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ON THE FUZZY STATISTICAL CONVERGENCE IN A FUZZY NORMED LINEAR SPACE

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Abstract. In this paper, we introduce the notions of the fuzzy statistical convergence of sequences, the fuzzy statistical Cauchy sequence on a fuzzy normed linear space. And we investigate some properties of the related completeness.

1. Introduction

Katsaras and Liu [7] first introduced the concepts of the fuzzy vector space and the fuzzy topological vector space. These ideas were modified by Kasaras [5], and in [6] he defined the fuzzy norm on a vector space. In [8] Krishna and Sarma discussed the generated fuzzy vector topology from an ordinary vector topology on a vector space. Also Krishna and Sarma [9] observed the convergence of sequence of fuzzy points. Rhie et al.[13] introduced the notion of fuzzy α-Cauchy sequence of fuzzy points and fuzzy completeness. Since the concept of the completeness is essential to describe the aspects of normed linear spaces relative to the closedness of a space, there may be rich applications for fuzzyfying Banach spaces if a new type of the fuzzy completeness is introduced in a fuzzy normed linear space. In this paper, we introduce the notions of the fuzzy statistical convergence of sequences, the fuzzy statistical Cauchy sequence on a fuzzy normed linear space and the related fuzzy completeness, as a generalization of those in ordinary normed linear spaces. And we investigate some related properties on the fuzzy statistical completeness.

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2. Preliminaries

Throughout this paper, X is a vector space over the field K(R or C). Fuzzy subsets of X are denoted by Greek letters in general. χ_A denotes the characteristic function of the set A.

DEFINITION 2.1. [7] For two fuzzy subset μ_1 and μ_2 of X, the fuzzy subset $\mu_1 + \mu_2$ is defined by

$$(\mu_1 + \mu_2)(x) = \sup_{x_1 + x_2 = x} min\{\mu_1(x_1), \mu_2(x_2)\}$$

And for a scalar t of K and a fuzzy subset μ of X, the fuzzy subset $t\mu$ is defined by

$$(t\mu)(x) = \begin{cases} \mu(x/t) & \text{if} \quad t \neq 0 \\ 0 & \text{if} \quad t = 0 \ \text{ and } \ x \neq 0 \\ \sup_{y \in X} \mu(y) & \text{if} \quad t = 0 \ \text{ and } \ x = 0. \end{cases}$$

DEFINITION 2.2. [5] $\mu \in I^X$ is said to be

1. *convex* if $t\mu + (1-t)\mu \leq \mu$ for each $t \in [0,1]$
2. *balanced* if $t\mu \leq \mu$ for each $t \in K$ with $| t | \leq 1$
3. *absorbing* if $sup_{t>0}\, t\mu(x) = 1$ for all $x \in X$.

DEFINITION 2.3. [5] Let (X, τ) be a topological space and $\omega(\tau) = \{f : (X,\tau) \to [0,1] \mid f \ \textit{is lower semicontinuous}\}$. Then $\omega(\tau)$ is a fuzzy topology on X. This topology is called the fuzzy topology generated by τ on X. The fuzzy usual topology on K means the fuzzy topology generated by the usual topology of K.

DEFINITION 2.4. [5] A *fuzzy linear topology* on a vector space X over K is a fuzzy topology on X such that the two mappings

$$\begin{aligned} + &: X \times X \to X, & (x,y) &\to x+y \\ \cdot &: K \times X \to X, & (t,x) &\to tx \end{aligned}$$

are continuous when K has the fuzzy usual topology. A linear space with a fuzzy linear topology is called a *fuzzy topological linear space* or a *fuzzy topological vector space*.

DEFINITION 2.5. [5] Let X be a fuzzy topological space and $x \in X$. A fuzzy set μ in X is called a neighborhood of x if there exists an open fuzzy set ρ with $\rho \leq \mu$ and $\rho(x) = \mu(x) > 0$. Warren has proved in [17] that a fuzzy set μ in X is open iff μ is a neighborhood of x for each $x \in X$ with $\mu(x) > 0$.

