JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **13**, No.2, December 2000

POSITIVE SOLUTIONS FOR CERTAIN ELLIPTIC SYSTEMS WITH THREE SPECIES INVOLVING SYMBIOTIC INTERACTIONS

Meejoung Kim* and Inkyung Ahn**

Abstract. Certain elliptic interacting system involving symbiotic is considered. The sufficient conditions for the existence of positive solutions of system is provided for four different types of interaction among three species by using the method of decomposing operator.

1. Introduction

Of concern is the existence of positive steady-state solutions to 3×3 elliptic interacting systems with density-dependent diffusions:

$$(1.1) \qquad \begin{cases} -\varphi_1(x,u)\Delta u = uf_1(u,v,w) & \\ -\varphi_2(x,v)\Delta v = vf_2(u,v) & \text{in } \Omega \\ -\varphi_3(x,w)\Delta w = wf_3(u,w) & \\ \kappa_1 \frac{\partial u}{\partial n} + \beta_1 u = 0 & \text{on } \partial\Omega \\ \kappa_2 \frac{\partial v}{\partial n} + \beta_2 v = 0 & \\ \kappa_3 \frac{\partial w}{\partial n} + \beta_3 w = 0. & \end{cases}$$

where $\Omega \subset \mathbf{R}^n$ is bounded with smooth boundary, $\beta_i, i = 1, 2, 3$, are positive constants and the nonlinear diffusions $\varphi_i, i = 1, 2, 3$, are strictly positive nondecreasing functions.

u, v, w may represent the densities of interacting populations arising from ecology, microbiology, immunology, etc. The functions f_i denote

Received by the editors on June 30, 2000.
1991 *Mathematics Subject Classifications* : 35J60, 35K57.
Key words and phrases: Positive solutions, Decomposing operator, Density-dependent diffusion, Nonlinear boundary conditions.

the relative growth rates of this populations. Two species are competing each other if each of their relative growth functions are decreasing in the other opposer. For predator-prey models, one of the functions involved will be increasing in the prey while the other decreasing in the predator. Two species are in cooperation if each of their relative growth rates is increasing in the other.

It is known that for 2×2 systems, the spectral properties of the linearization at marginal densities determine the positive coexistence in case of predations and competitions while it is dominated by the equilibria of the system in symbiotic interactions.

In this paper we study the coexistence of positive solutions of the system for 4 non-equivalent models involving cooperating interactions. We will show that the above principle for 2×2 systems carries over to 3×3. The methods employed is based on the method of decomposing operator.

2. Lemmas and Notations

In this section, we state some known lemmas and notations which will be useful in section 3. The proof of lemmas in this section can be found in [1] in details. Throughout this paper, we will consider problems in the space $\mathbb{X} = C(\bar{\Omega})$, where is a bounded region in $\mathbb{R}^n$ and let $r(T)$ denote the spectral radius of a linear operator T. First we define the classes $F \subset C(\bar{\Omega} \times \mathbb{R}^+)$ and $G \subset C(\bar{\Omega} \times \mathbb{R}^+)$ as follow:

DEFINITION 1. *Let $f = f(x, \xi), f \in F$ if and only if $f \in C(\bar{\Omega} \times \mathbb{R}^+)$ satisfies (F1) $f \in C^1$ in ξ, $f_\xi(x, \xi) < 0$ in $\Omega \times \mathbb{R}^+$, and for some $N \in \mathbb{R}^+$, $|f_\xi(x, \xi)| \leq N, (x, \xi) \in \Omega \times \mathbb{R}^+$.*

(F2) $f(x, \xi)$ is concave down in ξ on (x, ξ) where $f(x, \xi) < 0$.

(F3) $f(x, 0) > 0$ and $f(x, \xi) < 0$, where $(x, \xi) \in \Omega \times (c_0, +\infty)$

DEFINITION 2. *Let $\varphi = \varphi(x, \xi)$. Then $\varphi \in G$ if and only if $\varphi \in C(\bar{\Omega} \times \mathbb{R})$ which, in addition, satisfies*

(G1) $\varphi(x,\xi) \geq \delta > 0$ *for some constant* δ *and* $\xi \in \mathbb{R}^+$, $x \in \Omega$.

