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REMARKS ON SEPARATION AXIOMS ON GENERALIZED TOPOLOGICAL SPACES

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Abstract. We introduce to the notions of GT_1, GT_2, G-regular and G-normal on a GTS. And we investigate characterizations for such notions and relationships among GT_1, GT_2, GT_3 and G_4.

1. Introduction

In [1], Császár introduced the notions of generalized neighborhood systems and generalized topological spaces and the notion of continuous functions on generalized neighborhood systems and generalized topological spaces. He introduced the notion of product [5] of generalized topologies and investigated some properties of the product of generalized topologies. In this paper, we introduce to the notions of GT_1, GT_2,G-regular and G-normal on a GTS. We investigate properties for such notions, in particular, the GTS $(\prod_{k\in J} X_k, \mu)$ with the product $\mu = \mathbf{P}_{k\in J} g_k$ is GT_2 if and only if each GTS (X_k, g_k) is GT_2. And we investigate relationships among GT_1, GT_2, GT_3 and G_4.

2. Preliminaries

We recall some notions and notations defined in [1]. Let X be a nonempty set and g be a collection of subsets of X. Then g is called a *generalized topology* (briefly GT) on X iff $\emptyset \in g$ and $M_i \in g$ for $i \in I \neq \emptyset$ implies $\cup_{i\in I} M_i \in g$. We call the pair (X, g) a *generalized topological space* (briefly GTS) on X. The elements of g are called *g-open* sets and the complements are called *g-closed* sets. Set $gO(X) = \{U \subseteq X : U \in g\}$ and $gO(x) = \{U \in g : x \in U\}$. If (X, g) is a GTS and $A \subseteq X$, then the

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interior of A (denoted by $i_g(A)$) is the union of all $G \subseteq A$, $G \in g$, and the *closure* of A (denoted by $c_g(A)$) is the intersection of all g-closed sets containing A.

Let g and $g^{'}$ be generalized topologies on X and Y, respectively. Then a function $f : (X, g) \to (Y, g')$ is said to be

(1) $(g, g^{'})$-*continuous* [1] if $G^{'} \in g^{'}$ implies that $f^{-1}(G^{'}) \in g$;

(2) $(g, g^{'})$-*open* [6] if $G \in g$ implies that $f(G) \in g'$.

LEMMA 2.1. ([4]) *Let g be a GT on X and $A \subseteq X$. Then $x \in c_g(A)$ if and only if $V \cap A \neq \emptyset$ for every $V \in gO(x)$.*

Let $exp(X)$ denote the power set of X and let $\mathcal{B} \subseteq exp(X)$ satisfy $\emptyset \in \mathcal{B}$. Then all unions of some elements of $\mathcal{B}$ constitute a GT $\mu(\mathcal{B})$, and $\mathcal{B}$ is said to be a *base* [2] for $\mu(\mathcal{B})$. Let g be a GT on a nonempty set X, in general, let $\mathcal{M}_g$ denote the union of all elements of g.

Let J be an index set and μ_k a GT on X_k for $k \in J$. Let $\prod_{k\in J} X_k$ the Cartesian product of the sets X_k. We denote by p_k the projection $p_k : \prod_{k\in J} X_k \to X_k$. Let us consider all sets of the form $\prod_{k\in J} M_k$ where $M_k \in \mu_k$ and, with the exception of a finite number of indices k, $M_k = \mathcal{M}_{\mu_k}$. We denote by $\mathcal{B}$ the collection of all these sets. Since $\emptyset \in \mathcal{B}$, $\mathcal{B}$ is a base for a GT $\mu = \mu(\mathcal{B})$. We call $\mu = \mu(\mathcal{B})$ the *product* [4] of the GT's μ_k and denote it by $\mathbf{P}_{k\in J}\mu_k$.