Theorem 2.6. [5] *Let μ be a neighborhood of $z_0 = x_0 + y_0$ in a fuzzy topological vector space X. Then for each real number θ with $0 < \theta < \mu(z_0)$, there exist open neighborhoods μ_1, μ_2 of the points x_0, y_0. respectively, such that $\mu_1 + \mu_2 \leq \mu$ and $min\{\mu_1(x_0), \mu_2(y_0)\} > \theta$. In case $x_0 = y_0 = 0$, there exists an open neighborhood μ_3 of zero with $\mu_3(0) > \theta$. $\mu_3 \leq \mu$. and $\mu_3 + \mu_3 \leq \mu$.*

Definition 2.7. [5] Let x be a point in a fuzzy topological space X. A family F of neighborhoods of x is called a base for the system of all neighborhoods of x if for each neighborhood μ of x and each $0 < \theta < \mu(x)$, there exists $\mu_1 \in F$ with $\mu_1 \leq \mu$ and $\mu_1(x) > \theta$.

Definition 2.8. [6] A *fuzzy seminorm* on X is a fuzzy set ρ in X which is convex, balanced and absorbing. If in addition $inf_{t>0}\, t\rho(x) = 0$ for every nonzero x, then ρ is called a *fuzzy norm*.

Theorem 2.9. [6] *If ρ is a fuzzy seminorm on X, then the family $B_\rho = \{\theta \wedge (t\rho) \mid 0 < \theta \leq 1,\ t > 0\}$ is a base of zero for a fuzzy linear topology τ_ρ, where $\theta \wedge (t\rho)$ is the function $X \to [0, 1]$ such that $\theta \wedge t\rho(x) = min\{\theta, \rho(\frac{x}{t})\}$.*

Definition 2.10. [6] Let ρ be a seminorm on a linear space. The fuzzy topology τ_ρ in Theorem 2.9 is called the fuzzy topology induced by the fuzzy seminorm ρ. And a linear space equipped with a fuzzy seminorm (resp. fuzzy norm) is called a fuzzy seminormed(resp. fuzzy normed) linear space.

3. Fuzzy statistical convergence and fuzzy statistical completeness.

In [3] H. Fast introduced an extension of the usual concept of sequential limits which he called statistical convergence and also studied the concept as a summability mathod. In [2, 16], one may find a resent trend for this topics.

Definition 3.1. [16] The natural density of a set K of positive integers is defined by $\delta(K) = \lim_{n\to\infty} \frac{1}{n} \mid \{k \in K : k \leq n\} \mid$, where $\mid \{k \in K : k \leq n\} \mid$ denotes the number of elements of K not exceeding n.

Remarks. It is clear that for a finite set K, we have $\delta(K) = 0$. The natural density may not exist for each set K and is different from zero which means $\delta(K) > 0$. Besides that, $\delta(K^c) = 1-\delta(K)$ where K^c means the complement of K.

NOTATION. For facilitation, we use the following notation: if $< x_k >$ is a sequence such that x_k satisfies property P for all k except a set of natural density zero (equivalently for all k in a positive integer set with natural density one), then we say that x_k satisfies P for "almost all k", and we abbreviate this by "a.a. k."

DEFINITION 3.2. [14] The sequence $< x_k >$ on a normed linear space statistically convergent to the vector x provided that for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \mid \{k \in K :\| x_k - x \|\geq \epsilon\} \mid = 0,$$

i.e.,

$$\| x_k - x \| < \epsilon \quad a.a.\ k. \tag{1}$$

In this case we write $st - \lim x_k = x$.

EXAMPLE. *Define* $x_k = x$ *if* k *is a square and* $x_k = 0$ *otherwise. Then* $\mid \{k \leq n : x_k \neq 0\} \mid \leq \sqrt{n}$, *so* $st - \lim x_k = 0$. *Note that we could have assigned any values whatsoever to* x_k *when* k *is a square, and we would still have* $st - \lim x_k = 0$. *It is clear if the inequality in* (1) *holds for all but finitely many* k, *then* $\lim x_k = x$. *It follows that* $\lim x_k = x$ *implies* $st - \lim x_k = x$.

