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INTEGRAL TRANSFORMS OF FUNCTIONALS ON A FUNCTION SPACE OF TWO VARIABLES

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Abstract. We establish the various relationships among the integral transform $\mathcal{F}_{\alpha,\beta}F$, the convolution product $(F*G)_\alpha$ and the first variation δF for a class of functionals defined on $K(Q)$, the space of complex-valued continuous functions on $Q=[0,S]\times[0,T]$ which satisfy $x(s,0)=x(0,t)=0$ for all $(s,t)\in Q$. And also we obtain Parseval's and Plancherel's relations for the integral transform of some functionals defined on $K(Q)$.

1. Introduction and definitions

In a unifying paper [15], Lee defined an integral transform $\mathcal{F}_{\alpha,\beta}$ of analytic functionals on an abstract Wiener space. For certain values of the parameters α and β and for certain classes of functionals, the Fourier-Wiener transform [3], the Fourier-Feynman transform [4] and the Gauss transform are special cases of this integral transform $\mathcal{F}_{\alpha,\beta}$. In [6], Chang, Kim and Yoo established an interesting relationship between the integral transform and the convolution product for functionals on an abstract Wiener space. Recently [12] Kim, Kim and Skoug studied the relationships among the integral transform, the convolution product and the first variation for functionals defined on $K[0,T]$, the space of complex-valued continuous functions on $[0,T]$ which vanish at zero.

Let $K(Q)$ be the space of complex-valued continuous functions defined on $Q=[0,S]\times[0,T]$ and satisfying $x(s,0)=x(0,t)=0$ for all $(s,t)\in Q$. Let α and β be nonzero complex numbers. In this paper, we establish the various relationships among the integral transform $\mathcal{F}_{\alpha,\beta}F$, the convolution product $(F*G)_\alpha$ and the first variation δF for a class of

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functionals defined on $K(Q)$. Also we obtain Parseval's and Plancherel's relations for the integral transform of some functionals defined on $K(Q)$.

Let $C(Q)$ denote Yeh-Wiener space; that is, the space of all real-valued continuous functions $x(s,t)$ on Q with $x(s,0) = x(0,t) = 0$ for all $(s,t) \in Q$. Yeh [18] defined a Gaussian measure m_Y on $C(Q)$ (later modified in [20]) such that as a stochastic process $\{x(s,t) : (s,t) \in Q\}$ has mean $E[x(s,t)] = 0$ and covariance $E[x(s,t)x(u,v)] = \min\{s,u\}\min\{t,v\}$.

Let $\mathcal{M}$ denote the class of all Yeh-Wiener measurable subsets of $C(Q)$ and we denote the Yeh-Wiener integral of a Yeh-Wiener integrable functional F by

$$\int_{C(Q)} F(x)\, m_Y(dx). \tag{1.1}$$

Next we state the definitions of the integral transform $\mathcal{F}_{\alpha,\beta}F$, the convolution product $(F * G)_\alpha$ and the first variation δF for functionals defined on $K(Q)$.

Definition 1.1. Let F be a functional defined on $K(Q)$. Then the integral transform $\mathcal{F}_{\alpha,\beta}F$ of F is defined by

$$\mathcal{F}_{\alpha,\beta}F(y) = \int_{C(Q)} F(\alpha x + \beta y)\, m_Y(dx), \quad y \in K(Q) \tag{1.2}$$

if it exists [6, 12, 13, 15].

It is obvious that (1.2) implies that

$$\mathcal{F}_{\alpha,\beta}F(cy) = \mathcal{F}_{\alpha,c\beta}F(y)$$

for all real number c and for all $y \in K(Q)$.

Definition 1.2. Let F and G be functionals defined on $K(Q)$. Then the convolution product $(F * G)_\alpha$ of F and G is defined by

$$\begin{aligned}(F * G)_\alpha(y)\\ = \int_{C(Q)} F\Big(\frac{y+\alpha x}{\sqrt{2}}\Big) G\Big(\frac{y-\alpha x}{\sqrt{2}}\Big)\, m_Y(dx), \quad y \in K(Q)\end{aligned} \tag{1.3}$$

if it exists [6, 10, 12, 19, 21].

Definition 1.3. Let F be a functional defined on $K(Q)$ and let $w \in K(Q)$. Then the first variation δF of F is defined by

$$\delta F(y|w) = \frac{\partial}{\partial t} F(y + tw)|_{t=0}, \quad y \in K(Q) \tag{1.4}$$

if it exists [2, 5, 14, 16].

