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h-STABILITY OF THE NONLINEAR PERTURBED DIFFERENCE SYSTEMS

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Abstract. In this paper, we investigate h-stability of the nonlinear perturbed difference system by using comparison principle.

1. Introduction

Discrete Volterra systems arise mainly in the process of modeling of some real phenomena or by applying a numerical method to a Volterra integral equation. Medina and Pinto [9] introduced the notion of h-stability which is an important extension of the notion of exponential asymptotic stability. In the study of the stability properties of difference systems, the notion of h-stability is very useful because, when we study the asymptotic stability it is not easy to work with non-exponential types of stability. To study the various stability notions of nonlinear difference systems, the comparison principle [7] and variation of constants formula by Agarwal [1] play a fundamental role.

Media and Pinto [9-11] applied the h-stability to obtain a uniform treatment for the various stability notions in difference systems and gave new insights about stability for weakly stable difference systems(at least, for systems with stabilities weaker than those given by exponential stability and uniform Lipschitz stability). Also, Choi and Koo [3] obtained results for hS of nonlinear difference systems via n_∞-similarity. The stability problem for Volterra difference systems was studied by Elaydi [6], Elaydi and Murakami [6], Raffoul [12], Zouyousefain and Leela [13], Choi and Koo [2], and others.

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In this paper, we investigate h-stability of the nonlinear difference systems by using comparison principle.

2. Preliminaries

We consider the nonlinear difference system

$$x(n+1) = f(n, x(n)), \tag{2.1}$$

where $f : N(n_0) \times \mathbb{R}^m \to \mathbb{R}^m$, $N(n_0)=\{n_0, n_0+1, \cdots, n_0+k, \cdots\}$ (n_0 a nonnegative integer), $\mathbb{R}^m$ is the m-dimensional real euclidean space. We assume that $f_x = \partial f/\partial x$ exists and is continuous and invertible on $N(n_0) \times \mathbb{R}^m$, $f(n, 0) = 0$. Let $x(n) = x(n, n_0, x_0)$ be the unique solution of (2.1) with $x(n_0, n_0, x_0) = x_0$. Also, we consider its associated variational system

$$v(n+1) = f_x(n, 0)v(n) \tag{2.2}$$

and

$$z(n+1) = f_x(n, x(n, n_0, x_0))z(n) \tag{2.3}$$

of (2.1). The fundamental matrix $\Phi(n, n_0, 0)$ of (2.2) is given by

$$\Phi(n, n_0, 0) = \frac{\partial}{\partial x_0} x(n, n_0, 0)$$

and the fundamental matrix $\Phi(n, n_0, x_0)$ of (2.3) is given by

$$\Phi(n, n_0, x_0) = \frac{\partial}{\partial x_0} x(n, n_0, x_0)$$

(See [9])
The symbol $| \cdot |$ will be used to denote any convenient vector norm on $\mathbb{R}^m$.

We now recall the main definitions [11] that we need in the sequel.

Definition 2.1. The zero solution of (2.1), or more briefly system (2.1), is called (hS) h-stable if there exist $c \geq 1, \delta > 0$ and a positive bounded function $h : N(n_0) \to \mathbb{R}$ such that

$$| x(n, n_0, x_0) | \leq c \mid x_0 \mid h(n)h^{-1}(n_0)$$

for $n \geq n_0$ and $| x_0 | < \delta$ (here $h^{-1}(n) = 1/h(n)$),
(hSV) h-stable in variation if the zero solution of system (2.3) is hS.

The notion of n_∞-similarity in M was introduced by Choi and Koo [3].

Let M denote the set of all $m \times m$ invertible matrices $A(n)$ defined on $N(n_0)$ and N be the subset of M consisting of those nonsingular bounded matrices $S(n)$ such that $S^{-1}(n)$ is also bounded.

Definition 2.2. A matrix $A(n) \in M$ is n_∞-similar to a matrix $B(n) \in M$ if there exists an $m \times m$ matrix $F(n)$ absolutely summable over $N(n_0)$, i.e.,

$$\sum_{l=n_0}^{\infty} | F(l) | < \infty$$

such that

$$S(n+1)B(n) - A(n)S(n) = A(n)F(n)$$

for some $S(n) \in N$.

