

REGULARLY QUASI-ORDERED SPACES AND NORMALLY QUASI-ORDERED SPACES

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ABSTRACT. Generalizing normally quasi-ordered spaces, we introduce a concept of regularly quasi-ordered spaces and study their categorical properties. We obtain well behaved reflective subcategories of the category **Rqos** of regularly quasi-ordered spaces and continuous isotones, namely the full subcategory of **Rqos** determined by T_0 -objects among others, and this result can be extended to that in the category **Nqos** of normally quasi-ordered spaces and continuous isotones.

1. Introduction and main results

In 1940's, Nachbin extended the measure theory and function theory on topological spaces to those on topological partially ordered spaces([7]). After that, there have been many attempts to study topological partially ordered spaces by various authors([3], [4], [6], [8], [9], [11]).

It is well known that the category **Pord** of partially ordered sets and isotones is mono-topological but the category **Qord** of quasi-ordered sets and isotones is topological and therefore **Qord** is more convenient than **Pord**. So we are interested in topological quasi-ordered spaces determined by quasi-order relations instead of partial order relations.

A *topological quasi-ordered space* (X, τ, \leq) is a set X endowed with both a topology τ and a quasi-order \leq . A topological space (X, τ) may be considered as a topological quasi-ordered space when it is realized that X is a quasi-ordered set endowed with the discrete order and a quasi-ordered set (X, \leq) may be considered as a topological quasi-ordered space with discrete or indiscrete topology. Therefore the study of topological quasi-ordered spaces not only includes the study of general

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topology and quasi-ordered sets, but also reveals many generalizations of results in functional analysis for those in function spaces and measure theory.

In this paper, we study some properties of regularly quasi-ordered spaces and normally quasi-ordered spaces, and also categorical properties of the category \mathbf{Rqos} of regularly quasi-ordered spaces and the category \mathbf{Nqos} of normally quasi-ordered spaces. Moreover we obtain well behaved subcategories of them. In particular, we show that \mathbf{Rqos} and \mathbf{Nqos} are order compatible hereditary. Using some results about normality of quasi-order in Nachbin([7]), we suggest a generalization of Urysohn's Theorem on topological spaces to topological quasi-ordered spaces. We also show that the categories $\mathbf{T}_0\mathbf{Rqos}$, $\mathbf{T}_1\mathbf{Rqos}$, $\mathbf{T}_2\mathbf{Rqos}$ and \mathbf{Rpos} are epireflective subcategories, initially and finally dense in \mathbf{Rqos} , $\mathbf{R}_0\mathbf{Rqos}$, $\mathbf{R}_1\mathbf{Rqos}$ and $\mathbf{T}_0^{-1}\mathbf{Rqos}$, respectively. Similarly, by replacing \mathbf{Rqos} with \mathbf{Nqos} , we show that $\mathbf{T}_0\mathbf{Nqos}$, $\mathbf{T}_1\mathbf{Nqos}$, $\mathbf{T}_2\mathbf{Nqos}$ and \mathbf{Npos} are epireflective subcategories, initially and finally dense in \mathbf{Nqos} , $\mathbf{R}_0\mathbf{Nqos}$, $\mathbf{R}_1\mathbf{Nqos}$ and $\mathbf{T}_0^{-1}\mathbf{Nqos}$, respectively.

For the terminology not introduced in the paper, we refer to Adamek, Herrlich and Strecker [1] for the category theory and Bourbaki [2] for topology and Davey and Priestley [5] for the order theory. Also we assume throughout this paper that a subcategory of a category is full and isomorphism closed.

2. Regularly quasi-ordered spaces

A *continuous quasi-ordered space* (X, τ, \leq) is a topological quasi-ordered space with a continuous order \leq , i.e., for any $x \not\leq y$ in X , there are neighborhoods U of x and V of y such that $u \not\leq v$ for all $u \in U$ and $v \in V$ ([7]).

It is well known ([7, 9]) that a continuous partially ordered space is a T_2 -space but every topological space is a continuous quasi-ordered space when it is endowed with the indiscrete order. Furthermore, in a continuous quasi-ordered space (X, τ, \leq) and any $x \in X$, $\uparrow x = \{a \in X \mid x \leq a\}$ and $\downarrow x = \{a \in X \mid a \leq x\}$ are closed.

