

THE STABILITY OF LINEAR MAPPINGS IN BANACH MODULES ASSOCIATED WITH A GENERALIZED JENSEN MAPPING

SUNG JIN LEE*

ABSTRACT. Let X and Y be vector spaces. It is shown that a mapping $f : X \rightarrow Y$ satisfies the functional equation

$$(\ddagger) \quad dk \, f\left(\frac{\sum_{j=1}^{dk} x_j}{dk}\right) = \sum_{j=1}^{dk} f(x_j)$$

if and only if the mapping $f : X \rightarrow Y$ is Cauchy additive, and prove the Cauchy-Rassias stability of the functional equation $(\ddagger)$ in Banach modules over a unital C^* -algebra. Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^* -algebras. As an application, we show that every almost homomorphism $h : \mathcal{A} \rightarrow \mathcal{B}$ of $\mathcal{A}$ into $\mathcal{B}$ is a homomorphism when $h((k-1)^n u y) = h((k-1)^n u)h(y)$ for all unitaries $u \in \mathcal{A}$, all $y \in \mathcal{A}$, and $n = 0, 1, 2, \dots$.

Moreover, we prove the Cauchy-Rassias stability of homomorphisms in C^* -algebras.

1. Introduction

In 1940, S. M. Ulam [20] raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?

Let X and Y be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Hyers [3] showed that if $\epsilon > 0$ and $f : X \rightarrow Y$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \rightarrow Y$ such that

$$\|f(x) - T(x)\| \leq \epsilon$$

Received March 8, 2011; Accepted May 16, 2011.

2010 *Mathematics Subject Classifications*: Primary 39B52, 46L05, 47B48, 17A36.

Key words and phrases: Cauchy-Rassias stability, C^* -algebra homomorphism, Banach module over C^* -algebra.

for all $x \in X$.

Consider $f : X \rightarrow Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$(*) \quad \|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Th.M. Rassias [15] showed that there exists a unique $\mathbb{R}$ -linear mapping $T : X \rightarrow Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2-2^p} \|x\|^p$$

for all $x \in X$. The inequality $(*)$ that was introduced for the first time by Th.M. Rassias [12] is called *Cauchy-Rassias inequality* and the stability of the functional equation *Cauchy-Rassias stability*. This inequality has provided a lot of influence in the development of what is known as *Hyers-Ulam-Rassias stability* of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was taken up by a number of mathematicians (cf. [4], [6], [9], [14]–[19]). Th.M. Rassias [13] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Z. Gajda [1] following the same approach as in Th.M. Rassias [15], gave an affirmative solution to this question for $p > 1$.

Găvruta [2] generalized the Rassias' result: Let G be an abelian group and Y a Banach space. Denote by $\varphi : G \times G \rightarrow [0, \infty)$ a function such that

$$\tilde{\varphi}(x, y) = \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in G$. Suppose that $f : G \rightarrow Y$ is a mapping satisfying

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \rightarrow Y$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{2} \tilde{\varphi}(x, x)$$

for all $x \in G$. C. Park [7] applied the Găvruta's result to linear functional equations in Banach modules over a C^* -algebra.

Jun and Lee [5] proved the following: Denote by $\varphi : X \setminus \{0\} \times X \setminus \{0\} \rightarrow [0, \infty)$ a function such that

$$\tilde{\varphi}(x, y) = \sum_{j=0}^{\infty} 3^{-j} \varphi(3^j x, 3^j y) < \infty$$

for all $x, y \in X \setminus \{0\}$. Suppose that $f : X \rightarrow Y$ is a mapping satisfying

$$\|2f(\frac{x+y}{2}) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in X \setminus \{0\}$. Then there exists a unique additive mapping $T : X \rightarrow Y$ such that

$$\|f(x) - f(0) - T(x)\| \leq \frac{1}{3}(\tilde{\varphi}(x, -x) + \tilde{\varphi}(-x, 3x))$$

for all $x \in X \setminus \{0\}$. C. Park and W. Park [11] applied the Jun and Lee's result to the Jensen's equation in Banach modules over a C^* -algebra.

Throughout this paper, assume that d, k are positive integers with $k \geq 2$.