For the example of n_∞-similarity, see [3].

Remark 2.3. The notion of t_∞-similarity is an equivalence relation in the set of all $m \times m$ continuous matrices on $\mathbb{R}^+$ but the n_∞-similarity is not an equivalence relation in general.

We give some related properties that we need in the sequal.

Lemma 2.4. [10] *The linear defference system*

$$x(n+1) = A(n)x(n), \quad x(n_0) = x_0, \tag{2.4}$$

where $A(n)$ is an $m \times m$ matrix, is hS if and only if there exist $c \geq 1$ and a positive bounded function h defined on $N(n_0)$ such that for every $x_0 \in \mathbb{R}^m$,

$$| \phi(n, n_0, 0) | \leq ch(n)h(n_0)^{-1} \tag{2.5}$$

for $n \geq n_0$, where ϕ is the fundamental matrix of (2.4).

We consider the nonlinear difference system

$$x(n+1) = f(n, x(n))$$

and its perturbed difference system

$$y(n+1) = f(n, y(n)) + g(n, y(n)) \tag{2.6}$$

where $f, g : N(n_0) \times \mathbb{R}^m \to \mathbb{R}^m$, and $f(n,0) = g(n,0) = 0$. Let $y(n) = y(n, n_0, y_0)$ denote the solution of (2.6) satisfying the initial condition $y(n_0, n_0, y_0) = y_0$.

Theorem 2.5. [3] *Assume that $f_x(n,0)$ is n_∞-similar to $f_x(n, x(n, n_0, x_0))$ for $n \geq n_0 \geq 0$ and $| x_0 | \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of* (2.2) *is hS, then the solution $z = 0$ of* (2.3) *is hS.*

THEOREM 2.6. [8] *Assume that* $x(n, n_0, x_0)$ *and* $x(n, n_0, y_0)$ *are the solutions of* (2.1) *through* (n_0, x_0) *and* (n_0, y_0), *respectively, existing for* $n \geq n_0$, *such that* x_0, y_0 *belong to a convex subset of* $\mathbb{R}^m$. *Then, for* $n \geq n_0$,

$$x(n, n_0, y_0) - x(n, n_0, x_0) = [\int_0^1 \Phi(n, n_0, s(y_0 - x_0))ds](y_0 - x_0).$$

LEMMA 2.7. [3] *Let* $k(n, r)$ *be a nonincreasing function in* r *for any fixed* $n \in N(n_0)$. *Suppose that for* $n \geq n_0$,

$$v(n) - \sum_{l=n_0}^{n-1} k(l, v(l)) < u(n) - \sum_{l=n_0}^{n-1} k(l, u(l))$$

If $v(n_0) < u(n_0)$, *then* $v(n) < u(n)$ *for all* $n \geq n_0$.

3. Main results

In this section, we investigate hS for the nonlinear difference systems via n_∞-similarity.

THEOREM 3.1. [12] *If the zero solution of* (2.1) *is hS, then the zero solution of* (2.2) *is hS.*

Proof. It is analogous to that of Theorem 5 in [2]. □

THEOREM 3.2. *Suppose that* $f_x(n, 0)$ *is* n_∞-*similar to* $f_x(n, x(n, n_0, x_0))$ *for* $n \geq n_0 \geq 0$ *and* $|x_0| \leq \delta$ *for some constant* $\delta > 0$. *Then, the solution* $v = 0$ *of* (2.2) *is hS if and only if the solution* $z = 0$ *of* (2.3) *is hS.*

Proof. First suppose $v = 0$ of (2.2) is hS. Then, by Theorem 2.5, the solution $z = 0$ of (2.3) is hS.