The class of continuous quasi-ordered spaces and continuous isotones forms a category which will be denoted by \mathbf{Wqos} ([9]).

DEFINITION 2.1. (1) For a topological quasi-ordered space (X, τ, \leq) , the order \leq is said to be *upper(lower, resp.) regular* if for any closed increasing(decreasing, resp.) subset F of X and $a \notin F$, there are a

decreasing(an increasing, *resp.*) neighborhood U of a and an increasing(a decreasing, *resp.*) neighborhood V of F such that $U \cap V = \emptyset$.

If \leq is both upper and lower regular, then it is called a *regular order*.

(2) A *regularly quasi-ordered space* is a topological quasi-ordered space with a both continuous and regular quasi-order.

EXAMPLE 2.2. (1) For the two-point chain $(\{0, 1\}, I, \leq)$ with the indiscrete topology, \leq is regular but not continuous. Conversely, for the two-point chain $(\{0, 1\}, \tau, \leq)$ with $\tau = \{\emptyset, \{0\}, \{0, 1\}\}$, \leq is continuous but not regular.

(2) Consider $X = \{a, b, b'\}$, the topology $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, b'\}, X\}$ and the graph $G_{\leq} = \Delta_X \cup \{(a, b), (a, b'), (b, b'), (b', b)\}$, where $\Delta_X = \{(x, x) \mid x \in X\}$. Then (X, τ, \leq) is a regularly quasi-ordered space.

One has the following immediately from Definition 2.1.

REMARK 2.3. Let $(X, \tau, =)$ be a topological quasi-ordered space with the discrete order. Then one has:

(1) $(X, \tau, =) \in \mathbf{Wqos}$ if and only if (X, τ) is a Hausdorff space, because the graph $G_{=}$ of the discrete order is Δ_X .

(2) $(X, \tau, =)$ is a regularly quasi-ordered space if and only if (X, τ) is a regular space, because for any $A \subseteq X$, A is both increasing and decreasing. But we note that (X, τ) in Example 2.2 (2) is not a regular space.

(3) If (X, τ, \leq) is a regularly quasi-ordered space and $\tau \subseteq \tau'$, then (X, τ', \leq) need not be a regularly quasi-ordered space again(see [6]).

(4) There is a continuous quasi-ordered space which is not a regularly quasi-ordered space. Indeed, consider the set R of real numbers and the discrete order $=$ on R . For any non-zero real number x , let N_x be the filter generated by the set of open intervals in R containing x , and N_0 the filter generated by $\{(-p, p) - \{\frac{1}{n} \mid n \in N\} \mid p > 0\}$ containing 0. Then there is a unique topology τ on R such that N_x is precisely the neighborhood filter of x with respect to τ . Since (R, τ) is a Hausdorff space but not a regular space, $(R, \tau, =)$ is a continuous but not regularly quasi-ordered space.

Let \mathbf{Rqos} be the full subcategory of \mathbf{Wqos} determined by regularly quasi-ordered spaces and \mathbf{Rpos} its full subcategory determined by a partial order relation instead of a quasi-order relation. Then $\mathbf{Rpos} \subset \mathbf{Rqos} \subset \mathbf{Wqos}$.

THEOREM 2.4. For any topological quasi-ordered space (X, τ, \leq) , \leq is upper(lower, *resp.*) regular if and only if for each $x \in X$ and an

open decreasing(increasing, resp.) neighborhood U of x , there exists a decreasing(an increasing, resp.) neighborhood V of x such that the closure \bar{V} of V is contained in U .

Proof. Let $x \in X$ and U be an open decreasing neighborhood of x . Then the complement $\mathbb{C}U$ of U is closed increasing and $x \notin \mathbb{C}U$. Since \leq is upper regular, there are a decreasing neighborhood V of x and an increasing neighborhood W of $\mathbb{C}U$ such that $V \cap W = \emptyset$. So there is an open G such that $\mathbb{C}U \subseteq G \subseteq W$. Since $V \subseteq \mathbb{C}W \subseteq \mathbb{C}G \subseteq U$, the closure \bar{V} of V is contained in U .