Let $\{\theta_1, \theta_2, \dots, \theta_n\}$ be an orthonormal set of real-valued functions in $L_2(Q)$. Furthermore assume that each θ_j is of bounded variation in the sense of Hardy and Krause on Q. Then for each $y \in K(Q)$ and $j = 1, 2, \dots$, the Riemann-Stieltjes integral $\langle \theta_j, y\rangle \equiv \int_Q \theta_j(s,t)\,dy(s,t)$ exists. Furthermore

$$\begin{aligned} |\langle \theta_j, y\rangle| =& |\theta_j(S,T)y(S,T) - \int_0^T y(S,t)\,d\theta_j(S,t) \\ &- \int_0^S y(s,T)\,d\theta_j(s,T) + \int_Q y(s,t)\,d\theta_j(s,t)| \le C_j\|y\|_\infty \end{aligned} \tag{1.5}$$

with

$$\begin{aligned} C_j = |\theta_j(S,T)| &+ \mathrm{Var}(\theta_j(S,\cdot),[0,T]) \\ &+ \mathrm{Var}(\theta_j(\cdot,T),[0,S]) + \mathrm{Var}(\theta_j,Q). \end{aligned} \tag{1.6}$$

Next we describe the class of functionals that we work with in this paper. For $0 \le \sigma < 1$, let $E_\sigma(Q)$ be the space of all functionals $F : K(Q) \to \mathbb{C}$ of the form

$$F(y) = f(\langle \vec\theta, y\rangle) = f(\langle \theta_1, y\rangle, \dots, \langle \theta_n, y\rangle) \tag{1.7}$$

for some positive integer n, where $f(\vec\lambda) = f(\lambda_1, \dots, \lambda_n)$ is an entire function of the n complex variables $\lambda_1, \dots, \lambda_n$ of exponential type; that is to say,

$$|f(\vec\lambda)| \le A_F \exp\Big\{B_F \sum_{j=1}^n |\lambda_j|^{1+\sigma}\Big\} \tag{1.8}$$

for some positive constants A_F and B_F.

We finish this section by introducing a well-known Yeh-Wiener integration formula for functionals $f(\langle\vec\theta, x\rangle)$:

$$\int_{C(Q)} f(\langle \vec\theta, x\rangle)\,m_Y(dx) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} f(\vec u)\exp\Big\{-\frac{1}{2}\|\vec u\|^2\Big\}\,d\vec u \tag{1.9}$$

where $\|\vec u\|^2 = \sum_{j=1}^n u_j^2$ and $d\vec u = du_1 \cdots du_n$.

2. Integral transform, convolution product and first variation of functionals in $E_\sigma(Q)$

We first show that if F is an element of $E_\sigma(Q)$, then the integral transform $\mathcal{F}_{\alpha,\beta}F$ of F exists and is an element of $E_\sigma(Q)$.

THEOREM 2.1. *Let $F \in E_\sigma(Q)$ be given by (1.7). Then the integral transform $\mathcal{F}_{\alpha,\beta}F$ exists, belongs to $E_\sigma(Q)$ and is given by the formula*

$$\mathcal{F}_{\alpha,\beta}F(y) = h(\langle \vec{\theta}, y \rangle) \tag{2.1}$$

for $y \in K(Q)$, where

$$h(\vec{\lambda}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\alpha \vec{u} + \beta \vec{\lambda}) \exp\Big\{ -\frac{1}{2} \|\vec{u}\|^2 \Big\} \, d\vec{u}. \tag{2.2}$$

Proof. For each $y \in K(Q)$, using the Yeh-Wiener integration formula (1.9), we obtain

$$\begin{aligned} \mathcal{F}_{\alpha,\beta}F(y) = & \int_{C(Q)} f(\alpha \langle \vec{\theta}, x \rangle + \beta \langle \vec{\theta}, y \rangle) \, m_Y(dx) \\ = & (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\alpha \vec{u} + \beta \langle \vec{\theta}, y \rangle) \exp\Big\{ -\frac{1}{2} \|\vec{u}\|^2 \Big\} \, d\vec{u} \\ = & h(\langle \vec{\theta}, y \rangle) \end{aligned}$$

where h is given by (2.2). By [8, Theorem 3.15], $h(\vec{\lambda})$ is an entire function. Moreover by the inequality (1.8) we have

$$|h(\vec{\lambda})| \le (2\pi)^{-n/2} \int_{\mathbb{R}^n} A_F \exp\Big\{ B_F \sum_{j=1}^{n} |\alpha u_j + \beta \lambda_j|^{1+\sigma} - \frac{1}{2} \|\vec{u}\|^2 \Big\} \, d\vec{u}.$$

