JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 16, No. 1, June 2003

GENERALIZED ANTI–DERIVATIONS ON BANACH ALGEBRAS

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Abstract. We investigate generalized Baxter equations on Banach algebras. This is applied to understand generalized anti-derivations on Banach $*$-algebras.

1. Introduction

Let E_1 and E_2 be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Consider $h : E_1 \to E_2$ to be a mapping such that $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|h(x+y) - h(x) - h(y)\| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in E_1$. Th.M. Rassias [2] showed that there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|h(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}||x||^p$$

for all $x \in E_1$.

Sablik [3] investigated the generalized Abel's functional equation

$$\psi(xh(y) + yg(x)) = \varphi(x) + \varphi(y).$$

*Supported by Korea Research Foundation Grant KRF-2002-041-C00014.
Received by the editors on June 20, 2003.
2000 *Mathematics Subject Classifications*: Primary 39B52, 47B48.
Key words and phrases: Banach $*$-algebra, Baxter functional equation, anti-derivation.

When $h = g$, we get the original Abel's functional equation. In [1, 4], the authors investigated the functional equation

$$(\ddagger) \qquad h(xh(y)^k + yh(x)^n) = th(x)h(y).$$

The equation (‡) with $k = n = t = 1$ becomes the equation

$$h(xh(y) + yh(x)) = h(x)h(y),$$

which is the equation defining *anti-derivations* in the case where f is a bounded linear mapping defined on an algebra. Together with the equation

$$h(xh(y) + yh(x) - xy) = h(x)h(y),$$

they are special cases of the generalized Baxter equation

$$(\dagger) \qquad h(xh(y) + yh(x) - \kappa xy) = \omega h(x)h(y).$$

See [3] for details.

Throughout this paper, let $\mathcal{B}$ be a complex Banach algebra with norm $\|\cdot\|$.

The main purpose of this paper is to investigate generalized Baxter equations on Banach algebras, and generalized anti-derivations on Banach $*$-algebras.

2. Generalized Baxter functional equations on Banach algebras

In this section, assume that $h : \mathcal{B} \to \mathcal{B}$ is an additive mapping with $h(0) = 0$.

Theorem 1. *Let $\kappa, \omega \in \mathcal{B}$ be given. Assume that there exists a function $\varphi : \mathcal{B} \times \mathcal{B} \to [0, \infty)$ such that*

$$\text{(i)} \qquad \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x, 2^j y) < \infty,$$

$$\text{(ii)} \qquad \|h(ix) - ih(x)\| \le \varphi(x, x),$$

$$\text{(iii)} \qquad \|h(xh(y) + yh(x) - \kappa xy) - \omega h(x)h(y)\| \le \varphi(x, y)$$

for all $x, y \in \mathcal{B}$. If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{B}$, then the additive mapping $h : \mathcal{B} \to \mathcal{B}$ satisfies the generalized Baxter equation (†) and is $\mathbb{C}$-linear.

Proof. By the same reasoning as the proof of [2, Theorem], the additive mapping $h : \mathcal{B} \to \mathcal{B}$ is $\mathbb{R}$-linear.

Since $h : \mathcal{B} \to \mathcal{B}$ is additive, it follows from (ii) that

$$\|h(ix) - ih(x)\| = 2^{-n}\|h(2^n ix) - ih(2^n x)\| \le \frac{1}{2^n}\varphi(2^n x, 2^n x),$$

which tends to zero as $n \to \infty$ by (i) for all $x \in \mathcal{B}$. So $h(ix) = ih(x)$ for all $x \in \mathcal{B}$. For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. Thus

$$\begin{aligned} h(\lambda x) &= h(sx + itx) = sh(x) + th(ix) = sh(x) + ith(x) \\ &= \lambda h(x) \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and all $x \in \mathcal{B}$. So

$$h(\zeta x + \eta y) = h(\zeta x) + h(\eta y) = \zeta h(x) + \eta h(y)$$

for all $\zeta, \eta \in \mathbb{C}$ and all $x, y \in \mathcal{B}$. Hence the additive mapping $h : \mathcal{B} \to \mathcal{B}$ is $\mathbb{C}$-linear.