Conversely, let F be a closed increasing set and $a \notin F$. Then $\mathbb{C}F$ is open decreasing and $a \in \mathbb{C}F$. By the assumption, there is a decreasing neighborhood V of a with $\bar{V} \subseteq \mathbb{C}F$. So $\mathbb{C}V$ is increasing and $F \subseteq \mathbb{C}\bar{V} \subseteq \mathbb{C}V$. Hence $\mathbb{C}V$ is an increasing neighborhood of F , and clearly $V \cap \mathbb{C}V = \emptyset$. Thus \leq is an upper regular quasi-order.

We have dually the result for a lower regular quasi-order. \square

A subspace of a regular space is again regular, but the following example exhibits that it is not the case for a regular quasi-order.

EXAMPLE 2.5. Let X be the three point chain $\{0, 1, 2\}$ endowed with the topology given by the only non-trivial open set $\{0, 2\}$. Then the given order is clearly a regular quasi-order, for increasing or decreasing closed sets in X are trivial, i.e., \emptyset or X . The order on the subspace A of X consisting of $\{0, 1\}$ is not regular as indicated in Example 2.2 (1).

DEFINITION 2.6. A topological quasi-ordered subspace (A, τ_A, \leq_A) of (X, τ, \leq) is said to be *compatibly quasi-ordered* if for each τ_A -closed increasing(decreasing, resp.) set F in A , there are a τ -closed increasing(decreasing, resp.) set F^* in X such that $F = F^* \cap A$.

PROPOSITION 2.7. Any compatibly quasi-ordered subspace of a regularly quasi-ordered space is also a regularly quasi-ordered space.

Proof. Let (A, τ_A, \leq_A) be a compatibly quasi-ordered subspace of a regularly quasi-ordered space (X, τ, \leq) and F a τ_A -closed increasing subset in A and $a \in A - F$. Then there exists a τ -closed increasing subset F^* in X such that $F = F^* \cap A$. Since $a \notin F^*$ and (X, τ, \leq) is a regularly quasi-ordered space, there are an increasing neighborhood U of F^* and a decreasing neighborhood V of a in X such that $U \cap V = \emptyset$. So $U \cap A$ and $V \cap A$ are an increasing neighborhood of F and a decreasing neighborhood of a , respectively, in A . Moreover, $(U \cap A) \cap (V \cap A) = \emptyset$, and hence \leq_A is upper regular.

Dually one has the proof for the lower regular case.

Since a subspace of a continuous quasi-ordered space is clearly a continuous quasi-ordered space, (A, τ_A, \leq_A) is also a regularly quasi-ordered space. \square

3. Normally quasi-ordered spaces

In this section, we deal with normal quasi-ordered spaces which generalize normal spaces. The following definition is due to Nachbin [7].

DEFINITION 3.1. For a topological quasi-ordered space (X, τ, \leq) , the order \leq is said to be *normal* if for any closed increasing subset F_1 and closed decreasing subset F_2 of X with $F_1 \cap F_2 = \emptyset$, there are an increasing neighborhood U_1 of F_1 and a decreasing neighborhood U_2 of F_2 such that $U_1 \cap U_2 = \emptyset$.

A normal quasi-order need not be regular. Indeed, consider $X = \{a, a', b, c\}$, the topology $\tau = \{\emptyset, \{a, a'\}, \{b\}, \{a, a', b\}, X\}$ and the graph $G_{\leq} = \Delta_X \cup \{(a, a'), (a', a), (a, b), (a', b), (b, c), (a, c), (a', c)\}$. Then \leq is normal but not regular. Also there are topological quasi-ordered spaces with normal orders which are not continuous. For a simple example, the two-point space $(\{0, 1\}, I, \leq)$ in Example 2.2 (1), \leq is normal but not continuous.

DEFINITION 3.2. A *normally quasi-ordered space* is a topological space with a both continuous and normal quasi-order.