DEFINITION 3.3. [14] The sequence $< x_k >$ on a normed linear space is statistical Cauchy sequence provided that for every $\epsilon > 0$, there exists a number $N(= N(\epsilon))$ such that

$$\| x_k - x_N \| < \epsilon \quad a.a.\ k, \tag{2}$$

i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \mid \{k \leq n :\| x_k - x_N \|\geq \epsilon\} \mid = 0,$$

THEOREM 3.4. [14] *Every statistical Cauchy sequence* $< x_k >$ *on a Banach space is a statistically convergent sequence.*

Now, we introduce the notions of the fuzzy statistical convergence of sequences, the fuzzy statistical Cauchy sequence on a fuzzy normed linear space. And we investigate some properties relative to the fuzzy statistical completeness.

DEFINITION 3.5. Let (X, ρ) be a fuzzy normed linear space. A sequence $< x_k > \subset X$ is said to fuzzy statistically converge to a point $x \in X$ if and only if for every $\epsilon > 0$ and for every neighborhood of zero μ with $\mu(0) > 1-\epsilon$, there exists a positive integer set K with natural density one such that $k \in K$ implies $\mu(x_k - x) > 1-\epsilon$, i.e., $\mu(x_k - x) > 1-\epsilon$

a.a. k. x is said to be a fuzzy statical limit of $< x_k >$ and denoted by $fst - limx_k = x$.

THEOREM 3.6. *Let (X, ρ) be a fuzzy normed linear space and $< x_k >, < y_k >$ two sequences in X. Then*
(a) If $fst - limx_k = x$ and $fst - limy_k = y$, then $fst - lim(x_k + y_k) = x + y$.
(b) If $t \in K$ and $fst - limx_k = x$, then $fst - limtx_k = tx$.

Proof. (a) Let $\epsilon > 0$ and μ be a neighborhood of zero with $\mu(0) > 1 - \epsilon$. Then there exists an open neighborhood μ_1 such that $\mu_1 \leq \mu$, $\mu_1 + \mu_1 \leq \mu$ and $\mu_1(0) > 1 - \epsilon$ by Theorem 2.6. Since $fst - limx_k = x$ and $fst - limy_k = y$, there exist two positive integer sets K_1, K_2 with natural density one such that

$$k \in K_1 \quad implies \quad \mu(x_k - x) > 1 - \epsilon \quad and$$

$$k \in K_2 \quad implies \quad \mu(y_k - y) > 1 - \epsilon.$$

Let $K = K_1 \cap K_2$ and $k \in K$. Then

$$\begin{aligned}\mu((x_k + y_k) - (x + y)) & \\ &\geq (\mu_1 + \mu_1)((x_k + y_k) - (x + y)) \\ &= (\mu_1 + \mu_1)((x_k - x) + (y_k - y) \\ &\geq \min \mu_1(x_k - x), \mu_1(y_k - y) > 1 - \epsilon\end{aligned}$$

Therefore $< x_k + y_k >$ fuzzy statistically converges to $x + y$.

(b) If $t = 0$, then it is clear. Let $t \neq 0$. Since for every $\epsilon > 0$ and for every neighborhood of zero μ with $\mu(0) > 1 - \epsilon$, $\frac{1}{t}\mu$ is also a neighborhood of zero with $\frac{1}{t}\mu(0) = \mu(0) = 1 > 1 - \epsilon$, and $< x_k >$ fuzzy statistically converges to x, there exists a positive integer set K with natural density one such that $k \in K$ implies $\mu(tx_k - tx) = \frac{1}{t}\mu(x_k - x) > 1 - \epsilon$. Therefore $< tx_k >$ fuzzy statistically converges to tx. This completes the proof. □

Now, we will prove that the fuzzy statistical limit is unique. For the proof, we begin with following two lemmas.