THEOREM 2.2. ([5]) *Let μ_k a GT on X_k for $k \in J$ and the product $\mu = \mathbf{P}_{k\in J}\mu_k$ a GT on $\prod_{k\in J} X_k$. Then the projection $p_k : (\prod_{k\in J} X_k, \mu) \to (X_k, \mu_k)$ is (μ, μ_k)-open.*

In general, the projection $p_k : (\prod_{k\in J} X_k, \mu) \to (X_k, \mu_k)$ is not (μ, μ_k)-continuous (See Example 2.5 in [5]).

3. GT_1, GT_2, G-regular, G-normal

DEFINITION 3.1. Let (X, g) be a GTS. Then X is called a *relative GT_1-space* (simply, *GT_1-space*) if for $x_1, x_2 \in \mathcal{M}_g$ with $x_1 \neq x_2$, there exist $U, V \in g$ such that $x_1 \in U$, $x_2 \notin U$ and $x_2 \in V$, $x_1 \notin V$.

THEOREM 3.2. *A GTS (X, g) is a GT_1-space if and only if for each $x \in \mathcal{M}_g$, $\{x\} \cup (X - \mathcal{M}_g)$ is g-closed.*

Proof. Assume that X is GT_1. For each $z \in \mathcal{M}_g - \{x\}$, there exists a g-open U_z such that $z \in U_z$, $x \notin U_z$. This implies that $\mathcal{M}_g - \{x\}$ is g-open. Hence $(X - \mathcal{M}_g) \cup \{x\}$ is g-closed.

For the converse, let $x_1, x_2 \in \mathcal{M}_g$ with $x_1 \neq x_2$. Then by hypothesis, $(X - \mathcal{M}_g) \cup \{x_1\}$ and $(X - \mathcal{M}_g) \cup \{x_2\}$ are g-closed. Set $U = \mathcal{M}_g - \{x_2\}$ and $V = \mathcal{M}_g - \{x_1\}$. Then U, V are g-open sets and $x_1 \in U$, $x_2 \notin U$ and $x_2 \in V$, $x_1 \notin V$. Hence X is a GT_1-space. □

Definition 3.3. Let (X, g) be a GTS. Then X is called a *relative GT_2-space* (simply, *GT_2-space*) if for $x_1, x_2 \in \mathcal{M}_g$ with $x_1 \neq x_2$, there exist $U, V \in g$ such that $x_1 \in U$, $x_2 \in V$ and $U \cap V = \emptyset$.

From definitions of GT_1 and GT_2, it is obvious that every GT_2-space is GT_1 but the converse may not be true as in the next example.

Example 3.4. *Let R be the set of all real numbers and let $\mathcal{B} = \{(n, n+2), (n+1, n+3) : n \in Z\}$. Consider a generalized topology $g = \{\cup S : S \subseteq \mathcal{B}\}$ on R. Then the GTS (R, g) is GT_1 but not GT_2.*

Lemma 3.5. *Let (X, g) be a GTS. Every g-closed set includes $X - \mathcal{M}_g$.*

Proof. For every g-open set U in X, $U \subseteq \mathcal{M}_g$, so that $X - \mathcal{M}_g \subseteq U^c$. □

Theorem 3.6. *Let (X, g) be a GTS. Then the following properties are equivalent:*

(1) X is GT_2.

(2) Let $x \in \mathcal{M}_g$. For each $z \in \mathcal{M}_g$ with $z \neq x$, there is a g-open set U containing x such that $z \notin c_g(U)$.

(3) For $x \in \mathcal{M}_g$, $\cap\{c_g(U) : U \in g$ and $x \in U\} = \{x\} \cup (X - \mathcal{M}_g)$.

(4) The set $\Delta \cup (\mathcal{M}_g \times \mathcal{M}_g)^c$ is g-closed in $X \times X$, where the diagonal $\Delta = \{(x, x) : x \in X\}$.

Proof. (1) ⇒ (2) For $x \in \mathcal{M}_g$, let $z \in \mathcal{M}_g$ with $z \neq x$. Then there exist disjoint g-open sets U and V containing x and z, respectively. From Lemma 2.1, $z \notin c_g(U)$.