EXAMPLE 3.3. Consider a set $X = \{a_1, a_2, b_1, b_2\}$, the topology $\tau = \{\emptyset, \{a_1, a_2\}, \{b_1, b_2\}, X\}$ and the graph $G_{\leq} = \{(a_i, a_j) \mid 1 \leq i, j \leq 2\} \cup \{(b_s, b_k) \mid 1 \leq s, k \leq 2\} \cup \{(a_i, b_s) \mid 1 \leq i \leq 2, 1 \leq s \leq 2\}$. Then (X, τ, \leq) is a normally quasi-ordered space.

Let $\mathbf{Nqos}(\mathbf{Npos}, \text{resp.})$ denote the full subcategory of \mathbf{Wqos} determined by normally quasi-ordered (normally partially-ordered, resp.) spaces. Then $\mathbf{Npos} \subset \mathbf{Nqos} \subset \mathbf{Wqos}$.

REMARK 3.4. (1) A normally quasi-ordered space is a regularly quasi-ordered space, but the converse does not hold, i.e., $\mathbf{Nqos} \subset \mathbf{Rqos}$ (see [6]).

(2) $(X, \tau, =) \in \mathbf{Nqos}$ if and only if (X, τ) is a normal space. But for the space (X, τ, \leq) in Example 3.3, (X, τ) is not a normal space.

(3) If $(X, \tau, \leq) \in \mathbf{Nqos}$ and $\tau \subseteq \tau'$, then (X, τ', \leq) need not be in \mathbf{Nqos} (see [7]).

THEOREM 3.5. *For any topological quasi-ordered space (X, τ, \leq) , \leq is normal if and only if for any closed increasing(decreasing, resp.) subset F of X and an open increasing(decreasing, resp.) neighborhood U of F , there is an increasing(a decreasing, resp.) neighborhood V of F with $\overline{V} \subseteq U$.*

Proof. Assume that \leq is normal and U is an open increasing neighborhood of a closed increasing set F , then $\mathbb{C}U$ is closed decreasing and $\mathbb{C}U \cap F = \emptyset$. Since \leq is normal, there are an increasing neighborhood V of F and a decreasing neighborhood W of $\mathbb{C}U$ such that $V \cap W = \emptyset$. Since there is an open set G such that $\mathbb{C}U \subseteq G \subseteq W$, $V \subseteq \mathbb{C}W \subseteq \mathbb{C}G \subseteq U$, and hence $\overline{V} \subseteq U$.

Conversely, for any closed increasing subset F_1 and closed decreasing subset F_2 with $F_1 \cap F_2 = \emptyset$, $\mathbb{C}F_2$ is open increasing and $F_1 \subseteq \mathbb{C}F_2$. By the assumption there is an increasing neighborhood V of F_1 such that $\overline{V} \subseteq \mathbb{C}F_2$. And $\mathbb{C}V$ is also a decreasing neighborhood of F_2 , and clearly $V \cap \mathbb{C}V = \emptyset$. Thus \leq is normal. \square

Now, we suggest a good generalization of Urysohn's Separation Theorem on topological spaces to topological quasi-ordered spaces as follows. In the following, \mathfrak{R} denotes the real line endowed with the usual topology and the usual order.

THEOREM 3.6. *A topological quasi-ordered space (X, τ, \leq) is a normally quasi-ordered space if and only if it satisfies:*

- (1) *for any closed increasing subset F_1 and any closed decreasing subset F_2 of X with $F_1 \cap F_2 = \emptyset$, there is a continuous isotone $f : X \rightarrow \mathfrak{R}$ such that $f(F_1) \subseteq \{1\}$, $f(F_2) \subseteq \{0\}$ and $0 \leq f \leq 1$, and*
- (2) *for any $x \not\leq y$, there is a continuous isotone $g : X \rightarrow \mathfrak{R}$ such that $g(x) > g(y)$.*

Proof. It is known that \leq is normal if and only if (1) holds(see [7]). So it remains to show that a normal order \leq is continuous if and only if (2) holds.

