_\mu$. If $f$ is weakly quasi $(q_\psi, q_\mu)$-continuous, then $f$ is quasi $(q_\psi, q_\mu)$-continuous.*

*Proof.* Let $x \in X$ and $V$ any quasi $q_\mu$-open set containing $f(x)$. Then from $G$-quasi $q_\mu$-regularity, there exists a quasi $q_\mu$-open set $W$ such that $f(x) \in W \subseteq c_\mu(W) \cap \mathcal{M}_\mu \subseteq V$. For a quasi $q_\mu$-open set $W$ containing $f(x)$, since $f$ is weakly quasi $(q_\psi, q_\mu)$-continuous, there exists a quasi $q_\psi$-open set $U$ such that $f(U) \subseteq c_\mu(W)$. From $f(U) \subseteq \mathcal{M}_\mu$, it follows

$$f(U) \subseteq f(U) \cap \mathcal{M}_\mu \subseteq c_\mu(W) \cap \mathcal{M}_\mu \subseteq V.$$

Hence from Theorem 4.3 (6) of [7], $f$ is quasi $(q_\psi, q_\mu)$-continuous. □

COROLLARY 3.12. Let $f : (X, \psi_1, \psi_2) \to (Y, \mu_1, \mu_2)$ be a function on biGTS's. If $Y$ is $G$-quasi $q_\mu$-regular and bi-strong, and if $f$ is weakly quasi $(q_\psi, q_\mu)$-continuous, then $f$ is quasi $(q_\psi, q_\mu)$-continuous.

*Proof.* Since $f(\mathcal{M}_\psi) \subseteq Y = \mathcal{M}_\mu$, by Theorem 3.11, it is obvious. □

DEFINITION 3.13. Let $f : (X, \psi_1, \psi_2) \to (Y, \mu_1, \mu_2)$ be a function on biGTS's. Then a function $f : X \to Y$ is said to be *quasi* $(q_\psi, q_\mu)$*-open* if for every quasi $q_\psi$-open set $G$ in $X$, $f(G)$ is quasi $q_\mu$-open in $Y$.

Obviously we get the following lemmas.

LEMMA 3.14. *Let $f : (X, \psi_1, \psi_2) \to (Y, \mu_1, \mu_2)$ be quasi $(q_\psi, q_\mu)$-open on two biGTS's $(X, \psi_1, \psi_2), (Y, \mu_1, \mu_2)$. If $f$ is surjective and $X$ is bi-strong, then $Y$ is also bi-strong.*

LEMMA 3.15. *Let $f : (X, \psi_1, \psi_2) \to (Y, \mu_1, \mu_2)$ be quasi $(q_\psi, q_\mu)$-countious on two biGTS's $(X, \psi_1, \psi_2), (Y, \mu_1, \mu_2)$. If $Y$ is bi-strong, then $X$ is also bi-strong.*

THEOREM 3.16. *Let $f : (X, \psi_1, \psi_2) \to (Y, \mu_1, \mu_2)$ be a function on biGTS's. If $Y$ is $G$-quasi $q_\mu$-regular and if $f$ is quasi $(q_\psi, q_\mu)$-open, then every weakly quasi $(q_\psi, q_\mu)$-continuous is quasi $(q_\psi, q_\mu)$-continuous.*

*Proof.* It follows from Theorem 3.11. □

Let $(X, \psi)$ and $(Y, \mu)$ be generalized topological spaces. Then $f : (X, \psi) \to (Y, \mu)$ is said to be $(\psi, \mu)$*-open* [4] if for any $\psi$-open set $U$ in $X$, $f(U)$ is $\mu$-open in $Y$.

COROLLARY 3.17. Let $f : (X, \psi_1) \to (Y, \mu_1)$ be a function on GTS's. If $Y$ is $G$-regular and if $f$ is $(\psi_1, \mu_1)$-open, then every weakly $(\psi_1, \mu_1)$-continuous is $(\psi_1, \mu_1)$-continuous.

*Proof.* Consider two biGTS's $(X, \psi_1, \psi_1)$ and $(Y, \mu_1, \mu_1)$ on $X$ and $Y$, respectively. Then from Remark 3.3, Remark 3.7 and Theorem 3.16, the corollary is obtained □

## Acknowledgements

I thank the referee for some useful comments on the paper.


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