Conversely, suppose the solution $z = 0$ of (2.3) is hS. Let $x(n) = x(n, n_0, x_0)$ be any solution of (2.1). Then, by Theorem 2.6, we have

$$x(n, n_0, x_0) = [\int_0^1 \Phi(n, n_0, sx_0)ds]x_0$$

By Lemma 2.4, since the solution $z = 0$ of (2.3) is hS, there exist $c \geq 1$ and a positive bounded function h on $N(n_0)$ such that

$$| \Phi(n, n_0, x_0) | \leq ch(n)h(n)^{-1}$$

for $n \geq n_0 \geq 0$, where $\Phi(n, n_0, x_0)$ is the fundamental matrix of (2.3). From (2.5), we have

$$| x(n, n_0, x_0) | \leq \int_0^1 | \Phi(n, n_0, sx_0) | \, ds \, | x_0 |$$
$$\leq c \, | x_0 | \, h(n)h(n_0)^{-1}.$$

This implies that the zero solution of (2.1) is hS. Therefore, by Theorem 3.1, the solution $v = 0$ of (2.2) is hS and so the proof is complete. □

COROLLARY 3.3. *Under the same conditions of Theorem 3.2, the zero solution of* (2.1) *is hSV.*

THEOREM 3.4. *Suppose that* $f_x(n, 0)$ *is* n_∞*-similar to* $f_x(n, x(n, n_0, x_0))$ *for* $n \geq n_0 \geq 0$ *and* $| x_0 | \leq \delta$ *for some constant* $\delta > 0$ *and the solution* $x = 0$ *of* (2.1) *is hS. Also, suppose that*

$$| g(n, z) | \leq \lambda(n) | z | \qquad for \;\; n \geq n_0, \quad | z | < \infty,$$

where $\frac{h(n)}{h(n+1)}\lambda(n) \in l_1(N(n_0))$, *i.e.,* $\sum_{n=n_0}^{\infty} \frac{h(n)}{h(n+1)}\lambda(n) < \infty$, *then the zero solution* $y = 0$ *of* (2.6) *is hS.*

Proof. Using the discrete analogue of Alekseev's formula [10], the solutions of (2.1) and (2.6) with the same initial value are related by

$$y(n, n_0, y_0) = x(n, n_0, y_0) + \sum_{l=n_0}^{n-1} \int_0^1 \Phi(n, l+1, \mu(y(l), \tau)) d\tau \cdot g(l, y(l)),$$

where $\mu(y(n), \tau) = f(n, y(n)) + \tau g(n, y(n))$, $\tau \in [0, 1]$ and $\Phi(n, n_0, x_0)$ is the fundamental matrix of (2.3). In view of the assumptions, Theorem 3.1 and Theorem 3.2, the zero solution $z = 0$ of (2.3) is hS. Hence, we have

$$| y(n, n_0, y_0) |$$
$$\leq | x(n, n_0, y_0) | + \sum_{l=n_0}^{n-1} \int_0^1 |\Phi(n, l+1, \mu(y(l), \tau)) | \, d\tau \, | g(l, y(l)) |$$
$$\leq c \, | y_0 | h(n)h^{-1}(n_0) + c \sum_{l=n_0}^{n-1} h(n)h^{-1}(l+1)\lambda(l) \, | y(l) |.$$

Letting $u(n) = \frac{|y(n)|}{h(n)}$, we obtain

$$u(n) \leq cu(n_0) + c \sum_{l=n_0}^{n-1} h^{-1}(l+1)h(l)\lambda(l) \, | u(l) |.$$

Hence, by the discrete Bellman's inequality [1, 6], we obtain

$$\begin{aligned} \mid y(n) \mid &\leq ch(n)h^{-1}(n_0) \mid y_0 \mid \exp\left(c \sum_{s=n_0}^{n-1} \frac{h(s)}{h(s+1)} \lambda(s) \right) \\ &\leq c_1 h(n)h^{-1}(n_0) \mid y_0 \mid, \end{aligned}$$

where $c_1 = c \exp\left(c \sum_{s=n_0}^{\infty} \frac{h(s)}{h(s+1)} \lambda(s) \right)$ is a positive constant. The zero solution $y = 0$ of (2.3) is hS. This completes the proof. $\square$

Also, we examine the property of hS for the perturbed system

$$y(n+1) = f(n,y) + \sum_{l=n_0}^{n} g(l, y(l)), \quad y(n_0) = y_0. \tag{3.1}$$

where $g(n,0) = 0$.