Assume that \leq is continuous and $x \not\leq y$, $\uparrow x = \{a \in X \mid x \leq a\}$ and $\downarrow y = \{b \in X \mid b \leq y\}$ are disjoint closed increasing and closed decreasing sets, respectively. Moreover, since \leq is normal, there is a continuous isotone $f : X \rightarrow \mathfrak{R}$ such that $f(\uparrow x) \subseteq \{1\}$, $f(\downarrow y) \subseteq \{0\}$ by (1) and hence $f(x) = 1 > 0 = f(y)$.

Conversely, let $x \not\leq y$, then there is a continuous isotone $g : X \rightarrow \mathfrak{R}$ with $g(x) > g(y)$. Choose any real number ξ with $g(y) < \xi < g(x)$. Then $V = \{a \in X \mid g(a) > \xi\}$ is clearly an open increasing neighborhood of

x , $W = \{b \in X \mid g(b) < \xi\}$ an open decreasing neighborhood of y , and $V \cap W = \emptyset$. This completes the proof. \square

REMARK 3.7. Any subspace of a normally quasi-ordered space need not be normally quasi-ordered(see [6]). But any compatibly quasi-ordered subspace of a normally quasi-ordered space is also a normally quasi-ordered space, by the exactly same argument as that in Proposition 2.7.

4. Reflective subcategories between $\mathbf{Rqos}(\mathbf{Nqos}, \text{resp.})$ and $\mathbf{Rpos}(\mathbf{Npos}, \text{resp.})$

The category $\mathbf{T_0Rqos}(\mathbf{T_1Rqos}, \mathbf{T_2Rqos}, \text{resp.})$ denotes the full subcategory of \mathbf{Rqos} consisting of those objects (X, τ, \leq) such that (X, τ) is a T_0 -space (T_1 -space, T_2 -space, *resp.*). Similarly we define the full subcategories $\mathbf{T_0Nqos}$, $\mathbf{T_1Nqos}$ and $\mathbf{T_2Nqos}$ of \mathbf{Nqos} .

THEOREM 4.1. *The category $\mathbf{T_0Rqos}(\mathbf{T_0Nqos}, \text{resp.})$ is an epi-reflective subcategory of the category $\mathbf{Rqos}(\mathbf{Nqos}, \text{resp.})$. Moreover, $\mathbf{T_0Rqos}(\mathbf{T_0Nqos}, \text{resp.})$ is initially dense in $\mathbf{Rqos}(\mathbf{Nqos}, \text{resp.})$.*

Proof. For any $(X, \tau, \leq) \in \mathbf{Rqos}$, the $\mathbf{T_0Rqos}$ -reflection of (X, τ, \leq) can be constructed as follows:

Let $E : \mathbf{T_0Rqos} \hookrightarrow \mathbf{Rqos}$ be the embedding functor. For any $(X, \tau, \leq) \in \mathbf{Rqos}$, we define a relation $\mathcal{R} = \{(x, y) \in X \times X \mid \mathcal{N}(x) = \mathcal{N}(y)\}$, where $\mathcal{N}(x)$ is the neighborhood filter of x ($x \in X$). Then \mathcal{R} is an equivalence relation. On the quotient set X/\mathcal{R} , let $\tau_{\mathcal{R}}$ be the quotient topology on X/\mathcal{R} , i.e., the final topology with respect to the quotient map $q : X \rightarrow X/\mathcal{R}$ defined by $q(x) = [x]$, and the graph $G_{\leq_{\mathcal{R}}} = \{([x], [y]) \in X/\mathcal{R} \times X/\mathcal{R} \mid \text{there are } a, b \in X \text{ such that } a \leq b, q(a) = [x] \text{ and } q(b) = [y]\}$. Clearly $\leq_{\mathcal{R}}$ is a quasi-order and $(X/\mathcal{R}, \tau_{\mathcal{R}})$ is a T_0 -space. Moreover, since $q : (X, \tau, \leq) \rightarrow (X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}})$ is an onto open final map and every open set in (X, τ, \leq) is saturated with respect to $\tau_{\mathcal{R}}$, $(X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}}) \in \mathbf{T_0Rqos}$.