(2) ⇒ (3) For $x \in \mathcal{M}_g$, let $z \in \mathcal{M}_g$ with $z \neq x$. Then by (2), there is a g-open set U containing x such that $z \notin c_g(U)$, so by Lemma 3.5, $z \notin \cap\{c_g(U) : U \in g \text{ and } x \in U\} \supseteq \cup(X - \mathcal{M}_g)$. Thus we find that $\cap\{c_g(U) : U \in g \text{ and } x \in U\} = \{x\} \cup (X - \mathcal{M}_g)$.

(3) ⇒ (4) We show that $X \times X - (\Delta \cup (\mathcal{M}_g \times \mathcal{M}_g)^c)$ is g-open. For the proof, let (x, z) be any element in $X \times X - (\Delta \cup (\mathcal{M}_g \times \mathcal{M}_g)^c)$. Then $x, z \in \mathcal{M}_g$ and $z \neq x$. Since $z \notin \cap\{c_g(U) : U \in g \text{ and } x \in U\} = \{x\} \cup (X - \mathcal{M}_g)$, there exists some $U \in g$ such that $x \in U$ and $z \notin c_g(U)$. Since $U \cap (X - c_g(U)) = \emptyset$ and $X - c_g(U)$ is a g-open set containing z, $U \times (X - c_g(U))$ is a g-open set containing (x, z) such that

$U \times (X - c_g(U)) \cap (\Delta \cup (\mathcal{M}_g \times \mathcal{M}_g)^c) = \emptyset$. Thus $(x, z) \in U \times (X - c_g(U)) \subseteq X \times X - (\Delta \cup (\mathcal{M}_g \times \mathcal{M}_g)^c)$. Hence this implies $X \times X - (\Delta \cup (\mathcal{M}_g \times \mathcal{M}_g)^c)$ is g-open in $X \times X$

(4) $\Rightarrow$ (1) Let $x, z \in \mathcal{M}_g$ with $x \neq z$. Then $(x, z) \notin \Delta \cup (\mathcal{M}_g \times \mathcal{M}_g)^c$. Since $\Delta \cup (\mathcal{M}_g \times \mathcal{M}_g)^c$ is g-closed, by Lemma 2.1, there exists a g-open set $U \times V$ containing the point (x, z) such that $(U \times V) \cap (\Delta \cup (\mathcal{M}_g \times \mathcal{M}_g)^c) = \emptyset$. Hence we can say that there exist $U, V \in g$ such that $x \in U$, $z \in V$ and $U \cap V = \emptyset$. □

LEMMA 3.7. *Let (X, g) and (Y, g') be GTS's. If $f : (X, g) \to (Y, g')$ is (g, g')-open, then $f(\mathcal{M}_g) \subseteq \mathcal{M}_{g'}$.*

Proof. Since $\mathcal{M}_g \in g$, $f(\mathcal{M}_g) \in g'$ and so $f(\mathcal{M}_g) \subseteq \mathcal{M}_{g'}$. □

THEOREM 3.8. *Let $f : (X, g) \to (Y, g')$ be an injective, (g, g')-open and (g, g')-continuous function on GTS's (X, g) and (Y, g'). If Y is GT_2, then X is GT_2.*

Proof. Let $x_1, x_2 \in \mathcal{M}_g$ with $x_1 \neq x_2$. Then $f(x_1) \neq f(x_2)$ and from Lemma 3.7, we have $f(x_1), f(x_2) \in \mathcal{M}_{g'}$. Since Y is GT_2, there exist $U', V' \in g'$ such that $f(x_1) \in U'$, $f(x_2) \in V'$ and $U' \cap V' = \emptyset$. This implies $f^{-1}(U'), f^{-1}(V') \in g$, $x_1 \in f^{-1}(U'), x_2 \in f^{-1}(V')$ and $f^{-1}(U') \cap f^{-1}(V') = \emptyset$. Hence X is GT_2. □