THEOREM 3.5. *Suppose that $f_x(n,0)$ is n_∞-similar to $f_x(n, x(n, n_0, x_0))$ for $n \geq n_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$ and the solution $x = 0$ of (2.1) is hS with the nonincreasing function $h(n)$. Also, suppose that*

$$\mid \sum_{l=n_0}^{n} g(l,z) \mid \leq r(n, \mid z \mid) \quad for \ \ n \geq n_0, \quad \mid z \mid < \infty,$$

where $r : N(n_0) \times \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing in u for each fixed $n \in N(n_0)$ with $r(n,0) = 0$. Consider the scalar difference equation

$$u(n+1) = u(n) + cr(n, u(n)), \quad u(n_0) = u_0, \quad c > 1. \tag{3.2}$$

If the zero solution $u = 0$ of (3.2) is hS, then the zero solution $y = 0$ of (3.1) is also hS whenever $u_0 = c \mid y_0 \mid$.

Proof. Using the discrete analogue of Alekseev's formula [10], we have

$$\begin{aligned} &y(n, n_0, y_0) \\ &= x(n, n_0, y_0) + \sum_{l=n_0}^{n-1} \int_0^1 \Phi(n, l+1, \mu(y(l), \tau)) d\tau \cdot \sum_{k=n_0}^{l} g(k, y(k)), \end{aligned}$$

where $\mu(y(n), \tau) = f(n, y(n)) + \tau \sum_{l=n_0}^{n} g(l, y(l)), \tau \in [0,1]$ and $\Phi(n, n_0, x_0)$ is the fundamental matrix of (2.3). By the assumptions, Theorem 3.1 and Theorem 3.2, the zero solution $z = 0$ of (2.3) is hS. Hence, we obtain

$$
\begin{aligned}
&| y(n,n_0,y_0) | \\
&\le | x(n,n_0,y_0) | + \sum_{l=n_0}^{n-1} \int_0^1 | \Phi(n,l+1,\mu(y(l),\tau)) | d\tau \cdot \left| \sum_{k=n_0}^{l} g(k,y(k)) \right| \\
&\le c \mid y_0 \mid h(n)h^{-1}(n_0) + c \sum_{l=n_0}^{n-1} h(n)h^{-1}(l+1) r(l, \mid y(l) \mid) \\
&\le c \mid y_0 \mid + c \sum_{l=n_0}^{n-1} r(l, \mid y(l) \mid),
\end{aligned}
$$

since $h(n)$ is nonincreasing. Thus, we have

$$| y(n) | - c \sum_{l=n_0}^{n-1} r(l, | y(l) |) \le c \mid y_0 \mid = u_0 = u(n) - c \sum_{l=n_0}^{n-1} r(l, u(l)).$$

By Lemma 2.7, we get $y(n) < u(n)$ for all $n \ge n_0$. In view of the assumption, since $u = 0$ of (3.2) is hS, we obtain

$$
\begin{aligned}
| y(n) | < u(n) &\le c_1 u_0 h(n) h(n_0)^{-1} \\
&= c_1 c \mid y_0 \mid h(n) h(n_0)^{-1} \\
&= d \mid y_0 \mid h(n) h(n_0)^{-1}, \qquad d = c_1 c > 1
\end{aligned}
$$

Hence, the proof is complete. □

Remark 3.6. If we consider the linear difference system

$$x(n+1) = f(n, x(n)) = A(n)x(n) \tag{3.3}$$

and its perturbation

$$y(n+1) = A(n)y(n) + \sum_{l=n_0}^{n} g(l, y(l)), \tag{3.4}$$

where $A(n)$ is an $m \times m$ matrix defined on $N(n_0)$, then the zero solution $y = 0$ of (3.4) is hS under the same conditions in Theorem 3.5.

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