(G2) φ *is nondecreasing and concave down in* ξ.

LEMMA 2.1. *Let* $f \in F, \varphi \in G$. *Then the function* $f(x,u)/\varphi(x,u)$ *is decreasing in* $u > 0$.

LEMMA 2.2. *Let* $P > 0$ *be a constant and* $h \in C^\alpha(\bar{\Omega})$. *Consider equation*

$$-\varphi(x,u)\Delta u + Pu = h \text{ in } \Omega, \quad \kappa \partial u/\partial n + \beta u = 0 \text{ on } \partial\Omega, \tag{2.2}$$

where β, $\kappa \geq 0$ *is a constant. Let* $m > n$ *as in Definition 2. If* $0 \not\equiv h \geq 0$ *and* $\varphi \in G$ *then the equation (2.2) has a unique positive solution* $u \in C^{2,\alpha}(\Omega)$ *for some* $\alpha \in (0,1)$. *Moreover, if we define the solution operator* T *by* $u = Th$ *and denote it* $Th := (-\varphi(x,\cdot)\Delta \cdot + P\cdot)^{-1}h$, *then* T *is compact, continuous and monotone increasing in the positive cone* $\mathbb{K} \subset C(\bar{\Omega})^+$ *of the ordered Banach space* $C(\Omega)$.

¿From Lemma 2.2, the next two observations follow easily.

Observation 1. Let $a(x) \geq \delta_0 > 0$, and $b(x) \in L^\infty(\Omega)$. Also let P be positive constant such that $P + b(x) > 0$ for $a.e.\ x \in \Omega$. Then

(i) $\lambda_1(a(x)\Delta + b(x)) > 0 \Longleftrightarrow r[(-a(x)\Delta + P)^{-1}(P + b(x))] > 1$

(ii)$\lambda_1(a(x)\Delta + b(x)) < 0 \Longleftrightarrow r[(-a(x)\Delta + P)^{-1}(P + b(x))] < 1$

(iii) $\lambda_1(a(x)\Delta + b(x)) = 0 \Longleftrightarrow r[(-a(x)\Delta + P)^{-1}(P + b(x))] = 1$,

where λ_1 is the first eigenvalue under homogeneous Robin boundary condition.

Consider the following equation:

$$\begin{cases} -\varphi(x,u)\Delta u = uf(x,u) & \text{in } \Omega \\ \kappa \frac{\partial u}{\partial n} + \beta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where β, $\kappa \geq 0$ is a constant.

Observation 2. Suppose that $u_0 \geq v_0 \geq 0$ are upper and lower solutions of (2.3), respectively. Assume that $\varphi \in G$. Then there exists a maximal solution u of (2.3) such that $v_0 \leq u \leq u_0$.

We shall linearize the equation (2.3) at $u = 0$. Use Lemma 2.2 to define the solution operator S in $C(\bar{\Omega})$ by $Su = \bar{u}$, where $\bar{u}$ is the unique solution of

$$\begin{cases} -\varphi(x,\bar{u})\Delta\bar{u} + P\bar{u} = uf(x,u) + Pu & \text{in } \Omega \\ \kappa\frac{\partial\bar{u}}{\partial n} + \beta\bar{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

Also we define the solution operator S_L of linearization by $S_L w = v$, where v is the unique solution of

$$\begin{cases} -\varphi(x,0)\Delta v + Pv = vf(x,0) + Pw & \text{in } \Omega \\ \kappa\frac{\partial v}{\partial n} + \beta u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then we have the following lemma.

Lemma 2.3. *Suppose that $\varphi \in G$ and f satisfies (F1). The operator S is Frechet differentiable at $u = 0$, and $S'(0) = S_L$.*

We define the ordered interval

$$[[u_1, u_2]] := \{u \in C(\bar{\Omega}) : u_1 \le u \le u_2 \text{ for } u_1, u_2 \in C(\bar{\Omega}\}.$$

Let e be the unique solution of $\Delta e = 1$ in Ω, $\kappa_1\frac{\partial e}{\partial n} + \beta e = 0$ on $\partial\Omega$. Define the ordered Banach space $C_e(\bar{\Omega})$ by $C_e(\bar{\Omega}) = \cup_{\lambda\in R^+}\lambda[[-e,e]] = \cup_{\lambda\in R^+}[[-\lambda e, \lambda e]]$ with norm $||u||_e = \inf\{\lambda > 0 : -\lambda e \le u \le \lambda e\}$. Let $\mathbb{K}_e := C_e(\bar{\Omega})^+$.