But since

$$|\alpha u_j + \beta \lambda_j|^{1+\sigma} \le |2\alpha u_j|^{1+\sigma} + |2\beta \lambda_j|^{1+\sigma},$$

we have

$$|h(\vec{\lambda})| \le A_{\mathcal{F}_{\alpha,\beta}F} \exp\Big\{ B_{\mathcal{F}_{\alpha,\beta}F} \sum_{j=1}^{n} |\lambda_j|^{1+\sigma} \Big\}$$

where

$$A_{\mathcal{F}_{\alpha,\beta}F} = (2\pi)^{-n/2} A_F \Big(\int_{\mathbb{R}} \exp\Big\{ B_F |2\alpha u|^{1+\sigma} - \frac{u^2}{2} \Big\} \, du \Big)^n < \infty$$

and $B_{\mathcal{F}_{\alpha,\beta}F} = B_F (2|\beta|)^{1+\sigma}$. Hence $\mathcal{F}_{\alpha,\beta}F \in E_\sigma(Q)$. □

Next we show that the convolution product of functionals from $E_\sigma(Q)$ exists and is an element of $E_\sigma(Q)$. We may assume that F and G in Theorem 2.2 below can be expressed using the same positive integer n. For details, see Remark 1.4 in [12].

Theorem 2.2. *Let $F, G \in E_\sigma(Q)$ be given by* (1.7) *with corresponding entire functions f and g. Then the convolution $(F * G)_\alpha$ exists, belongs to $E_\sigma(Q)$ and is given by the formula*

$$(F * G)_\alpha(y) = k(\langle \vec{\theta}, y\rangle) \tag{2.3}$$

for $y \in K(Q)$ where

$$k(\vec{\lambda}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f\Big(\frac{\vec{\lambda} + \alpha\vec{u}}{\sqrt{2}}\Big) g\Big(\frac{\vec{\lambda} - \alpha\vec{u}}{\sqrt{2}}\Big) \exp\Big\{-\frac{1}{2}\|\vec{u}\|^2\Big\}\, d\vec{u}. \tag{2.4}$$

Proof. For each $y \in K(Q)$, using the Yeh-Wiener integration formula (1.9), we obtain

$$\begin{aligned}
&(F * G)_\alpha(y)\\
&= \int_{C(Q)} f\Big(\frac{\langle \vec{\theta}, y\rangle + \alpha\langle \vec{\theta}, x\rangle}{\sqrt{2}}\Big) g\Big(\frac{\langle \vec{\theta}, y\rangle - \alpha\langle \vec{\theta}, x\rangle}{\sqrt{2}}\Big)\, m_Y(dx)\\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} f\Big(\frac{\langle \vec{\theta}, y\rangle + \alpha\vec{u}}{\sqrt{2}}\Big) g\Big(\frac{\langle \vec{\theta}, y\rangle - \alpha\vec{u}}{\sqrt{2}}\Big) \exp\Big\{-\frac{1}{2}\|\vec{u}\|^2\Big\}\, d\vec{u}\\
&= k(\langle \vec{\theta}, y\rangle)
\end{aligned}$$

where k is given by (2.4). By [8, Theorem 3.15], $k(\vec{\lambda})$ is an entire function and

$$\begin{aligned}
|k(\vec{\lambda})| \le& (2\pi)^{-n/2} A_F A_G\\
&\int_{\mathbb{R}^n} \exp\Big\{(B_F + B_G)\sum_{j=1}^{n}\Big(\frac{|\lambda_j| + |\alpha u_j|}{\sqrt{2}}\Big)^{1+\sigma} - \frac{1}{2}\|\vec{u}\|^2\Big\}\, d\vec{u}.
\end{aligned}$$

By the same method as in Theorem 2.1, we have

$$|k(\vec{\lambda})| \le A_{(F*G)_\alpha} \exp\Big\{B_{(F*G)_\alpha} \sum_{j=1}^{n} |\lambda_j|^{1+\sigma}\Big\},$$

where $B_{(F*G)_\alpha} = (B_F + B_G)2^{(1+\sigma)/2}$ and

$$A_{(F*G)_\alpha} = (2\pi)^{-n/2} A_F A_G \Big(\int_{\mathbb{R}} \exp\Big\{(B_F + B_G)(\sqrt{2}|\alpha u|)^{1+\sigma} - \frac{u^2}{2}\Big\} du\Big)^n.$$

Hence $(F * G)_\alpha \in E_\sigma(Q)$. □

In the following theorem, we fix $w \in K(Q)$ and consider $\delta F(y|w)$ as a function of y, while in Theorem 2.4 we fix $y \in K(Q)$ and consider $\delta F(y|w)$ as a function of w.