For any $(Y, \tau', \leq') \in \mathbf{T_0Rqos}$ and $f : (X, \tau, \leq) \rightarrow E((Y, \tau', \leq'))$ in \mathbf{Rqos} , we have the relation $\mathcal{R} \subseteq \ker(f)$. By the property of topological quasi-ordered spaces(see [10]), there is a unique continuous isotone $\bar{f} : (X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}}) \rightarrow (Y, \tau', \leq')$ defined by $\bar{f}([x]) = f(x)$ with $E(\bar{f}) \circ q = f$. Thus $(q, (X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}}))$ is the $\mathbf{T_0Rqos}$ -reflection of (X, τ, \leq) .

For the second statement, we note that the graph $G_{\leq} = (q \times q)^{-1}(G_{\leq_{\mathcal{R}}})$, for \leq is a continuous order on (X, τ, \leq) and hence \leq is the initial order for q . Also the topology τ on X is initial for q , because for any open U

in X , $U = q^{-1}(q(U))$ and hence the reflection q is initial. Thus $\mathbf{T}_0\mathbf{Rqos}$ is initially dense in \mathbf{Rqos} .

With the similar process we have the result for $\mathbf{T}_0\mathbf{Nqos}$ and \mathbf{Nqos} . □

We define the following full subcategories of \mathbf{Rqos} :

DEFINITION 4.2. (1) $\mathbf{R}_0\mathbf{Rqos}$ consists of those objects X in \mathbf{Rqos} which satisfy the condition: for any $x, y \in X$, if there is an open neighborhood U of x with $y \notin U$, then there is an open neighborhood V of y with $x \notin V$.

(2) $\mathbf{R}_1\mathbf{Rqos}$ consists of those objects X in \mathbf{Rqos} which satisfy the condition: for any $x, y \in X$, if there is an open neighborhood U of x with $y \notin U$, then there are open neighborhoods U' and V' of x and y , respectively, such that $U' \cap V' = \emptyset$.

(3) $\mathbf{T}_0^{-1}\mathbf{Rqos}$ consists of those objects X in \mathbf{Rqos} which satisfy the condition: for any $x, y \in X$, $x \leq y$ and $y \leq x$ if and only if $\mathcal{N}(x) = \mathcal{N}(y)$.

From the above definitions, we have the following inclusion between them immediately.

REMARK 4.3. (1) $\mathbf{Rpos} \subset \mathbf{T}_2\mathbf{Rqos} \subset \mathbf{T}_1\mathbf{Rqos} \subset \mathbf{T}_0\mathbf{Rqos} \subset \mathbf{Rqos}$
 (2) $\mathbf{Rpos} \subset \mathbf{T}_0^{-1}\mathbf{Rqos} \subset \mathbf{R}_1\mathbf{Rqos} \subset \mathbf{R}_0\mathbf{Rqos} \subset \mathbf{Rqos}$

PROPOSITION 4.4. (1) $\mathbf{R}_0\mathbf{Rqos} = \{(X, \tau, \leq) \mid \mathbf{T}_0\mathbf{Rqos}\text{-reflection of } (X, \tau, \leq) \text{ is a } T_1\text{-space}\}$

(2) $\mathbf{R}_1\mathbf{Rqos} = \{(X, \tau, \leq) \mid \mathbf{T}_0\mathbf{Rqos}\text{-reflection of } (X, \tau, \leq) \text{ is a } T_2\text{-space}\}$

(3) $\mathbf{T}_0^{-1}\mathbf{Rqos} = \{(X, \tau, \leq) \mid \mathbf{T}_0\mathbf{Rqos}\text{-reflection of } (X, \tau, \leq) \text{ is a partially ordered space}\}$

Proof. (1) and (3) are similar to the proof of Propositions 2.14 and 2.4, respectively, in [9].