Lemma 2.4. *Let $f \in F$ and $\varphi \in G$. If $\lambda_1(\varphi(x,0)\Delta + f(x,0)) > 0$, then the equation (2.3) has a unique positive solution in $C^{2,\alpha}(\Omega)$. Moreover, if $\lambda_1(\varphi(x,0)\Delta + f(x,0)) \le 0$, then $u \equiv 0$ is the only nonnegative solution of (2.3).*

According to Lemma 2.4, the equation (2.3) has a unique positive solution. We denote it by $u_{\varphi,f}$. Let u_{φ_n,f_n} be the unique positive solution of

$$-\varphi_n(x,u)\Delta u = uf_n(x,u) \text{ in } \Omega, \kappa\frac{\partial u}{\partial n} + \beta u = 0 \text{ on } \partial\Omega.$$

LEMMA 2.5. *Assume $f \in F$ and $\varphi \in G$*
(i) $(\varphi, f) \mapsto u_{\varphi,f}$ is a continuous mapping of $G \times F \to C^{1,\alpha}(\Omega \times \mathbb{R}^+)$ for some $\alpha \in (0,1)$.
(ii) If $\frac{f_1}{\varphi_1} \geq \frac{f_2}{\varphi_2} \not\equiv \frac{f_1}{\varphi_1}, x \in \Omega$, then $u_{\varphi_1,f_1} > u_{\varphi_2,f_2}$ or $u_{\varphi_1,f_1} \equiv u_{\varphi_2,f_2} \equiv 0$.

Let $\mathbb{X}$ be a Banach space and let F be a strongly positive nonlinear compact operator $\mathbb{X}$ such that $F(0) = 0$.

LEMMA 2.6. *Assume $F'(0)$ exists with $r(F'(0)) > 1$. If for all $\mu \in (0,1]$ the solution to the equation $u = \mu F u$ has a priori bound, then F has a positive fixed point u such that $Fu = u$ in the positive cone $\mathbb{K}$ of $\mathbb{X}$.*

Proof. See the proof Theorem 13.2 in [2]. □

3. Existence Theorem

In this section, we study the existence of positive solutions of system involving cooperating interactions. Let $\lambda_{1,\beta_i}(A)$ denote the positive principal eigenvalues of a suitable differential operator A under the boundary conditions $\kappa_i \frac{\partial \cdot}{\partial n} + \beta_i \cdot = 0$, $i = 1,2,3$. First we impose the following hypotheses on the functions $f_i, \varphi_i, i = 1,2,3$:
(S1) $\varphi_1, \varphi_2, \varphi_3 \in C(\bar{\Omega} \times \mathbb{R}^+), \varphi_1(x,\cdot), \varphi_2(x,\cdot), \varphi_3(x,\cdot) \in G$.
(S2) $f_1(u,v,w), f_2(u,v), f_3(u,w)$ are of C^1-functions in u, v, w and their partial derivatives are uniformly bounded. Moreover, assume that $D_u f_1 < 0, D_v f_2 < 0, D_w f_3 < 0$, for $u, v, w > 0$. For fixed $u, v, w \in C(\bar{\Omega})^+$, $f_1(\cdot, v, w), f_2(u, \cdot), f_3(u, \cdot) \in F$
(S3) There is a constant $c_3 > 0$ such that $f_3(0, w) \leq 0$ on $w \in [c_3, \infty)$ and there is a constant $c_2 = c_2(M) > 0$ such that $f_2(0, v) < 0$ on $w \in (0, M]$ when $v > c_2$ and there is a constant $c_1 = c_1(M, N) > 0$ such that $f_1(u, v, w) <$ on $(u, w) \in (0, M] \times (0, N]$ when $u > c_1$.
(S4) $\lambda_{1,\beta_1}(\varphi_1(x,0)\Delta + f_1(0,0,0)) > 0$, $\lambda_{1,\beta_2}(\varphi_2(x,0)\Delta + f_2(0,0)) > 0$, $\lambda_{1,\beta_3}(\varphi_3(x,0)\Delta + f_3(0,0)) > 0$.

