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# WARING'S PROBLEM FOR LINEAR FRACTIONAL TRANSFORMATIONS

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Abstract. Waring's problem deals with representing any nonconstant function in a set of functions as a sum of $k$th powers of nonconstant functions in the same set. Consider $\sum_{i=1}^{p} f_i(z)^k = z$. Suppose that $k \geq 2$. Let $p$ be the smallest number of functions that give the above identity. We consider Waring's problem for the set of linear fractional transformations and obtain $p = k$.

## 1. Introduction

Waring's problem for a set $S$ of functions is the following question: "For a given integer $k$ satisfying $k \geq 2$, what is the smallest positive integer $n$ such that any nonconstant function $f$ in $S$ can be expressed in the form $f = {f_1}^k + {f_2}^k + \cdots + {f_n}^k$ for some choice of nonconstant functions $f_1, f_2, \ldots, f_k$ in $S$?" We allow complex coefficients in these problems.

Suppose that $k \geq 2$ and that $n \geq 2$. Consider the equation of the form

$$\sum_{i=1}^{n} f_i(z)^k = f(z),$$

where $f_1, f_2, \ldots, f_n$ and $f$ are nonconstant polynomials with complex coefficients. Suppose that

$$\sum_{i=1}^{n} f_i(z)^k = z. \tag{1.1}$$

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Then we get

$$\sum_{i=1}^{n} f_i(f(z))^k = f(z)$$

by the substitution of $f(z)$ for $z$. Thus any nonconstant polynomial $f(z)$ can be represented by the sum of $n$ $k$th powers of nonconstant polynomials. Therefore studying the equation (1.1) is important.

DEFINITION 1.1. Suppose that $k \geq 2$ and that $n \geq 2$. Let $f_1, f_2, \ldots, f_n$ be nonconstant functions in a set $S$ of functions satisfying

$$\sum_{i=1}^{n} f_i(z)^k = z. \tag{1.2}$$

$W_S(k)$ denotes the smallest number $n$ (which depends on $k$) satisfying the equation (1.2).

We denote the sets of linear polynomials, polynomials, entire functions, rational functions, and meromorphic functions by $L$, $P$, $E$, $R$ and $M$ respectively. By a meromorphic function we mean a meromorphic function in the whole complex plane. Newman and Slater showed that the identity function $z$ can be always represented as a sum of $k$ $k$th powers of nonconstant linear polynomials [7]. Therefore Waring's problems for $L$, $P$, $E$, $R$ and $M$ are solvable and $k$ is an upper bound for $W_L(k)$, $W_P(k)$, $W_E(k)$, $W_R(k)$ and $W_M(k)$. S. Hurwitz has conjectured that $W_P(k) = k$ [7]. Also, Heilbronn has conjectured that $k$ is minimal even if entire functions are allowed, i.e., $W_E(k) = k$ [3].

THEOREM 1.2 ([4], [7]). *We have*

$$W_P(k) > \frac{1}{2} + \sqrt{k + \frac{1}{4}}, \qquad k \geq 3. \tag{1.3}$$

THEOREM 1.3 ([4]). *We have*

$$W_E(k) \geq \frac{1}{2} + \sqrt{k + \frac{1}{4}}, \qquad k \geq 2. \tag{1.4}$$

THEOREM 1.4 ([2], [4]). *We have*

$$W_R(k) > \sqrt{k+1}, \qquad k \geq 2. \tag{1.5}$$

THEOREM 1.5 ([4]). *We have*

$$W_M(k) \geq \sqrt{k+1}, \qquad k \geq 2. \tag{1.6}$$

THEOREM 1.6 ([6]). *We have*

$$W_L(k) = k. \tag{1.7}$$

More details and results can be found in the survey paper; see [5].

DEFINITION 1.7. A *linear fractional transformation*, also called a *Möbius transformation* or a *bilinear transformation*, is a map

$$f(z) = \frac{az+b}{cz+d}, \ (ad - bc \neq 0). \tag{1.8}$$

We denote the set of linear fractional transformations by $T$. There is some interest in the representation. Since the class $T$ is closed under composition, we can deduce the representability of all nonconstant functions in $T$ from that of $z$.