In this paper, we solve the following functional equation

$$(1.i) \quad dk f\left(\frac{\sum_{j=1}^{dk} x_j}{dk}\right) = \sum_{j=1}^{dk} f(x_j),$$

which is called a *generalized Jensen functional equation*. Each solution of the functional equation (1.i) is called a *generalized Jensen mapping*. We moreover prove the Cauchy-Rassias stability of the functional equation (1.i) in Banach modules over a unital C^* -algebra. The main purpose of this paper is to investigate homomorphisms between C^* -algebras and between Poisson C^* -algebras, and to prove their Cauchy-Rassias stability.

2. A generalized Jensen's mapping

Throughout this section, assume that X and Y are linear spaces.

LEMMA 2.1. *An odd mapping $f : X \rightarrow Y$ satisfies (1.i) for all $x_1, x_2, \dots, x_{dk} \in X$ if and only if f is Cauchy additive.*

Proof. Assume that $f : X \rightarrow Y$ satisfies (1.i) for all $x_1, x_2, \dots, x_{dk} \in X$. Putting $x_2 = \dots = x_{dk} = 0$ in (1.i), we get

$$(2.1) \quad dk f\left(\frac{x_1}{dk}\right) = f(x_1)$$

for all $x_1 \in X$. Putting $x_3 = \dots = x_{dk} = 0$ in (1.i), it follows from (2.1) that

$$f(x_1 + x_2) = dk f\left(\frac{x_1 + x_2}{dk}\right) = f(x_1) + f(x_2)$$

for all $x_1, x_2 \in X$. Thus f is Cauchy additive.

The converse is obviously true. $\square$

3. Cauchy-Rassias stability of the generalized Jensen's mapping in Banach modules over a C^* -algebra

Throughout this section, assume that $\mathcal{A}$ is a unital C^* -algebra with norm $|\cdot|$ and unitary group $\mathcal{U}(\mathcal{A})$, and that X and Y are left Banach modules over $\mathcal{A}$ with norms $\|\cdot\|$ and $\|\cdot\|$, respectively.

Given a mapping $f : X \rightarrow Y$, we set

$$D_u f(x_1, \dots, x_{dk}) := dk f\left(\frac{\sum_{j=1}^{dk} u x_j}{dk}\right) - \sum_{j=1}^{dk} u f(x_j)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x_1, \dots, x_{dk} \in X$.

THEOREM 3.1. *Let $f : X \rightarrow Y$ be an odd mapping for which there is a function $\varphi : X^{dk} \rightarrow [0, \infty)$ such that*

(3.i)

$$\tilde{\varphi}(x_1, \dots, x_{dk}) := \sum_{j=0}^{\infty} \frac{1}{(k-1)^j} \varphi((k-1)^j x_1, \dots, (k-1)^j x_{dk}) < \infty,$$

(3.ii)

$$\|D_u f(x_1, \dots, x_{dk})\| \leq \varphi(x_1, \dots, x_{dk})$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x_1, \dots, x_{dk} \in X$. Then there exists a unique $\mathcal{A}$ -linear generalized Jensen's mapping $L : X \rightarrow Y$ such that

$$(3.iii) \quad \|f(x) - L(x)\| \leq \frac{1}{d(k-1)} \tilde{\varphi}(\underbrace{(k-1)x, \dots, (k-1)x}_{d \text{ times}}, \underbrace{-x, \dots, -x}_{d(k-1) \text{ times}})$$

for all $x \in X$.

Proof. Note that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$ since f is an odd mapping. Let $u = 1 \in \mathcal{U}(\mathcal{A})$. Putting $x_1 = \cdots = x_d = (k-1)x$ and $x_{d+1} = \cdots = x_{dk} = -x$ in (3.ii), we have

$$(3.1) \quad \begin{aligned} & \| -d f((k-1)x) - d(k-1)f(-x) \| \\ & \leq \varphi(\underbrace{(k-1)x, \dots, (k-1)x}_{d \text{ times}}, \underbrace{-x, \dots, -x}_{d(k-1) \text{ times}}) \end{aligned}$$

for all $x \in X$. So

$$\begin{aligned} & \| f(x) - \frac{1}{k-1} f((k-1)x) \| \\ & \leq \frac{1}{d(k-1)} \varphi(\underbrace{(k-1)x, \dots, (k-1)x}_{d \text{ times}}, \underbrace{-x, \dots, -x}_{d(k-1) \text{ times}}) \end{aligned}$$