Theorem 2.3. *Let $F \in E_\sigma(Q)$ be given by (1.7) and let $w \in K(Q)$. Then*

$$\delta F(y|w) = p(\langle \vec{\theta}, y\rangle) \tag{2.5}$$

for $y \in K(Q)$ where

$$p(\vec{\lambda}) = \sum_{j=1}^{n} \langle \theta_j, w\rangle f_j(\vec{\lambda}). \tag{2.6}$$

Furthermore, as a function of $y \in K(Q)$, $\delta F(y|w)$ is an element of $E_\sigma(Q)$.

Proof. For $y \in K(Q)$,

$$\begin{aligned}\delta F(y|w) =& \frac{\partial}{\partial t} f(\langle \vec{\theta}, y\rangle + t\langle \vec{\theta}, w\rangle)|_{t=0} \\ =& \sum_{j=1}^{n} \langle \theta_j, w\rangle f_j(\langle \vec{\theta}, y\rangle) = p(\langle \vec{\theta}, y\rangle)\end{aligned}$$

where p is given by (2.6). Since $f(\vec{\lambda})$ is an entire function, $f_j(\vec{\lambda})$ and so $p(\vec{\lambda})$ are entire functions. By the Cauchy integral formula, we have

$$f_j(\lambda_1, \cdots, \lambda_j, \cdots, \lambda_n) = \frac{1}{2\pi i}\int_{|\zeta - \lambda_j|=1} \frac{f(\lambda_1, \cdots, \zeta, \cdots, \lambda_n)}{(\zeta - \lambda_j)^2}\, d\zeta.$$

By the inequality (1.8), for any ζ with $|\zeta - \lambda_j| = 1$, we have

$$\begin{aligned}&\Big|\frac{f(\lambda_1, \cdots, \zeta, \cdots, \lambda_n)}{(\zeta - \lambda_j)^2}\Big| \\ \le& A_F \exp\{B_F[|\lambda_1|^{1+\sigma} + \cdots + |\zeta|^{1+\sigma} + \cdots + |\lambda_n|^{1+\sigma}]\} \\ \le& A_F \exp\Big\{2^{1+\sigma} B_F\Big[\sum_{j=1}^{n} |\lambda_j|^{1+\sigma} + 1\Big]\Big\}.\end{aligned}$$

Hence

$$|f_j(\vec{\lambda})| \le A_F \exp\{2^{1+\sigma} B_F\} \exp\Big\{2^{1+\sigma} B_F \sum_{j=1}^{n} |\lambda_j|^{1+\sigma}\Big\},$$

and so

$$|p(\vec{\lambda})| \le \sum_{j=1}^{n} |\langle \theta_j, w\rangle||f_j(\vec{\lambda})| \le A_{\delta F(\cdot|w)} \exp\Big\{B_{\delta F(\cdot|w)} \sum_{j=1}^{n} |\lambda_j|^{1+\sigma}\Big\}$$

where

$$A_{\delta F(\cdot|w)} = A_F \exp\{2^{1+\sigma} B_F\}\|w\|_\infty \sum_{j=1}^{n} C_j < \infty$$

with C_j given by (1.6) and $B_{\delta F(\cdot|w)} = 2^{1+\sigma} B_F$. □

THEOREM 2.4. *Let $y \in K(Q)$ and let $F \in E_\sigma(Q)$ be given by (1.7). Then*

$$\delta F(y|w) = q(\langle \vec{\theta}, w \rangle) \tag{2.7}$$

for $w \in K(Q)$ where

$$q(\vec{\lambda}) = \sum_{j=1}^{n} \lambda_j f_j(\langle \vec{\theta}, y \rangle). \tag{2.8}$$

Furthermore, as a function of w, $\delta F(y|w)$ is an element of $E_\sigma(Q)$.