The assumption (S4) implies that $u_0, v_0, w_0 > 0$. This means that the three species can survive by themselves in the absence of the other species. We will use the following notations for simplicity.
$u \rightarrow v$ if u preys on v, i.e., $D_v f_1 \geq 0, D_u f_2 \leq 0$.
$u \leftrightarrow v$ if u, v compete, i.e., $D_v f_1 \leq 0, D_u f_2 \leq 0$.
$u \cdots v$ if u, v cooperate, i.e., $D_v f_1 \geq 0, D_u f_2 \geq 0$.
For example, $u \rightarrow v \leftrightarrow w$ represents a model in which u preys on v and v, w compete.

¿From Lemma 2.4, we can define the operator $S, T : C(\bar{\Omega}) \rightarrow C(\bar{\Omega})$ as follows. For $u \in C(\bar{\Omega}), Su$ is the unique solution of the equation

$$-\varphi_2(x, v)\Delta v = v f_2(u, v), \quad \kappa_2 \frac{\partial v}{\partial n} + \beta_2 v = 0 \text{ on } \partial\Omega \tag{3.4}$$

if $\lambda_{1,\beta_2}(\varphi_2(x, \cdot)\Delta + f_2(u, 0)) > 0$ and $Su \equiv 0$ otherwise. Tu is determined similarly from the equation

$$-\varphi_3(x, w)\Delta w = w f_3(u, w), \quad \kappa_3 \frac{\partial w}{\partial n} + \beta_3 w = 0 \text{ on } \partial\Omega \tag{3.5}$$

provided that $\lambda_{1,\beta_3}(\varphi_3(\cdot)\Delta + f_3(x, 0, w)) > 0$ and $Tu \equiv 0$ otherwise.

Let $U_1 = \{u \in C(\bar{\Omega}) : Su > 0\}$, $U_2 = \{u \in C(\bar{\Omega}) : Tu > 0\}$. By Lemma 2.5, it is easy to see that the operators S, T are continuous operators and that if u, w cooperator or u is a prey of w , then T is a strictly increasing operator in the sense that $u_1 \not\equiv u_2 \geq u_1$ and $u_2 \in U_2$ implies $Tu_2 > Tu_1$ and that if u, w compete or u preys on w, then T is a strictly decreasing operator in sense that $u_1 \not\equiv u_2 \geq u_1$ and $u_1 \in U_2$ implies $Tu_2 < Tu_1$. Similarly conclusion is true for S with respect to U_1. Define the operator $\mathbb{A} : C(\bar{\Omega}) \rightarrow C(\bar{\Omega})$ by the equation

$$\begin{cases} -\varphi_1(x, z)\Delta z + Pz = u f_1(u, Su, Tu) + Pu \\ \kappa_1 \frac{\partial z}{\partial n} + \beta_1 z = 0 \text{ on } \partial\Omega. \end{cases}$$

Denote it by

$$z = \mathbb{A}u = (-\varphi_1(x, \cdot)\Delta \cdot + P \cdot)^{-1}[u f_1(u, Su, Tu) + Pu]$$

where $P > 0$ is a constant. Then $\bar{u}$ is a fixed point of $\mathbb{A}$ in $\mathbb{K}$ iff $(\bar{u}, S\bar{u}, T\bar{u})$ is a positive solution of the system (1.1). Also it is not hard to see that in our four different models, there is an a-priori bound for the positive fixed point of A. For $u_i, v_i, w_i, i = 1, 2$, denote
$<(u_1, v_1, w_1), (u_2, v_2, w_2)>:= \{(u, v, w) \in C(\bar{\Omega}) \oplus C(\bar{\Omega}) \oplus C(\bar{\Omega}) :$
$(u_1, v_1, w_1) < (u, v, w) < (u_2, v_2, w_2) \ \in \ \Omega\}$.