for all $x \in X$. Hence

$$(3.2) \quad \begin{aligned} & \| \frac{1}{(k-1)^n} f((k-1)^n x) - \frac{1}{(k-1)^{n+1}} f((k-1)^{n+1} x) \| \\ & = \frac{1}{(k-1)^n} \| f((k-1)^n x) - \frac{1}{k-1} f((k-1)(k-1)^n x) \| \\ & \leq \frac{1}{d(k-1)^{n+1}} \varphi(\underbrace{(k-1)^{n+1} x, \dots, (k-1)^{n+1} x}_{d \text{ times}}, \underbrace{-(k-1)^n x, \dots, -(k-1)^n x}_{d(k-1) \text{ times}}) \end{aligned}$$

for all $x \in X$ and all positive integers n . By (3.2), we have

$$(3.3) \quad \begin{aligned} & \| \frac{1}{(k-1)^m} f((k-1)^m x) - \frac{1}{(k-1)^n} f((k-1)^n x) \| \leq \sum_{l=m}^{n-1} \frac{1}{d(k-1)^{l+1}} \\ & \times \varphi(\underbrace{(k-1)^{l+1} x, \dots, (k-1)^{l+1} x}_{d \text{ times}}, \underbrace{-(k-1)^l x, \dots, -(k-1)^l x}_{d(k-1) \text{ times}}) \end{aligned}$$

for all $x \in X$ and all positive integers m and n with $m < n$. This shows that the sequence $\{ \frac{1}{(k-1)^n} f((k-1)^n x) \}$ is a Cauchy sequence for all $x \in X$.

Since Y is complete, the sequence $\{\frac{1}{(k-1)^n}f((k-1)^n x)\}$ converges for all $x \in X$. So we can define a mapping $L : X \rightarrow Y$ by

$$L(x) := \lim_{n \rightarrow \infty} \frac{1}{(k-1)^n} f((k-1)^n x)$$

for all $x \in X$. Since $f(-x) = -f(x)$ for all $x \in X$, we have $L(-x) = L(x)$ for all $x \in X$. Also, we get

$$\begin{aligned} & \|D_1 L(x_1, \dots, x_{dk})\| \\ &= \lim_{n \rightarrow \infty} \frac{1}{(k-1)^n} \|D_1 f((k-1)^n x_1, \dots, (k-1)^n x_{dk})\| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{(k-1)^n} \varphi((k-1)^n x_1, \dots, (k-1)^n x_{dk}) = 0 \end{aligned}$$

for all $x_1, \dots, x_{dk} \in X$. By Lemma 2.1, L is Cauchy additive. Putting $m = 0$ and letting $n \rightarrow \infty$ in (3.2), we get (3.iii).

Now, let $L' : X \rightarrow Y$ be another generalized Jensen's mapping satisfying (3.iii). Then we have

$$\begin{aligned} \|L(x) - L'(x)\| &= \frac{1}{(k-1)^n} \|L((k-1)^n x) - L'((k-1)^n x)\| \\ &\leq \frac{1}{(k-1)^n} (\|L((k-1)^n x) - f((k-1)^n x)\| \\ &\quad + \|L'((k-1)^n x) - f((k-1)^n x)\|) \\ &\leq \frac{2}{d(k-1)^{n+1}} \underbrace{\tilde{\varphi}((k-1)^{n+1} x, \dots, (k-1)^{n+1} x,}_{d \text{ times}} \\ &\quad \underbrace{-(k-1)^n x, \dots, -(k-1)^n x,}_{d(k-1) \text{ times}} \end{aligned}$$

which tends to zero as $n \rightarrow \infty$ for all $x \in X$. So we can conclude that $L(x) = L'(x)$ for all $x \in X$. This proves the uniqueness of L .

By the assumption, for each $u \in \mathcal{U}(\mathcal{A})$, we get

$$\begin{aligned} \|D_u L(x, \underbrace{0, \dots, 0}_{dk-1 \text{ times}})\| &= \lim_{n \rightarrow \infty} \frac{1}{(k-1)^n} \|D_u f((k-1)^n x, \underbrace{0, \dots, 0}_{dk-1 \text{ times}})\| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{(k-1)^n} \varphi((k-1)^n x, \underbrace{0, \dots, 0}_{dk-1 \text{ times}}) = 0 \end{aligned}$$

for all $x \in X$. So

$$dkL\left(\frac{ux}{dk}\right) = uL(x)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x \in X$. Since L is Cauchy additive,

$$L(ux) = dkL\left(\frac{ux}{dk}\right) = uL(x)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x \in X$.