(2) For any $(X, \tau, \leq) \in \mathbf{R}_1\mathbf{Rqos}$, it is enough to show that the $\mathbf{T}_0\mathbf{Rqos}$ -reflection $(X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}})$ is a T_2 -space. Suppose that $[x] \neq [y]$ in $(X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}})$. Then $\mathcal{N}(x) \neq \mathcal{N}(y)$ so that we may assume that there is an open neighborhood U of x with $y \notin U$. Since $(X, \tau, \leq) \in \mathbf{R}_1\mathbf{Rqos}$, there are open neighborhoods V and W of x and y , respectively, such that $V \cap W = \emptyset$. So $q(V)$ and $q(W)$ are disjoint open neighborhoods of $[x]$ and $[y]$, respectively. Hence $(X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}})$ is a T_2 -space.

For the converse, take any (X, τ, \leq) in the right hand side. Suppose that for x, y in X , there is an open neighborhood U of x with $y \notin U$. Then $\mathcal{N}(x) \neq \mathcal{N}(y)$, and hence $[x] \neq [y]$. Since $(X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}})$ is a

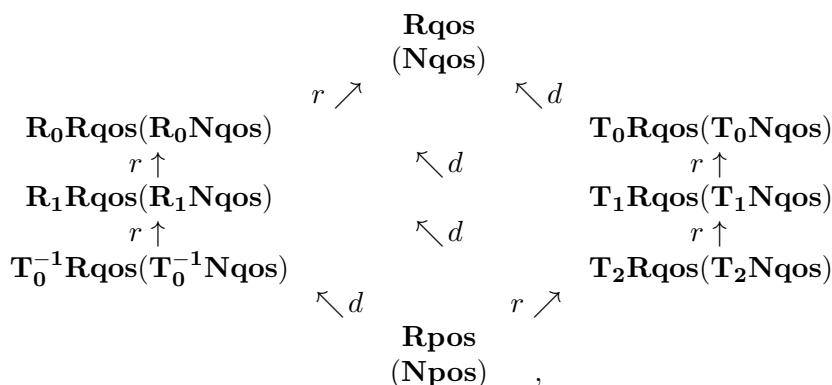
T_2 -space, there are disjoint open neighborhoods V and W of $[x]$ and $[y]$, respectively. So $q^{-1}(V)$ and $q^{-1}(W)$ are disjoint open neighborhoods of x and y , respectively. Hence $(X, \tau, \leq) \in \mathbf{R}_1\mathbf{Rqos}$. \square

COROLLARY 4.5. $\mathbf{T}_1\mathbf{Rqos}$, $\mathbf{T}_2\mathbf{Rqos}$ and \mathbf{Rpos} are epireflective subcategories and initially dense in $\mathbf{R}_0\mathbf{Rqos}$, $\mathbf{R}_1\mathbf{Rqos}$ and $\mathbf{T}_0^{-1}\mathbf{Rqos}$, respectively.

Using the fact that \mathbf{Rpos} is finally dense in \mathbf{Wqos} , we have $\mathbf{T}_0\mathbf{Rqos}$, $\mathbf{T}_1\mathbf{Rqos}$, $\mathbf{T}_2\mathbf{Rqos}$ and \mathbf{Rpos} are finally dense in \mathbf{Rqos} , $\mathbf{R}_0\mathbf{Rqos}$, $\mathbf{R}_1\mathbf{Rqos}$ and $\mathbf{T}_0^{-1}\mathbf{Rqos}$, respectively.

Replacing \mathbf{Rqos} by \mathbf{Nqos} in the above Definition 4.2 through Corollary 4.5, we have the analogous results, namely $\mathbf{T}_0\mathbf{Nqos}$, $\mathbf{T}_1\mathbf{Nqos}$, $\mathbf{T}_2\mathbf{Nqos}$ and \mathbf{Npos} are epireflective subcategories, initially dense and finally dense in \mathbf{Nqos} , $\mathbf{R}_0\mathbf{Nqos}$, $\mathbf{R}_1\mathbf{Nqos}$ and $\mathbf{T}_0^{-1}\mathbf{Nqos}$, respectively.

Collecting the above facts, we have the following diagram of reflective subcategories between the category $\mathbf{Rqos}(\mathbf{Nqos}, \text{ resp.})$ and $\mathbf{Rpos}(\mathbf{Npos}, \text{ resp.})$:



where d is the inclusion functor with initial and final dense reflection and r is the inclusion functor with epireflection.

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