THEOREM 3.1. *Assume that (S1)-(S4) for* $f_i, \varphi_i, i = 1, 2, 3$.

Case 1. $v \cdots u \cdots w$.

If the system (1.1) has a positive equilibrium (C_1, C_2, C_3), *then it has a positive solution in* $<(u_0, v_0, w_0), (C_1, C_2, C_3)>$.

Case 2. $v \cdots u \leftarrow w$.

Assume that $\lambda_{1,\beta_1}(\varphi_1(x, \cdot)\Delta + f_1(0, v_0, w_0)) > 0$ *and the subsystem* $-\varphi_1(x, u)\Delta u = uf_1(u, v, 0)$, $-\varphi_2(x, v)\Delta v = vf_2(u, v)$ *has a positive equilibrium* (C_1, C_2), *i.e.,* $f_1(C_1, C_2, 0) = f_2(C_1, C_2) = 0$. *Then system (1.1) has a positive solution in* $<(0, v_0, w_0), (C_1, C_2, \infty)>$.

Case 3. $v \cdots u \leftrightarrow w$.

Assume $-\varphi_1(x, u)\Delta u = uf_1(u, v, 0)$, $-\varphi_2(x, v)\Delta v = vf_2(u, v)$ *has a positive equilibrium* (C_1, C_2) *and* $\lambda_{1,\beta_1}(\varphi_1(x, \cdot)\Delta + f_1(0, v_0, w_0)) > 0$, $\lambda_{1,\beta_3}(\varphi_3(x, \cdot)\Delta + f_3(C_1, 0)) > 0$. *Then system (1.1) has a positive solution in* $<(0, v_0, 0), (C_1, C_2, w_0)>$.

Case 4. $v \cdots u \rightarrow w$.

Let $C = \max_{x\in\bar{\Omega}} w_0(x)$. *Assume* $-\varphi_1(x, u)\Delta u = uf_1(u, v, C)$, $-\varphi_2(x, v)\Delta v = vf_2(u, v)$ *has a positive equilibrium* (C_1, C_2) *and* $\lambda_{1,\beta_3}(\varphi_3(x, \cdot)\Delta + f_3(C_1, 0)) > 0$. *Then system (1.1) has a positive solution in* $<(u_0, v_0, 0), (C_1, C_2, w_0)>$.

Proof. Case 1. In this case $D_v f_1, D_w f_1, D_u f_2, D_u f_3 > 0$. Note that S and T are both increasing. Then $-\varphi_1(x, u_0)\Delta u_0 = u_0 f_1(u_0, 0, 0) < u_0 f_1(u_0, Su_0, Tu_0)$ and this implies $u_0 < \mathbb{A}u_0$. The equation

$$-\varphi_2(x, v)\Delta v = vf_2(C_1, v) \text{ in } \Omega, \ \ \kappa_2 \frac{\partial v}{\partial n} + \beta_2 v = 0 \text{ on } \partial\Omega \tag{3.6}$$

has a unique positive solution $v_1 = SC_1$. The constant C_2 is an upper solution of (3.6) and the function ϵv_1 is a lower solution to (3.6) for $\epsilon \in (0,1]$. When ϵ is small, we have $\epsilon v_1 < C_2$. Therefore by Observation 2 there exists a positive solution $v^{\sim}$ of (3.6) in $[[\epsilon v_1, C_2]]$. By the uniqueness $v^{\sim} = v_1$. This shows that $SC_1 \leq C_2$. Similarly, $TC_1 \leq C_3$. So $-\varphi_1(x, C_1)\Delta C_1 \geq C_1 f_1(C_1, SC_1, TC_1)$, thus $C_1 \geq \mathbb{A}C_1$. By the similar argument, we see that $u_0 \leq C_1$. By Corollary 6.2 [2] $\mathbb{A}$ has a positive fixed point $\bar{u}$ in $[[u_0, C_1]]$. Therefore $(\bar{u}, S\bar{u}, T\bar{u})$ is a positive solution of (1.1). Also $C_1 \not\equiv \bar{u} \leq C_1$ implies $v_0 = S0 < SC_1 \leq C_2$ and $w_0 = T0 < T\bar{u} < TC_1 \leq C_3$ by the strict monotonocity of the operators S, T. Since for all $v \in < v_0, C_2 >$ and $w \in < w_0, C_3 >$, $-\varphi_1(x, u_0)\Delta u_0 = u_0 f_1(u_0, 0, 0) < u_0 f_1(u_0, v, w)$ and $C_1 f_1(C_1, v, w) < C_1 f_1(C_1, C_2, C_3) = -\varphi_1(x, C_1)\Delta C_1$, we conclude that $u_0 < \bar{u} < C_1$. Thus $(\bar{u}, S\bar{u}, T\bar{u})$ in $< (u_0, v_0, w_0), (C_1, C_2, C_3) >$.