By the same reasoning as in the proofs of [8] and [10], one can show that the unique generalized Jensen's mapping $L : \mathcal{A} \rightarrow \mathcal{B}$ is an $\mathcal{A}$ -linear mapping. $\square$

COROLLARY 3.2. *Let θ and $p < 1$ be positive real numbers. Let $f : X \rightarrow Y$ be an odd mapping such that*

$$\|D_u f(x_1, \dots, x_{dk})\| \leq \theta \sum_{j=1}^{dk} \|x_j\|^p$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x_1, \dots, x_{dk} \in X$. Then there exists a unique $\mathcal{A}$ -linear generalized Jensen's mapping $L : X \rightarrow Y$ such that

$$\|f(x) - L(x)\| \leq \frac{(k-1) + (k-1)^p}{(k-1) - (k-1)^p} \theta \|x\|^p$$

for all $x \in X$.

Proof. Define $\varphi(x_1, \dots, x_{dk}) = \theta \sum_{j=1}^{dk} \|x_j\|^p$, and apply Theorem 3.1. $\square$

THEOREM 3.3. *Let $f : X \rightarrow Y$ be an odd mapping for which there is a function $\varphi : X^{dk} \rightarrow [0, \infty)$ such that*

(3.iv)

$$\tilde{\varphi}(x_1, \dots, x_{dk}) := \sum_{j=1}^{\infty} (k-1)^j \varphi\left(\frac{x_1}{(k-1)^j}, \dots, \frac{x_{dk}}{(k-1)^j}\right) < \infty,$$

(3.v)

$$\|D_u f(x_1, \dots, x_{dk})\| \leq \varphi(x_1, \dots, x_{dk})$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x_1, \dots, x_{dk} \in X$. Then there exists a unique $\mathcal{A}$ -linear generalized Jensen's mapping $L : X \rightarrow Y$ such that

$$(3.vi) \quad \|f(x) - L(x)\| \leq \frac{1}{d(k-1)} \tilde{\varphi}(\underbrace{(k-1)x, \dots, (k-1)x}_{d \text{ times}}, \underbrace{-x, \dots, -x}_{d(k-1) \text{ times}})$$

for all $x \in X$.

Proof. Replacing x by $\frac{x}{k-1}$ in (3.1), we have

$$\|f(x) - (k-1)f(\frac{x}{k-1})\| \leq \frac{1}{d} \varphi(\underbrace{x, \dots, x}_{d \text{ times}}, \underbrace{-\frac{x}{k-1}, \dots, -\frac{x}{k-1}}_{d(k-1) \text{ times}})$$

for all $x \in X$. So

$$\begin{aligned} & \|(k-1)^n f(\frac{x}{(k-1)^n}) - (k-1)^{n+1} f(\frac{x}{(k-1)^{n+1}})\| \\ (3.4) \quad &= (k-1)^n \|f(\frac{x}{(k-1)^n}) - (k-1)f(\frac{1}{k-1} \cdot \frac{x}{(k-1)^n})\| \\ &\leq \frac{(k-1)^n}{d} \varphi(\underbrace{\frac{x}{(k-1)^n}, \dots, \frac{x}{(k-1)^n}}_{d \text{ times}}, \underbrace{-\frac{x}{(k-1)^{n+1}}, \dots, -\frac{x}{(k-1)^{n+1}}}_{d(k-1) \text{ times}}) \end{aligned}$$

for all $x \in X$ and all positive integers n . By (3.4), we have

$$\begin{aligned} & \|(k-1)^m f(\frac{x}{(k-1)^m}) - (k-1)^n f(\frac{x}{(k-1)^n})\| \\ &\leq \sum_{l=m}^{n-1} \frac{(k-1)^l}{d} \varphi(\underbrace{\frac{x}{(k-1)^l}, \dots, \frac{x}{(k-1)^l}}_{d \text{ times}}, \underbrace{-\frac{x}{(k-1)^{l+1}}, \dots, -\frac{x}{(k-1)^{l+1}}}_{d(k-1) \text{ times}}) \\ (3.5) \quad & \end{aligned}$$