THEOREM 3.9. *Let $f : (X, g) \to (Y, g')$ be an injective (g, g')-open function on GTS's (X, g) and (Y, g'). If X is GT_2 and $f(\mathcal{M}_g) = \mathcal{M}_{g'}$, then Y is GT_2.*

Proof. Let $y_1, y_2 \in \mathcal{M}_{g'}$ with $y_1 \neq y_2$. Then from $f(\mathcal{M}_g) = \mathcal{M}_{g'}$, there exist $x_1, x_2 \in \mathcal{M}_g$ such that $f(x_1) = y_1, f(x_2) = y_2$. Since X is GT_2, there exist $U, V \in g$ such that $x_1 \in U$, $x_2 \in V$ and $U \cap V = \emptyset$. Thus $f(U), f(V) \in g'$, $y_1 \in f(U)$ and $y_2 \in f(V)$. And from injectivity of f, $f(U \cap V) = f(U) \cap f(V) = \emptyset$ and so Y is GT_2. □

THEOREM 3.10. *The GTS $(\prod_{k \in J} X_k, \mu)$ with the product $\mu = \mathbf{P}_{k \in J} g_k$ is GT_2 if and only if each GTS (X_k, g_k) is GT_2.*

Proof. Assume that each GTS (X_k, g_k) is GT_2 and that $x = (x_k)$, $y = (y_k) \in \mathcal{M}_\mu$ with $x \neq y$. Then $x_k \neq y_k$ for some $k \in J$, so there are disjoint g_k-open sets $U(x_k), U(y_k) \in g_k$.

Set

$U = \prod \{U_\beta \in g_\beta : \text{if } \beta = k, U_\beta = U(x_k);\ \text{otherwise, } U_\beta = \mathcal{M}_{g_\beta}\}$;

$V = \prod \{V_\beta \in g_\beta : \text{if } \beta = k, V_\beta = V(y_k);\ \text{otherwise, } V_\beta = \mathcal{M}_{g_\beta}\}$.

Then $U, V \in \mu$, $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

For the converse, let $(\prod_{k\in J} X_k, \mu)$ be GT_2 and for $k \in J$, let $x_k, y_k \in \mathcal{M}_{g_k}$ with $x_k \neq y_k$. Consider two points $x = (p_\beta), y = (q_\beta) \in \mathcal{M}_\mu$ satisfying the following: If $\beta = k$, $p_\beta = x_k$ and $q_\beta = y_k$; otherwise, $p_\beta = q_\beta = z_\beta$ for some $z_\beta \in \mathcal{M}_{g_\beta}$. Since $(\prod_{k\in J} X_k, \mu)$ is GT_2, there are disjoint μ-open sets U and V such that $x \in U, y \in V$. Then $x_k \in p_k(U), y_k \in p_k(V)$, $p_k(U) \cap p_k(V) = \emptyset$ and from Theorem 2.2, $p_k(U), p_k(V) \in g_k$. Hence (X_k, g_k) is GT_2. □

Definition 3.11. ([7]) Let (X, g) be a GTS. Then X is said to be *relative G-regular* (simply, *G-regular*) if for $x \in \mathcal{M}_g$ and g-closed set F with $x \notin F$, there exist $U, V \in g$ such that $x \in U$, $F \cap \mathcal{M}_g \subseteq V$ and $U \cap V = \emptyset$. And if X is GT_1 and *G-regular*, then it is said to be GT_3.

Example 3.12. *Let* $X = \{a, b, c, d, e\}$ *and* $g = \{\emptyset, \{a, b\}, \{c, d\}, \{b, c\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}$ *be a generalized topology on* X.

Then (X, g) *is* GT_2. *But for* $b \in \mathcal{M}_g$ *and* g*-closed set* $F = \{a, c, e\}$, *there are not disjoint* g*-sets* U *and* V *such that* $b \in U$ *and* $F \cap \mathcal{M}_g \subseteq V$, *so that* (X, g) *is not* GT_3.