Proof. Equations (2.7) and (2.8) are immediate from the first part in the proof of Theorem 2.3. Clearly $q(\vec{\lambda})$ is an entire function. Next, using the estimation for $|f_j|$ in the proof of Theorem 2.3 above, we obtain

$$\begin{aligned}|q(\vec{\lambda})| \leq& \sum_{j=1}^{n} |\lambda_j f_j(\langle \vec{\theta}, y \rangle)| \\ \leq& A_F e^{2^{1+\sigma} B_F} \exp\{2^{1+\sigma} B_F \|y\|_\infty^{1+\sigma} (C_1^{1+\sigma} + \cdots + C_n^{1+\sigma})\} \sum_{j=1}^{n} |\lambda_j|.\end{aligned}$$

Since $t \leq \exp\{t^{1+\sigma}\}$ for all $t \geq 0$,

$$\sum_{j=1}^{n} |\lambda_j| \leq \exp\Big\{ \Big(\sum_{j=1}^{n} |\lambda_j| \Big)^{1+\sigma} \Big\} \leq \exp\Big\{ n^{1+\sigma} \sum_{j=1}^{n} |\lambda_j|^{1+\sigma} \Big\}$$

and so

$$|q(\vec{\lambda})| \leq A_{\delta F(y|\cdot)} \exp\Big\{ B_{\delta F(y|\cdot)} \sum_{j=1}^{n} |\lambda_j|^{1+\sigma} \Big\}$$

where $B_{\delta F(y|\cdot)} = n^{1+\sigma}$ and

$$A_{\delta F(y|\cdot)} = A_F e^{2^{1+\sigma} B_F} \exp\{2^{1+\sigma} B_F \|y\|_\infty^{1+\sigma} (C_1^{1+\sigma} + \cdots + C_n^{1+\sigma})\} < \infty.$$

Hence, as a function of w, $\delta F(y|w) \in E_\sigma(Q)$. □

3. Further results

3.1. Relationships involving two concepts

In this subsection, we establish all of the various relationships involving exactly two of the three concepts of "integral transform", "convolution product" and "first variation" for functionals belonging to $E_\sigma(Q)$.

The seven distinct relationships, as well as alternative expressions for some of them, are given by equations (3.1) through (3.7) below.

In view of Theorem 2.1 through Theorem 2.4 above, all of the functionals that occur in this section are elements of $E_\sigma(Q)$. For example, let F and G be any functionals in $E_\sigma(Q)$. Then by Theorem 2.2, the functional $(F*G)_\alpha$ belongs to $E_\sigma(Q)$, and hence by Theorem 2.1, the functional $\mathcal{F}_{\alpha,\beta}(F*G)_\alpha$ also belongs to $E_\sigma(Q)$. By similar arguments, all of the functionals that arise in equations (3.1) through (3.11) below, exist and belong to $E_\sigma(Q)$.

Once we have shown the existence theorems (Theorems 2.1 through 2.4 above), the proofs of the Formulas 3.1 through 3.7 below are similar to those in [12]. Hence we just state the formulas without proofs.

Formula 3.1. The integral transform of the convolution product equals the product of the integral transforms:

$$\begin{aligned}\mathcal{F}_{\alpha,\beta}(F*G)_\alpha(y) =&\mathcal{F}_{\alpha,\beta}F\Big(\frac{y}{\sqrt{2}}\Big)\mathcal{F}_{\alpha,\beta}G\Big(\frac{y}{\sqrt{2}}\Big)\\ =&\mathcal{F}_{\alpha,\beta/\sqrt{2}}F(y)\mathcal{F}_{\alpha,\beta/\sqrt{2}}G(y)\end{aligned} \tag{3.1}$$

for all y in $K(Q)$.

Formula 3.2. A formula for the convolution product of the integral transform of functionals from $E_\sigma(Q)$:

$$\begin{aligned}&(\mathcal{F}_{\alpha,\beta}F*\mathcal{F}_{\alpha,\beta}G)_\alpha(y)\\ =&(2\pi)^{-3n/2}\int_{\mathbb{R}^{3n}} f\Big(\alpha\vec{r}+\frac{\beta}{\sqrt{2}}\langle\vec{\theta},y\rangle+\frac{\beta\alpha}{\sqrt{2}}\vec{u}\Big)\\ &g\Big(\alpha\vec{s}+\frac{\beta}{\sqrt{2}}\langle\vec{\theta},y\rangle-\frac{\beta\alpha}{\sqrt{2}}\vec{u}\Big)\exp\Big\{-\frac{\|\vec{u}\|^2+\|\vec{r}\|^2+\|\vec{s}\|^2}{2}\Big\}\,d\vec{u}\,d\vec{r}\,d\vec{s}\end{aligned} \tag{3.2}$$

for all y in $K(Q)$.