for all $x \in X$ and all positive integers m and n with $m < n$. This shows that the sequence $\{(k-1)^n f(\frac{x}{(k-1)^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{(k-1)^n f(\frac{x}{(k-1)^n})\}$ converges for all $x \in X$. So we can define a mapping $L : X \rightarrow Y$ by

$$L(x) := \lim_{n \rightarrow \infty} (k-1)^n f(\frac{x}{(k-1)^n})$$

for all $x \in X$. Also, we get

$$\begin{aligned}\|D_1 L(x_1, \dots, x_{dk})\| &= \lim_{n \rightarrow \infty} (k-1)^n \|D_1 f(\frac{x_1}{(k-1)^n}, \dots, \frac{x_{dk}}{(k-1)^n})\| \\ &\leq \lim_{n \rightarrow \infty} (k-1)^n \varphi(\frac{x_1}{(k-1)^n}, \dots, \frac{x_{dk}}{(k-1)^n}) = 0\end{aligned}$$

for all $x_1, \dots, x_{dk} \in X$. By Lemma 2.1, L is Cauchy additive. Putting $m = 0$ and letting $n \rightarrow \infty$ in (3.5), we get (3.vi).

The rest of the proof is similar to the proof of Theorem 3.1. $\square$

COROLLARY 3.4. *Let θ and $p > 1$ be positive real numbers. Let $f : X \rightarrow Y$ be an odd mapping such that*

$$\|D_u f(x_1, \dots, x_{dk})\| \leq \theta \sum_{j=1}^{dk} \|x_j\|^p$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x_1, \dots, x_{dk} \in X$. Then there exists a unique $\mathcal{A}$ -linear generalized Jensen's mapping $L : X \rightarrow Y$ such that

$$\|f(x) - L(x)\| \leq \frac{(k-1)^p + (k-1)}{(k-1)^p - (k-1)} \theta \|x\|^p$$

for all $x \in X$.

Proof. Define $\varphi(x_1, \dots, x_{dk}) = \theta \sum_{j=1}^{dk} \|x_j\|^p$, and apply Theorem 3.3. $\square$

4. Isomorphisms in unital C^* -algebras

Throughout this section, assume that $\mathcal{A}$ is a unital C^* -algebra with norm $\|\cdot\|$, unit e and unitary group $\mathcal{U}(\mathcal{A})$, and that $\mathcal{B}$ is a unital C^* -algebra with norm $\|\cdot\|$.

We investigate C^* -algebra isomorphisms in unital C^* -algebras.

THEOREM 4.1. *Let $h : \mathcal{A} \rightarrow \mathcal{B}$ be an odd bijective mapping satisfying $h((k-1)^n u y) = h((k-1)^n u) h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and $n = 0, 1, 2, \dots$, for which there is a function $\varphi : \mathcal{A}^{dk} \rightarrow [0, \infty)$ satisfying (3.i)*

such that

(4.i)

$$\|dk \, h(\frac{\sum_{j=1}^{dk} \mu x_j}{dk}) - \sum_{j=1}^{dk} \mu h(x_j)\| \leq \varphi(x_1, \dots, x_{dk}),$$

(4.ii)

$$\|h((k-1)^n u^*) - h((k-1)^n u)^*\| \leq \varphi(\underbrace{(k-1)^n u, \dots, (k-1)^n u}_{dk \text{ times}})$$

for all $u \in \mathcal{U}(\mathcal{A})$, all $x_1, \dots, x_{dk} \in \mathcal{A}$, all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and $n = 0, 1, 2, \dots$. Assume that

$$(4.iii) \quad \lim_{n \rightarrow \infty} \frac{h((k-1)^n e)}{(k-1)^n} \text{ is invertible.}$$

Then the odd bijective mapping $h : \mathcal{A} \rightarrow \mathcal{B}$ is a C^* -algebra isomorphism.