Theorem 3.13. ([7]) *Let* (X, g) *be a GTS. Then* X *is G-regular if and only if for* $x \in \mathcal{M}_g$ *and a* g*-open set* U *containing* x, *there is a* g*-open set* V *containing* x *such that* $x \in V \subseteq c_g(V) \cap \mathcal{M}_g \subseteq U$.

Proof. Assume that X is G-regular. Then for $x \in \mathcal{M}_g$ and a g-open set U containing x, x and the g-closed set U^c have disjoint g-open sets V, W with $x \in V$, $U^c \cap \mathcal{M}_g \subseteq W$. Since $V \subseteq W^c$ and W^c is g-closed, it follows $c_g(V) \subseteq W^c$. This implies $c_g(V) \cap (U^c \cap \mathcal{M}_g) \subseteq c_g(V) \cap W = \emptyset$, thus $c_g(V) \cap \mathcal{M}_g \subseteq U$.

For the converse, let F be any g-closed set and $x \notin F$ for $x \in \mathcal{M}_g$. Then since F^c is a g-open set containing x, by hypothesis, there is a g-open set V containing x such that $x \in V \subseteq c_g(V) \cap \mathcal{M}_g \subseteq F^c$, thus $c_g(V) \cap \mathcal{M}_g \cap F = \emptyset$, so that $\mathcal{M}_g \cap F \subseteq c_g(V)^c$. Hence X is G-regular. □

Definition 3.14. Let (X, g) be a GTS. Then X is said to be *relative G-normal* (simply, *G-normal*) if for g-closed sets F_1 and F_2 with $F_1 \cap F_2 = X - \mathcal{M}_g$, there exist $U, V \in g$ such that $F_1 \cap \mathcal{M}_g \subseteq U$, $F_2 \cap \mathcal{M}_g \subseteq V$ and $U \cap V = \emptyset$. And if X is GT_1 and *G-normal*, then it is said to be GT_4.

Recall that for nonempty set X, a GT μ is *normal* [3] iff, whenever F and F' are μ-closed sets such that $F \cap F' = \emptyset$, there exist μ-open sets G and G' satisfying $F \subseteq G$, $F' \subseteq G'$ and $G \cap G' = \emptyset$. If $\mathcal{M}_\mu = X$, *G-normality* is exactly the normality. And a G-normal space need not be G-regular, as seen by the next example.

EXAMPLE 3.15. *Let* $X = \{a, b, c, d\}$ *and a generalized topology* $g = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. *Then* $X - \mathcal{M}_g = \{d\}$ *and the* g*-closed sets are* X, $\{b, c, d\}$, $\{a, c, d\}$, $\{c, d\}$ *and* $\{d\}$.

Consider the following pairs of two g*-closed sets* F_1 *and* F_2 *satisfying* $F_1 \cap F_2 = X - \mathcal{M}_g$: X *and* $\{d\}$, $\{b, c, d\}$ *and* $\{d\}$, $\{a, c, d\}$ *and* $\{d\}$, $\{c, d\}$ *and* $\{d\}$. *Since* $\{d\} \cap \mathcal{M}_g = \emptyset$, *obviously* (X, g) *is G-normal. But* (X, g) *is not is G-regular since* $b \notin \{a, c, d\}$, *and the only superset of* $\{a, c, d\} \cap \mathcal{M}_g$ *is* $\{a, b, c\}$ *which contains* b.

THEOREM 3.16. *Let* (X, g) *be a GTS. Then* X *is G-normal if and only if for a* g*-closed setF and a* g*-open set* U *with* $F \cap \mathcal{M}_g \subseteq U$, *there is a* g*-open set* V *containing* x *such that* $F \subseteq V \subseteq c_g(V) \cap \mathcal{M}_g \subseteq U$.

Proof. It is similar to the proof of Theorem 3.13. □

Finally, the general relationship between the discussed spaces is given in the following diagram:

$$GT_4 \Rightarrow GT_3 \Rightarrow GT_2 \Rightarrow GT_1$$

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