Formula 3.3. The integral transform with respect to the first argument of the variation equals $1/\beta$ times the first variation of the integral transform:

$$\mathcal{F}_{\alpha,\beta}(\delta F(\cdot|w))(y)=\frac{1}{\beta}\delta\mathcal{F}_{\alpha,\beta}F(y|w)=\sum_{j=1}^{n}\langle\theta_j,w\rangle\mathcal{F}_{\alpha,\beta}F_j(y) \tag{3.3}$$

for all y and w in $K(Q)$.

Formula 3.4. The transform with respect to the second argument of the variation equals β times the first variation of the functional:

$$\mathcal{F}_{\alpha,\beta}(\delta F(y|\cdot))(w)=\beta\delta F(y|w) \tag{3.4}$$

for all y and w in $K(Q)$.

Formula 3.5. A formula for the first variation of the convolution product of functionals from $E_\sigma(Q)$:

$$\delta(F*G)_\alpha(y|w) = \sum_{j=1}^{n} \frac{\langle\theta_j, w\rangle}{\sqrt{2}} \left[(F_j * G)_\alpha(y) + (F * G_j)_\alpha(y)\right] \tag{3.5}$$

for all y and w in $K(Q)$.

Formula 3.6. A formula for the convolution product, with respect to the first argument of the variation, of the first variation of functionals from $E_\sigma(Q)$:

$$(\delta F(\cdot|w) * \delta G(\cdot|w))_\alpha(y) = \sum_{j=1}^{n}\sum_{l=1}^{n} \langle\theta_j, w\rangle\langle\theta_l, w\rangle (F_j * G_l)_\alpha(y) \tag{3.6}$$

for all y and w in $K(Q)$.

Formula 3.7. A formula for the convolution product, with respect to the second argument of the variation, of the first variation of functionals from $E_\sigma(Q)$:

$$(\delta F(y|\cdot) * \delta G(y|\cdot))_\alpha(w) = \frac{1}{2}\delta F(y|w)\delta G(y|w) - \frac{\alpha^2}{2}\sum_{j=1}^{n} F_j(y)G_j(y) \tag{3.7}$$

for all y and w in $K(Q)$.

3.2. Parseval's and Plancherel's relation

Let $H_0 = H_0(Q)$ be the space of real-valued functions f on Q which are absolutely continuous and whose derivative Df is in $L_2(Q)$. The inner product on H_0 is given by

$$\langle f, g\rangle = \int_Q (Df)(s)(Dg)(s)\,ds.$$

Then H_0 is a real separable infinite dimensional Hilbert space. Let $B_0 = B_0(Q)$ be the Yeh-Wiener space $C(Q)$ and equip B_0 with the sup norm. Then (H_0, B_0, m_Y) is an abstract Wiener space.

We restrict our attention, in this subsection, to the space $E_0(Q)$ rather than $E_\sigma(Q)$. Now it is well known, see for example [6, 15], that for all $F \in E_0(Q)$, all $y \in K(Q)$ and all complex numbers a, b and c,

$$\begin{aligned}&\int_{C(Q)}\int_{C(Q)} F(ax+by+cw)\, m_Y(dx)\, m_Y(dy)\\ &= \int_{C(Q)} F(\sqrt{a^2+b^2}z+cw)\, m_Y(dz)\end{aligned} \tag{3.8}$$

and that

$$\mathcal{F}_{\alpha,\beta}(\mathcal{F}_{\alpha',\beta'}F)(y) = F(y) = \mathcal{F}_{\alpha',\beta'}(\mathcal{F}_{\alpha,\beta}F)(y) \tag{3.9}$$

provided $\beta\beta' = 1$ and $\alpha^2 + (\beta\alpha')^2 = 0$.

THEOREM 3.8. *Let $F, G \in E_0(Q)$ and let α' be a complex number such that $\alpha^2 + (\beta\alpha')^2 = 0$. Then Parseval's relation*

$$\begin{aligned}&\int_{C(Q)} \mathcal{F}_{\alpha,\beta}F\Big(\frac{\alpha' y}{\sqrt{2}}\Big)\mathcal{F}_{\alpha,\beta}G\Big(\frac{\alpha' y}{\sqrt{2}}\Big)\, m_Y(dy)\\ &= \int_{C(Q)} F\Big(\frac{\alpha y}{\sqrt{2}}\Big)G\Big(-\frac{\alpha y}{\sqrt{2}}\Big)\, m_Y(dy)\end{aligned} \tag{3.10}$$

holds. In particular, if $\beta = i$, we have

$$\begin{aligned}&\int_{C(Q)} \mathcal{F}_{\alpha,i}F\Big(\frac{\alpha y}{\sqrt{2}}\Big)G\Big(\frac{\alpha y}{\sqrt{2}}\Big)\, m_Y(dy)\\ &= \int_{C(Q)} F\Big(\frac{\alpha y}{\sqrt{2}}\Big)\mathcal{F}_{\alpha,i}G\Big(\frac{\alpha y}{\sqrt{2}}\Big)\, m_Y(dy).\end{aligned} \tag{3.11}$$