Proof. We can consider a C^* -algebra as a Banach module over a unital C^* -algebra $\mathbb{C}$. So by Theorem 3.1, there exists a unique $\mathbb{C}$ -linear mapping $H : \mathcal{A} \rightarrow \mathcal{B}$ such that

$$(4.iv) \quad \|h(x) - H(x)\| \leq \frac{1}{d(k-1)} \tilde{\varphi}(\underbrace{(k-1)x, \dots, (k-1)x}_{d \text{ times}}, \underbrace{-x, \dots, -x}_{d(k-1) \text{ times}})$$

for all $x \in \mathcal{A}$. The mapping $H : \mathcal{A} \rightarrow \mathcal{B}$ is given by

$$(4.1) \quad H(x) = \lim_{n \rightarrow \infty} \frac{1}{(k-1)^n} h((k-1)^n x)$$

for all $x \in \mathcal{A}$.

By (3.i) and (4.ii), we get

$$\begin{aligned} H(u^*) &= \lim_{n \rightarrow \infty} \frac{h((k-1)^n u^*)}{(k-1)^n} = \lim_{n \rightarrow \infty} \frac{h((k-1)^n u)^*}{(k-1)^n} \\ &= (\lim_{n \rightarrow \infty} \frac{h((k-1)^n u)}{(k-1)^n})^* = H(u)^* \end{aligned}$$

for all $u \in \mathcal{U}(\mathcal{A})$. Since H is $\mathbb{C}$ -linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements (cf. [7, Theorem 4.1.7]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A})$),

$$\begin{aligned} H(x^*) &= H(\sum_{j=1}^m \overline{\lambda_j} u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j)^* \\ &= (\sum_{j=1}^m \lambda_j H(u_j))^* = H(\sum_{j=1}^m \lambda_j u_j)^* = H(x)^* \end{aligned}$$

for all $x \in \mathcal{A}$.

Since $h((k-1)^n uy) = h((k-1)^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \dots$,

$$(4.2) \quad \begin{aligned} H(uy) &= \lim_{n \rightarrow \infty} \frac{1}{(k-1)^n} h((k-1)^n uy) \\ &= \lim_{n \rightarrow \infty} \frac{1}{(k-1)^n} h((k-1)^n u)h(y) = H(u)h(y) \end{aligned}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. By the additivity of H and (4.2),

$$(k-1)^n H(uy) = H((k-1)^n uy) = H(u((k-1)^n y)) = H(u)h((k-1)^n y)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Hence

$$(4.3) \quad H(uy) = \frac{1}{(k-1)^n} H(u)h((k-1)^n y) = H(u) \frac{1}{(k-1)^n} h((k-1)^n y)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Taking the limit in (4.3) as $n \rightarrow \infty$, we obtain

$$(4.4) \quad H(uy) = H(u)H(y)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Since H is $\mathbb{C}$ -linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A})$), it follows from (4.4) that

$$\begin{aligned} H(xy) &= H\left(\sum_{j=1}^m \lambda_j u_j y\right) = \sum_{j=1}^m \lambda_j H(u_j y) = \sum_{j=1}^m \lambda_j H(u_j)H(y) \\ &= H\left(\sum_{j=1}^m \lambda_j u_j\right)H(y) = H(x)H(y) \end{aligned}$$

for all $x, y \in \mathcal{A}$.

By (4.2) and (4.4),

$$H(e)H(y) = H(ey) = H(e)h(y)$$

for all $y \in \mathcal{A}$. Since $\lim_{n \rightarrow \infty} \frac{h((k-1)^n e)}{(k-1)^n} = H(e)$ is invertible,

$$H(y) = h(y)$$

for all $y \in \mathcal{A}$.

Therefore, the odd bijective mapping $h : \mathcal{A} \rightarrow \mathcal{B}$ is a C^* -algebra isomorphism. $\square$

COROLLARY 4.2. Let $h : \mathcal{A} \rightarrow \mathcal{B}$ be an odd bijective mapping satisfying $h((k-1)^n uy) = h((k-1)^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \dots$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\begin{aligned} \|dkh(\frac{\sum_{j=1}^{dk} \mu x_j}{dk}) - \sum_{j=1}^{dk} \mu h(x_j)\| &\leq \theta \sum_{j=1}^{dk} \|x_j\|^p, \\ \|h((k-1)^n u^*) - h((k-1)^n u)^*\| &\leq dk(k-1)^{np}\theta \end{aligned}$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{A})$, $n = 0, 1, 2, \dots$, and all $x_1, \dots, x_{dk} \in \mathcal{A}$. Assume that

$$\lim_{n \rightarrow \infty} \frac{h((k-1)^n e)}{(k-1)^n} \text{ is invertible.}$$

Then the odd bijective mapping $h : \mathcal{A} \rightarrow \mathcal{B}$ is a C^* -algebra isomorphism.