LEMMA 3.7. *Let (X, ρ) be a fuzzy normed linear space, $x \in X$ and $\epsilon \in (0, 1)$. If for every $t > 0$, $t\rho(x) > \epsilon$, then x is the zero vector of the space X.*

Proof. Suppose that x is not zero vector. Since for every $t > 0$, $t\rho(x) > \epsilon$, $inf_{t>0} t\rho(x) \geq \epsilon > 0$. This contradicts to the fact that ρ is a fuzzy norm on X. Hence x is the zero vector of X. □

Lemma 3.8. *Let (X, ρ) be a fuzzy normed linear space, $x \in X$ and $\epsilon \in (0,1)$. If for each neighborhood of zero μ with $\mu(0) > \epsilon$, $\mu(x) > \epsilon$ then x is the zero vector of X.*

Proof. Fix a $\theta > \epsilon$. Since for every $t > 0$, $\theta \wedge t\rho$ is a neighborhood of zero and $\theta \wedge t\rho(0) = \theta \wedge \rho(0) = \theta \wedge 1 > \epsilon$ for all $t > 0$. This implies that for every $t > 0$, $t\rho(x) > \epsilon$. By the above lemma, x is the zero vector of X. □

Theorem 3.9. *The fuzzy statistical limit of a sequence $< x_k >$ is unique.*

Proof. Suppose that $< x_k >$ fuzzy statistically converges to x and x'. If $\epsilon > 0$ and μ is a neighborhood of zero with $\mu(0) > 1 - \epsilon$, then there exist two positive integer sets K_1 and K_2 with natural density one such that

$$k \in K_1 \quad implies \quad \mu(x_k - x) > 1 - \epsilon \quad and$$
$$k \in K_2 \quad implies \quad \mu(x_k - x') > 1 - \epsilon.$$

Since μ is a neighborhood of zero and $\mu(0) > 1 - \epsilon$, there exists a neighborhood of zero μ_1 such that $\mu_1(0) > 1-\epsilon$, $\mu_1 \leq \mu$ and $\mu_1 + \mu_1 \leq \mu$ by Theorem 2.6. Now, we have

$$\begin{aligned} \mu(x - x') &\geq (\mu_1 + \mu_1)(x - x') \\ &= (\mu_1 + \mu_1)((x - x_k) + (x_k - x')) \\ &\geq \min\{\mu_1(x_k - x), \mu_1(x_k - x')\} > 1 - \epsilon \ \ for\ all\ k \in K_1 \cap K_2 \end{aligned}$$

By the above lemma, we get $x - x' = 0$ equivalently $x = x'$. This completes the proof. □

Definition 3.10. Let (X, ρ) be a fuzzy normed linear space. A sequence $< x_k >$ is said to be a fuzzy statistical Cauchy sequence if and only if for every $\epsilon > 0$ and for every neighborhood of zero μ with $\mu(0) > 1 - \epsilon$, there exists a positive integer set K with natural density one such that $k, l \in K$ implies $\mu(x_k - x_l) > 1 - \epsilon$.

Theorem 3.11. *Let (X, ρ) be a fuzzy normed linear space. Then every fuzzy statistically convergent sequence in (X, ρ) is a fuzzy statistical Cauchy sequence.*

Proof. Let $< x_k >$ fuzzy statistically converge to a point $x \in X$. Then for every $\epsilon > 0$ and for every neighborhood of zero μ with $\mu(0) > 1 - \epsilon$, there exists a neighborhood of zero μ_1 such that $\mu_1 \leq \mu$, $\mu_1 + \mu_1 \leq \mu$ and $\mu(0) > 1 - \epsilon$ by Theorem 2.6. Since μ_1 is a neighborhood of zero and $\mu_1(0) > 1-\epsilon$, there exists a positive integer set K with natural density one such that

$$k \in K \text{ implies } \mu_1(x_k - x) > 1-\epsilon.$$

Now, we have

$$\begin{aligned}\mu(x_k - x_l) &\geq (\mu_1 + \mu_1)((x_k - x) + (x - x_l)) \\ &\geq \min\{\mu_1(x_k - x), \mu_1(x - x_l)\} > 1-\epsilon \text{ for all } k, l \in K.\end{aligned}$$

Therefore $< x_k >$ is a fuzzy statistical Cauchy sequence. This completes the proof. □

Now, we consider some relations between the fuzzy statistical completeness and ordinary completeness on a linear space.

DEFINITION 3.12. A fuzzy normed linear space (X, ρ) is said to be fuzzy statistically complete if and only if every fuzzy statistical Cauchy sequence fuzzy statistically converges to a point $x \in X$.