Moreover, formula (3.11) *induces Plancherel's relation of the form*

$$\int_{C(Q)} \Big|\mathcal{F}_{\alpha,i}F\Big(\frac{\alpha y}{\sqrt{2}}\Big)\Big|^2 m_Y(dy) = \int_{C(Q)} \Big|F\Big(\frac{\alpha y}{\sqrt{2}}\Big)\Big|^2 m_Y(dy). \tag{3.12}$$

Proof. From Formula 3.1 and Definition 1.1, it follows that the left hand side of (3.10) is equal to

$$\begin{aligned}&\int_{C(Q)} \mathcal{F}_{\alpha,\beta}(F*G)_\alpha(\alpha' y)\, m_Y(dy)\\ &= \int_{C(Q)}\int_{C(Q)} (F*G)_\alpha(\alpha x+\beta\alpha' y)\, m_Y(dx)\, m_Y(dy).\end{aligned}$$

But by (3.8) and the fact that $\alpha^2 + (\beta\alpha')^2 = 0$, the last integral is equal to $(F*G)_\alpha(0)$, which is equal to the right hand side of (3.10).

From (3.9), we know that $\mathcal{F}_{\alpha,i}(\mathcal{F}_{\alpha,-i}G)(y) = G(y)$ and so we have

$$\begin{aligned}
&\int_{C(Q)} \mathcal{F}_{\alpha,i}F\Big(\frac{\alpha y}{\sqrt{2}}\Big)G\Big(\frac{\alpha y}{\sqrt{2}}\Big)\, m_Y(dy)\\
&= \int_{C(Q)} \mathcal{F}_{\alpha,i}F\Big(\frac{\alpha y}{\sqrt{2}}\Big)\mathcal{F}_{\alpha,i}(\mathcal{F}_{\alpha,-i}G)\Big(\frac{\alpha y}{\sqrt{2}}\Big)\, m_Y(dy)\\
&= \int_{C(Q)} F\Big(\frac{\alpha y}{\sqrt{2}}\Big)\mathcal{F}_{\alpha,-i}G\Big(-\frac{\alpha y}{\sqrt{2}}\Big)\, m_Y(dy),
\end{aligned}$$

where the second equality is obtained by (3.10). But it is easy to see that $\mathcal{F}_{\alpha,-i}G(-\alpha y/\sqrt{2}) = \mathcal{F}_{\alpha,i}G(\alpha y/\sqrt{2})$ and this completes the proof of (3.11).

Finally, since $\overline{\mathcal{F}_{\alpha,i}F(\alpha y/\sqrt{2})} = \mathcal{F}_{\overline{\alpha},-i\overline{\alpha}/\alpha}\overline{F}(\alpha y/\sqrt{2})$, by (3.11), we have

$$\begin{aligned}
&\int_{C(Q)} \Big|\mathcal{F}_{\alpha,i}F\Big(\frac{\alpha y}{\sqrt{2}}\Big)\Big|^2\, m_Y(dy)\\
&= \int_{C(Q)} F\Big(\frac{\alpha y}{\sqrt{2}}\Big)\mathcal{F}_{\alpha,i}\mathcal{F}_{\overline{\alpha},-i\overline{\alpha}/\alpha}\overline{F}\Big(\frac{\alpha y}{\sqrt{2}}\Big)\, m_Y(dy).
\end{aligned}$$

But by (3.8), it is easy to see that $\mathcal{F}_{\alpha,i}\mathcal{F}_{\overline{\alpha},-i\overline{\alpha}/\alpha}\overline{F}(\alpha y/\sqrt{2}) = \overline{F(\alpha y/\sqrt{2})}$ and this completes the proof of (3.12). □

3.3. Classes of functionals

Several classes of functionals were introduced during the study of Fourier-Wiener transform and integral transform on Wiener spaces. For example,

(i) Cameron and Martin [3] and Yeh [19] introduced the spaces E_0 and E_m on classical Wiener space.
(ii) Lee [15] and Chang, Kim and Yoo [6] introduced the spaces $\mathcal{E}_0$ and $\mathcal{E}_a$ on abstract Wiener space.
(iii) Kim, Kim and Skoug [12] introduced the spaces E_0 and E_σ on classical Wiener space.