## 2. The representation of a function by linear fractional transformations

Now we prove our theorems.

THEOREM 2.1. *Suppose that $k \geq 2$ and that $n \geq 2$. Let $f_1, f_2, \dots, f_n$ be nonconstant linear fractional transformations satisfying*

$$\sum_{i=1}^{n} f_i(z)^k = z. \tag{2.1}$$

*Suppose that at least one of the $f_i$ is not a linear polynomial and that $p$ is the smallest number $n$ satisfying the equation* (2.1)*. Then, $p \geq 2k$.*

We do not need to consider the case that all functions $f_i$ are linear polynomials because of Theorem 1.6.

*Proof.* Let $p$ be the smallest number $n$ satisfying the equation (2.1). Suppose that

$$\sum_{j=1}^{p} f_j(z)^k = z, \tag{2.2}$$

where each $f_j$ is a linear fractional transformation. Suppose that $f_1, \dots,$ $f_q$ are linear polynomials (if $q = 0$, then there are no linear polynomials) while the remaining $f_j$ have finite poles. Any $f_j$ with a finite pole can be written as $(az + b)/(z - z_0)$. Divide the functions $f_j$ with finite poles into groups so that those in a group have the same finite pole. Consider any such group, say labeled so that it consists of $f_m, \dots, f_v$, where $q < m \leq v$, with pole at $z_0$. Since all the other functions appearing in the equation (2.2) have no pole at $z_0$, it must be the case that

$$\sum_{j=m}^{v} f_j(z)^k \tag{2.3}$$

has no pole at $z_0$. For $m \leq j \leq v$, we can write

$$f_j(z) = (a_j z + b_j)/(z - z_0).$$

Then we get

$$\sum_{j=m}^{v} f_j(z)^k = \frac{1}{(z - z_0)^k} \sum_{j=m}^{v} (a_j z + b_j)^k.$$

It follows that

$$\sum_{j=m}^{v} (a_j z + b_j)^k,$$

which is a polynomial of degree at most $k$, must have a zero of order at least $k$ at $z_0$. Hence for some constant $C$, we must have

$$\sum_{j=m}^{v} (a_j z + b_j)^k = C(z - z_0)^k. \tag{2.4}$$

Incidentally, since no individual function $f_j$ is constant, this requires that in this group we have at least two functions (and possibly many more), that is, $v - m \geq 1$. Hence we get

$$\sum_{j=m}^{v} f_j(z)^k = \sum_{j=m}^{v} \frac{(a_j z + b_j)^k}{(z - z_0)^k} = C$$

and so any other such group adds up to a constant as well. Since the right hand side of the equation (2.2) is $z$, it follows that we must have some linear polynomials as well, that is, we have $q \geq 1$. Thus, denoting the sum of all those groups by a constant $d$, we see that

$$\sum_{j=1}^{q} f_j(z)^k = z - d,$$

and replacing $z - d$ by $z$ and noting that each $f_j(z + d)$ is a linear polynomial for $1 \leq j \leq q$, we find that $q \geq k$ by Theorem 1.6. So $p \geq k$, and further, if there are functions $f_j$ with a finite pole, then $p \geq k + 2$.

But let us now ask how many functions we need for the equation (2.4) to hold. Since we may replace $z - z_0$ by $z$ and assume that $C = 1$ without really changing the problem, we are asking how large, for a given $k$, the number $n$ needs to be so that we can have

$$\sum_{j=1}^{n}(a_j z + b_j)^k = z^k. \tag{2.5}$$

Because $v - m \geq 1$ in the equation (2.4), we can suppose that $n \geq 2$. Now, we suppose that $n < k$ and will obtain a contradiction. According to the minimality of $n$, all the $(a_j z + b_j)^k$ are linearly independent. Thus we can have $b_j = 0$ for at most one $j$. Then $(b_j + a_j z)^k = {b_j}^k \left(1 + \frac{a_j}{b_j} z\right)^k$ if $b_j \neq 0$. Suppose that ${b_j}^k = \beta_j$ and that $\frac{a_j}{b_j} = \alpha_j$ for each $j$.