Case 2. In this case $D_v f_1, D_u f_2, D_u f_3 > 0, D_w f_1 < 0$. Note that S and T both are increasing. Since $\mathbb{A}'(0) = (-\varphi_1(x,\cdot)\Delta + P)^{-1}(f_1(0, v_0, w_0) + P)$ and $\lambda_1(\varphi_1(x,\cdot)\Delta + f_1(0, v_0, w_0)) > 0$, we have $r(\mathbb{A}'(0)) > 1$, by Observation 1. Note that the restriction of $\mathbb{A}$ on $C_e(\bar{\Omega})$ is strongly positive. By Krein-Rutman theorem, $r(\mathbb{A}'(0))$ is an eigenvalue of $\mathbb{A}'(0)$ with a positive eigenvector and $\mathbb{A}'(0)$ has no other eigenvalues with positive eigenvectors. Let $\mathbb{A}_\theta = \theta\mathbb{A}$ and $R = C_1$. Suppose $\mathbb{A}_\theta u = u$ with $u \in \partial B_R(0) \cap \mathbb{K}$ for some $\theta \in (0,1]$. Then $-\varphi_1(x, u/\theta)\Delta u = \theta u f_1(u, Su, Tu) + P(\theta - 1)u$. Since u, Su, Tu are in $W^{2,p}(\Omega)$ for any $p > 0$ and f is C^1, we see that $u \in W^{3,p}(\Omega)$ for any $p > 0$. By the Sobolev imbedding theorem $u \in C^2(\bar{\Omega})$. Now $u \leq C_1$ implies $Su \leq C_2$. If u attains its maximum at $x_0 \in \Omega$, let $u(x_0) = \max_{x\in\bar{\Omega}} u(x) = C_1 > 0$. Since $Tu > T0 = w_0 > 0$, we have $0 \leq -\varphi_1(x, u(x_0)/\theta)\Delta u(x_0) \leq \theta u(x_0) f(u(x_0), Su(x_0), Tu(x_0)) < \theta u(x_0) f(u(x_0), C_2, w_0(x_0)) \leq \theta C_1 f(C_1, C_2, 0) = 0$, a contradiction. If u attains its maximum only on $\partial\Omega$, let $u(x_0) = \max_{x\in\bar{\Omega}} u(x) = C_1 > 0$, for some $x_0 \in \partial\Omega$. Since $\partial\Omega$ is smooth, we can choose a ball $B \subset \Omega$

such that $f(u(x_0), Su(x_0), Tu(x_0)) < f(C_1, C_2, 0) = 0$, can be chosen so small $f(u(x), Su(x), Tu(x)) < 0$ that on $\bar{B}$. We have $-\varphi_1(x, u(x)/\theta)$ $\Delta u(x) \leq \theta u(x) f_1(u(x), Su(x), Tu(x)) < 0$ on $\bar{B}$. Applying the Hopf lemma to u in the domain B we conclude that $\frac{\partial u(x_0)}{\partial n} > 0$ with respect to $\partial\Omega$. Then $\kappa_1 \frac{\partial u(x_0)}{\partial n} + \beta_1 u(x_0) > 0$, contradicting to the boundary condition. In any case it shows that $\mathbb{A}_\theta$ has no fixed point on $\partial B_R(0) \cap \mathbb{K}$ for $\theta \in (0,1]$. Therefore $\mathbb{A}u \neq \lambda u$, for all $\lambda \in [1, \infty), u \in \partial B_R(0) \cap \mathbb{K}$. By Proposition 13.2 [2], the operator $\mathbb{A}$ has a positive fixed point $\bar{u} \in [[0, C_1]]$. So $(\bar{u}, S\bar{u}, T\bar{u})$ is a positive solution of (1.1). $T\bar{u} > T0 = w_0$. $C_1 \not\equiv \bar{u} \leq C_1$ implies $v_0 = S0 < S\bar{u} < SC_1 \leq C_2$. Using the similar argument as in the proof of Case 1, we have that $\bar{u} < C_1$.