Now by (iii) and the additivity of h,

$$\begin{aligned} &\|h(xh(y) + yh(x) - \kappa xy) - \omega h(x)h(y)\| \\ &\quad = 2^{-2n}\|h(2^n xh(2^n y) + 2^n yh(2^n x) - 2^{2n}\kappa xy) - \omega h(2^n x)h(2^n y)\| \\ &\qquad \le \frac{1}{2^{2n}}\varphi(2^n x, 2^n y) \le \frac{1}{2^n}\varphi(2^n x, 2^n y), \end{aligned}$$

which tends to zero as $n \to \infty$ by (i) for all $x, y \in \mathcal{B}$. Hence the additive mapping $h : \mathcal{B} \to \mathcal{B}$ satisfies the generalized Baxter equation (†), as desired. □

Corollary 2. *Let* $\kappa, \omega \in \mathcal{B}$ *be given. Assume that there exist constants* $\theta \geq 0$ *and* $p \in [0,1)$ *such that*

$$\|h(ix) - ih(x)\| \leq 2\theta\|x\|^p,$$
$$\|h(xh(y) + yh(x) - \kappa xy) - \omega h(x)h(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in \mathcal{B}$. *If* $h(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in \mathcal{B}$, *then the additive mapping* $h : \mathcal{B} \to \mathcal{B}$ *satisfies the generalized Baxter equation* (†) *and is* $\mathbb{C}$*-linear.*

Proof. Define $\varphi(x,y) = \theta(\|x\|^p + \|y\|^p)$, and apply Theorem 1. □

Now we are going to investigate generalized anti-derivations on Banach $*$-algebras, which is a bounded linear $*$-mapping with $\kappa = 0$ in (†).

Theorem 3. *Let* $\mathcal{B}$ *be a Banach* $*$*-algebra, and let* $\kappa, \omega \in \mathcal{B}$ *be given. Let* $h : \mathcal{B} \to \mathcal{B}$ *be a continuous mapping for which there exists a function* $\varphi : \mathcal{B} \times \mathcal{B} \to [0, \infty)$ *satisfying* (i) *and* (ii) *such that*

$$\|h(xh(y) + yh(x)) - \omega h(x)h(y)\| \leq \varphi(x,y),$$
$$\text{(iv)} \qquad \|h(x^*) - h(x)^*\| \leq \varphi(x,x)$$

for all $x, y \in \mathcal{B}$. *Then the additive mapping* $h : \mathcal{B} \to \mathcal{B}$ *is a generalized anti-derivation.*

Proof. Since $h : \mathcal{B} \to \mathcal{B}$ is continuous, $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{B}$. By the same reasoning as the proof of Theorem 1, the additive mapping $h : \mathcal{B} \to \mathcal{B}$ satisfies the generalized Baxter equation (†) with $\kappa = 0$ and is $\mathbb{C}$-linear. But $h : \mathcal{B} \to \mathcal{B}$ is a continuous mapping, which is $\mathbb{C}$-linear. So $h : \mathcal{B} \to \mathcal{B}$ is bounded.

It follows from (iv) that

$$\|h(x^*) - h(x)^*\| = 2^{-n}\|h(2^n x^*) - h(2^n x)^*\| \leq \frac{1}{2^n}\varphi(2^n x, 2^n x),$$

which tends to zero as $n \to \infty$ by (i) for all $x \in \mathcal{B}$. So $h(x^*) = h(x)^*$ for all $x \in \mathcal{B}$. Hence the additive mapping $h : \mathcal{B} \to \mathcal{B}$ is a generalized anti-derivation, as desired. □

Corollary 4. *Let $\kappa, \omega \in \mathcal{B}$ be given. Assume that there exist constants $\theta \geq 0$ and $p \in [0,1)$ such that*

$$\|h(ix) - ih(x)\| \leq 2\theta\|x\|^p,$$
$$\|h(xh(y) + yh(x)) - \omega h(x)h(y)\| \leq \theta(\|x\|^p + \|y\|^p),$$
$$\|h(x^*) - h(x)^*\| \leq 2\theta\|x\|^p$$

for all $x, y \in \mathcal{B}$. If $h : \mathcal{B} \to \mathcal{B}$ is a continuous mapping, then the additive mapping $h : \mathcal{B} \to \mathcal{B}$ is a generalized anti-derivation.

Proof. Define $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$, and apply Theorem 3. □

References

1. N. Brillouët-Belluot, *On the solutions of the functional equation* $f(f(y)^k x + f(x)^l y) = \lambda f(x)f(y)$, Publ. Math. Debrecen **41** (1987), 213–223.
2. Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
3. M. Sablik, *On a generalized functional equation of Abel*, Publ. Math. Debrecen **60** (2002), 29–46.
4. M. Sablik and P. Urban, *On the solutions of the equation* $f(xf(y)^k + yf(x)^l) = f(x)f(y)$, Demonstratio Math. **18** (1985), 863–867.

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