Proof. Define $\varphi(x_1, \dots, x_{dk}) = \theta \sum_{j=1}^{dk} \|x_j\|^p$, and apply Theorem 4.1. $\square$

THEOREM 4.3. Let $h : \mathcal{A} \rightarrow \mathcal{B}$ be an odd bijective mapping satisfying $h((k-1)^n uy) = h((k-1)^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and $n = 0, 1, 2, \dots$, for which there is a function $\varphi : \mathcal{A}^{dk} \rightarrow [0, \infty)$ satisfying (3.i), (4.ii), and (4.iii) such that

$$(4.v) \quad \|dkh(\frac{\sum_{j=1}^{dk} \mu x_j}{dk}) - \sum_{j=1}^{dk} \mu h(x_j)\| \leq \varphi(x_1, \dots, x_{dk}),$$

for all $x_1, \dots, x_{dk} \in \mathcal{A}$ and $\mu = 1, i$. If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then the odd bijective mapping $h : \mathcal{A} \rightarrow \mathcal{B}$ is a C^* -algebra isomorphism.

Proof. Put $\mu = 1$ in (4.v). By the same reasoning as in the proof of Theorem 4.1, there exists a unique generalized Jensen's mapping $H : \mathcal{A} \rightarrow \mathcal{B}$ satisfying (4.iv). By the same reasoning as in the proof of [15, Theorem], the additive mapping $H : \mathcal{A} \rightarrow \mathcal{B}$ is $\mathbb{R}$ -linear.

Put $\mu = i$ in (4.v). By the same method as in the proof of Theorem 4.1, one can obtain that

$$H(ix) = \lim_{n \rightarrow \infty} \frac{h((k-1)^n ix)}{(k-1)^n} = \lim_{n \rightarrow \infty} \frac{ih((k-1)^n x)}{(k-1)^n} = iH(x)$$

for all $x \in \mathcal{A}$.

For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So

$$\begin{aligned} H(\lambda x) &= H(sx + itx) = sH(x) + tH(ix) \\ &= sH(x) + itH(x) = (s + it)H(x) = \lambda H(x) \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and all $x \in \mathcal{A}$. So

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbb{C}$, and all $x, y \in \mathcal{A}$. Hence the additive mapping $H : \mathcal{A} \rightarrow \mathcal{B}$ is $\mathbb{C}$ -linear.

The rest of the proof is the same as in the proof of Theorem 4.1. $\square$

Now we prove the Cauchy-Rassias stability of C^* -algebra homomorphisms in unital C^* -algebras.

THEOREM 4.4. *Let $h : \mathcal{A} \rightarrow \mathcal{B}$ be an odd mapping for which there exists a function $\varphi : \mathcal{A}^{dk} \rightarrow [0, \infty)$ satisfying (3.i), (4.i) and (4.ii) such that*

$$\begin{aligned} &\|h((k-1)^n u (k-1)^n v) - h((k-1)^n u)h((k-1)^n v)\| \\ (4.vi) \quad &\leq \varphi((k-1)^n u, (k-1)^n v, \underbrace{0, \dots, 0}_{dk-2 \text{ times}}) \end{aligned}$$

for all $u, v \in \mathcal{U}(\mathcal{A})$ and $n = 0, 1, 2, \dots$. Then there exists a unique C^* -algebra homomorphism $H : \mathcal{A} \rightarrow \mathcal{B}$ satisfying (4.iv).

Proof. By the same reasoning as in the proof of Theorem 4.1, there exists a unique $\mathbb{C}$ -linear involutive mapping $H : \mathcal{A} \rightarrow \mathcal{B}$ satisfying (4.iv).