The class $E_\sigma(Q)$ introduced in Section 1 of this paper and work with in this paper is the Yeh-Wiener space version of the class E_σ on Wiener space. And it is very natural class of functionals in which to study the relationships that exist among the integral transform, the convolution product and the first variation. (See Remark 1.5 in [12].)

Next we briefly summarize relationships between the above classes of functionals.

Remark 3.9. (i) The class $\mathcal{E}_0$ is a subclass of $\mathcal{E}_a$. (See [6].)

(ii) When B is the classical Wiener space $C_0[0,1]$, $\mathcal{E}_0$ is the space E_0 and $\mathcal{E}_a$ contains the space E_m. (See [6].)

(iii) If $\sigma = 0$, then the class $E_\sigma(Q) = E_0(Q)$ corresponds to the Yeh-Wiener space version of the class E_0.

(iv) For $0 < \sigma_1 < \sigma_2 < 1$, $E_0 \subset E_{\sigma_1} \subset E_{\sigma_2} \subset L_2(C_0[0,T])$ (see [12]) and $E_0(Q) \subset E_{\sigma_1}(Q) \subset E_{\sigma_2}(Q) \subset L_2(C(Q))$, where all the inclusions are proper.

Now we introduce one more class of functionals $E_a(Q)$, the Yeh-Wiener space version of the space $\mathcal{E}_a$.

Let $E_a(Q)$ be the space of all functionals $F : K(Q) \to \mathbb{C}$ which satisfy the following conditions:

(i) $F(x+\lambda y)$ is an entire function of the complex variable λ for all x and y in $K(Q)$, and

(ii) there exist positive constants c and d depending only on F such that

$$|F(y)| \le c\exp\{d\|y\|_\infty\} \tag{3.13}$$

for all $y \in K(Q)$.

It is well known, see for example [6,15], that equation (3.8) holds for all $F \in E_0(Q)$ or $F \in E_a(Q)$. But we do not know whether (3.8) holds for $F \in E_\sigma(Q)$ or not. This is why we restrict our attention to the functional $F \in E_0(Q)$ in Theorem 3.8 above.

Our final example shows that neither $E_\sigma(Q)$ nor $E_a(Q)$ contains the other as a subset.

Example 3.10. Let $\theta(s,t) = 1/\sqrt{ST}$ and let $0 < \sigma < 1$. Then $F(y) = \exp\{\langle\theta, y\rangle^{1+\sigma}\}$ belongs to $E_\sigma(Q)$. Suppose that $E_\sigma(Q) \subseteq E_a(Q)$. Then we must have

$$\exp\{\langle\theta, y\rangle^{1+\sigma}\} \le c\exp\{d\|y\|_\infty\}$$

for some fixed positive constants c and d, that is, we must have

$$\Big(\frac{|y(S,T)|}{\sqrt{ST}}\Big)^{1+\sigma} \le \ln c + d\|y\|_\infty$$

for all $y \in K(Q)$. Pick, for each positive integer n, $y_n \in K(Q)$ such that $y_n(S,T) = \|y_n\|_\infty = n$. Then we must have

$$\Big(\frac{n}{\sqrt{ST}}\Big)^{1+\sigma} \le \ln c + dn$$

for all n. But this is impossible since $\sigma > 0$, and so $E_\sigma(Q) \not\subseteq E_a(Q)$.

On the other hand, let $\{\theta_j\}_{j=1}^{\infty}$ be a complete orthonormal sequence of functions in $L_2(Q)$, each of bounded variation on Q. Let

$$F(y) = \exp\Big\{\sum_{j=1}^{\infty} \frac{\langle \theta_j, y\rangle}{2^j C_j}\Big\}$$

with C_j given by (1.6). Then F is not an element of $E_\sigma(Q)$ for $0 \le \sigma < 1$ because it depends upon $\langle \theta_m, y\rangle$ for every $m = 1, 2, \ldots$ and so it can not be written in the form (1.7) for any positive integer n. But by the inequality (1.5), we have

$$|F(y)| \le \exp\{\|y\|_\infty\}$$

and so F belongs to $E_a(Q)$. Hence $E_a(Q) \not\subseteq E_\sigma(Q)$.

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