LEMMA 3.13. *Let $(X, \|\cdot\|)$ be a normed linear space and B the closed unit ball of X. Then every fuzzy statistical Cauchy sequence on the fuzzy normed linear space (X, χ_B) is a statistical Cauchy sequence with respect to the ordinary norm.*

Proof. Let $\eta > 0$ be given. Since $\theta \wedge \frac{\eta}{2}\chi_B(0) > 1-\epsilon$ if $\theta > 1-\epsilon$, for every $\epsilon > 0$ with $\theta > 1-\epsilon$, $\theta \wedge \frac{\eta}{2}\chi_B$ is a neighborhood of zero with $\theta \wedge \frac{\eta}{2}\chi_B(0) > 1-\epsilon$. Hence there exists a positive integer set K with natural density one such that $k, l \in K$ implies

$$\begin{aligned} & \theta \wedge \tfrac{\eta}{2}\chi_B(x_k - x_l) > 1-\epsilon \\ \Longrightarrow \quad & \tfrac{\eta}{2}\chi_B(x_k - x_l) > 1-\epsilon \ \ a.a.\ k \\ \Longrightarrow \quad & \chi_B(\tfrac{2}{\eta}(x_k - x_l)) > 1-\epsilon \ \ a.a.\ k \\ \Longrightarrow \quad & \chi_B(\tfrac{2}{\eta}(x_k - x_l)) = 1 \ \ a.a.\ k \\ \Longrightarrow \quad & \| x_k - x_l \| \leq \tfrac{\eta}{2} < \eta \ \ a.a.\ k \end{aligned}$$

Therefore $< x_k >$ is a statistical Cauchy sequence in $(X, \|\cdot\|)$. □

THEOREM 3.14. *$(X, \|\cdot\|)$ be a Banach space. Then the fuzzy normed linear space (X, χ_B) is fuzzy statistically complete where B is the closed unit ball of X.*

Proof. Let $< x_k >$ be a fuzzy statistical Cauchy sequence in (X, χ_B). Then it is a statistical Cauchy sequence with respect to the ordinary norm $\|\cdot\|$ by the above lemma. Since $(X, \|\cdot\|)$ is complete, there exists an $x \in X$ such that $< x_k >$ statistically converges to x by Theorem 3.4. Now, we show that $< x_k >$ fuzzy statistically converges to this x in

(X, χ_B). Let $\epsilon > 0$ and μ be a neighborhood of zero with $\mu(0) > 1-\epsilon$. Then there exist $1-\epsilon < \theta \leq 1$, $\eta > 0$ such that $\theta \wedge \eta\chi_B \leq \mu$ because that $\{\theta \wedge t\chi_B \ : t > 0,\ 0 < \theta \leq 1\}$ is a base at zero. For this $\eta > 0$, there exists a positive integer set K with natural density one such that

$$\begin{aligned} & k \in K \quad \text{implies} \quad \| x_k - x \| < \eta \\ \Longrightarrow \quad & \theta \wedge \eta\chi_B(x_k - x) \ > 1 - \epsilon \quad a.a. \ \ k \\ \Longrightarrow \quad & \mu(x_k - x) > 1 - \epsilon \quad a.a. \ \ k \end{aligned}$$

That is $< x_k >$ fuzzy statistically converges to x, therefore (X, χ_B) is fuzzy statistically complete. This completes the proof. □

COROLLARY 3.15. *The field $K(\mathbb{R} or \mathbb{C})$ with the fuzzy topology generated by the usual topology on K is a fuzzy statistically complete fuzzy normed linear space.*

DEFINITION 3.16. [5] Two fuzzy seminorms ρ_1, ρ_2 on x are said to be equivalent if $\tau_{\rho_1}, \tau_{\rho_2}$.

THEOREM 3.17. [13] *Let $(X, \|\cdot\|)$ be a normed linear space. If ρ is a lower semicontinuous fuzzy norm on X, and have the bounded support: $\{x \in X |\ \rho(x) > 0\}$ is bounded, then ρ is equivalent to the fuzzy norm χ_B where B is the closed unit ball of X.*

By Theorem 3.14 and the above theorem we get the following theorem.

THEOREM 3.18. *If X is a Banach space and ρ is a lower semicontinuous fuzzy norm on X having the bounded support, then the fuzzy normed linear space (x, ρ) is fuzzy statistically complete.*

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