By (4.vi),

$$\begin{aligned} &\frac{1}{(k-1)^{2n}} \|h((k-1)^n u (k-1)^n v) - h((k-1)^n u)h((k-1)^n v)\| \\ &\leq \frac{1}{(k-1)^{2n}} \varphi((k-1)^n u, (k-1)^n v, \underbrace{0, \dots, 0}_{dk-2 \text{ times}}) \\ &\leq \frac{1}{(k-1)^n} \varphi((k-1)^n u, (k-1)^n v, \underbrace{0, \dots, 0}_{dk-2 \text{ times}}), \end{aligned}$$

which tends to zero by (3.i) as $n \rightarrow \infty$. By (4.1),

$$\begin{aligned} H(uv) &= \lim_{n \rightarrow \infty} \frac{h((k-1)^n u (k-1)^n v)}{(k-1)^{2n}} \\ &= \lim_{n \rightarrow \infty} \frac{h((k-1)^n u) h((k-1)^n v)}{(k-1)^{2n}} \\ &= \lim_{n \rightarrow \infty} \frac{h((k-1)^n u)}{(k-1)^n} \frac{h((k-1)^n v)}{(k-1)^n} = H(u)H(v) \end{aligned}$$

for all $u, v \in \mathcal{U}(\mathcal{A})$. Since H is $\mathbb{C}$ -linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A})$),

$$\begin{aligned} H(xv) &= H\left(\sum_{j=1}^m \lambda_j u_j v\right) = \sum_{j=1}^m \lambda_j H(u_j v) = \sum_{j=1}^m \lambda_j H(u_j) H(v) \\ &= H\left(\sum_{j=1}^m \lambda_j u_j\right) H(v) = H(x) H(v) \end{aligned}$$

for all $x \in \mathcal{A}$ and all $v \in \mathcal{U}(\mathcal{A})$. By the same method as given above, one can obtain that

$$H(xy) = H(x)H(y)$$

for all $x, y \in \mathcal{A}$. So the mapping $H : \mathcal{A} \rightarrow \mathcal{B}$ is a C^* -algebra homomorphism. $\square$

REFERENCES

- [1] Z. Gajda, *On stability of additive mappings*, Internat. J. Math. and Math. Sci. **14** (1991), 431–434.
- [2] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
- [3] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.
- [4] D. H. Hyers and Th. M. Rassias, *Approximate homomorphisms*, Aequationes Math. **44** (1992), 125–153.
- [5] K. Jun and Y. Lee, *A generalization of the Hyers-Ulam-Rassias stability of Jensen's equation*, J. Math. Anal. Appl. **238** (1999), 305–315.
- [6] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras Elementary Theory*, Academic Press, New York, 1983.
- [7] C. Park, *On the stability of the linear mapping in Banach modules*, J. Math. Anal. Appl. **275** (2002), 711–720.
- [8] C. Park, *Modified Trif's functional equations in Banach modules over a C^* -algebra and approximate algebra homomorphisms*, J. Math. Anal. Appl. **278** (2003), 93–108.

- [9] C. Park, *Linear $*$ -derivations on JB^* -algebras*, Acta Math. Scientia **25** (2005), 449–454.
- [10] C. Park and J. Hou, *Homomorphisms between C^* -algebras associated with the Trif functional equation and linear derivations on C^* -algebras*, J. Korean Math. Soc. **41** (2004), 461–477.
- [11] C. Park and W. Park, *On the Jensen's equation in Banach modules*, Taiwanese J. Math. **6** (2002), 523–531.
- [12] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
- [13] Th. M. Rassias, *Problem 16; 2* Report of the 27th International Symp. on Functional Equations, Aequationes Math. **39** (1990), 292–293; 309.
- [14] Th. M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. **62** (2000), 23–130.
- [15] Th. M. Rassias, *The problem of S.M. Ulam for approximately multiplicative mappings*, J. Math. Anal. Appl. **246** (2000), 352–378.
- [16] Th. M. Rassias, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. **251** (2000), 264–284.
- [17] Th. M. Rassias and P. Šemrl, *On the behavior of mappings which do not satisfy Hyers-Ulam stability*, Proc. Amer. Math. Soc. **114** (1992), 989–993.
- [18] Th. M. Rassias and P. Šemrl, *On the Hyers-Ulam stability of linear mappings*, J. Math. Anal. Appl. **173** (1993), 325–338.
- [19] T. Trif, *On the stability of a functional equation deriving from an inequality of Popoviciu for convex functions*, J. Math. Anal. Appl. **272** (2002), 604–616.
- [20] S. M. Ulam, *Problems in Modern Mathematics*, Wiley, New York, 1960.

*

DEPARTMENT OF MATHEMATICS
 DAEJIN UNIVERSITY
 KYUNGGI 487-711, REPUBLIC OF KOREA
 E-mail: hyper@daejin.ac.kr