Suppose that $b_n = 0$ and $b_j \neq 0$ for $1 \leq j \leq n-1$. Then

$$\begin{aligned}
\sum_{j=1}^{n}(b_j + a_j z)^k &= {a_n}^k z^k + \sum_{j=1}^{n-1} \beta_j (1 + \alpha_j z)^k \\
&= {a_n}^k z^k + \sum_{j=1}^{n-1} \beta_j \left( \sum_{r=0}^{k} \binom{k}{r} {\alpha_j}^r z^r \right) \\
&= {a_n}^k z^k + \sum_{r=0}^{k} \binom{k}{r} z^r \left( \sum_{j=1}^{n-1} \beta_j {\alpha_j}^r \right).
\end{aligned}$$

Since the right hand side of the equation (2.5) is equal to $z^k$, we get, in particular, the system of equations

$$\sum_{j=1}^{n-1} {\alpha_j}^r \beta_j = 0 \qquad \text{for } 0 \leq r \leq k-1. \tag{2.6}$$

Because $n < k$, we use $n-1$ equations for $0 \leq r \leq n-2$. Now consider $\beta_j$ for $1 \leq j \leq n-1$ as unknowns. Then the coefficients form a square matrix $M_1$ whose determinant is given by

$$|M_1| = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ {\alpha_1}^2 & {\alpha_2}^2 & \cdots & {\alpha_{n-1}}^2 \\ \vdots & \vdots & \ddots & \vdots \\ {\alpha_1}^{n-2} & {\alpha_2}^{n-2} & \cdots & {\alpha_{n-1}}^{n-2} \end{vmatrix}.$$

Since the determinant of $M_1$ is the van der Monde determinant [1], we get

$$|M_1| = \prod_{i<j} (\alpha_j - \alpha_i).$$

Since all the $(a_j z + b_j)^k$ are linearly independent, we have $\alpha_i \neq \alpha_j$ for $i \neq j$ and we get $|M_1| \neq 0$. Hence the system (2.6) of homogeneous linear equations has only the trivial solution and so ${b_j}^k = \beta_j = 0$ for all $j$ with $1 \leq j \leq n-1$. This is a contradiction.

Suppose that $b_j \neq 0$ for each $j$. Then

$$\sum_{j=1}^{n} (b_j + a_j z)^k = \sum_{r=0}^{k} \binom{k}{r} z^r \left( \sum_{j=1}^{n} \beta_j {\alpha_j}^r \right).$$

Because the right hand side of the equation (2.5) is equal to $z^k$, we get

$$\sum_{j=1}^{n} {\alpha_j}^r \beta_j = 0 \qquad \text{for } 0 \leq r \leq k-1. \tag{2.7}$$

By using $n$ equations for $0 \leq r \leq n-1$, we have a coefficient matrix $M_2$ whose determinant is given by

$$|M_2| = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ {\alpha_1}^2 & {\alpha_2}^2 & \cdots & {\alpha_n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ {\alpha_1}^{n-1} & {\alpha_2}^{n-1} & \cdots & {\alpha_n}^{n-1} \end{vmatrix}.$$

Since $|M_2| \neq 0$, the system (2.7) has only the trivial solution and so ${b_j}^k = \beta_j = 0$ for all $j$. This is a contradiction. Therefore we get $n \geq k$.

Hence, if not all the $f_j$ are linear polynomials in the equation (2.2), the number of linear polynomials is greater than or equal to $k$ and the number of linear fractional transformations with finite poles is also greater than or equal to $k$. Therefore we get $p \geq 2k$. □

Theorem 2.2. *We have*

$$W_T(k) = k.$$

*Proof.* We get this result by Theorem 1.6 and Theorem 2.1. □

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