Case 3. In this case $D_v f_1, D_u f_2 > 0, D_u f_3, D_w f_1 < 0$. Note that S is increasing and T is decreasing. Since $\lambda_{1,\beta_1}(\varphi_1(x, \cdot)\Delta + f_1(0, v_0, w_0)) > 0, r(\mathbb{A}'(0)) > 1$, we can show that there exists $\underline{u} > 0$ with $\underline{u} < C_1$ and $\mathbb{A}\underline{u} > \underline{u}$. Since $SC_1 \leq C_2$ and $TC_1 \geq 0$, we have,

$$-\Delta C_1 = 0 = \frac{C_1 f_1(C_1, C_2, 0)}{\varphi(x, C_1)} \geq \frac{C_1 f_1(C_1, SC_1, TC_1)}{\varphi(x, C_1)},$$

and this show $C_1 \geq \mathbb{A}C_1$. By Corollary 6.2 [2] $\mathbb{A}$ has a positive fixed point $\bar{u} \in [[\underline{u}, C_1]]$. Therefore $(\bar{u}, S\bar{u}, T\bar{u})$ is a positive solution of (1.1). Also $C_1 \not\equiv \bar{u} \leq C_1$ implies $0 < v_0 = S0 < S\bar{u} < SC_1 \leq C_2$ and $< TC_1 < T\bar{u} < T0 = w_0$. It follows that $\bar{u} < C_1$ by the argument used in the proof of Case 2. Thus $(\bar{u}, S\bar{u}, T\bar{u}) \in < (0, v_0, 0), (C_1, C_2, w_0) >$.

Case 4. In this case $D_v f_1, D_w f_1, D_u f_2 > 0, D_u f_3 < 0$ and so S is increasing and T is decreasing. Similarly as in Case 2, the operator has a positive fixed point $\bar{u} \in [[0, C_1]]$ and $\bar{u} \neq C_1$. Therefore $v_0 = S0 < S\bar{u} < SC_1 \leq C_2$ and $w_0 = T0 > T\bar{u} > TC_1 \geq 0$. Thus $u_0 < \bar{u} < C_1$ and $(\bar{u}, S\bar{u}, T\bar{u})$ is a positive solution of system (1.1). $\square$

References

1. I. Ahn, Positive solutions for predator-prey equations with nonlinear diffusion rates, J. Korean Math. Soc. 31, No. 4 (1994), 545-558.
2. H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev. 18 (1976), 620-709.
3. J. Blat and K.J. Brown, Bifurcation of steady-state solutions in the predator-prey and competition systems, Proc. Roy. Soc. Edinburgh, (A), 97(1984), 21-34.
4. E.N. Dancer, On positive solutions of some pairs of differential equations, Trans. Amer. Math. Soc. 284(1984), 729-743.
5. L. Li, Coexistence theorems of steady-state for predator-prey interacting systems, Trans. Amer. Math. Soc. 305 (1988), 143-166.
6. L. Li and R. Logan, Positive solutions to general elliptic competition models, J. of Diff. Integ Eqs. 4 (1991), 817-834.
7. Nela Lakos, Existence of steady-state solutions for an one predator-two prey system, SIAM J. Math. Anal., 21(1990), 647-659.
8. S. Zheng, A reaction-diffusion system of a predator-prey-mutualist model, Math. Biosci., 78(1986), 217-245.

*

Department of Mathematics
Korea University
Seoul, 136-701, Korea
E-mail: kmj@semi.korea.ac.kr

**

Department of Mathematics
Korea University
Chochiwon, Chungnam, 339-700, Korea
E-mail: ahn@gauss.korea.ac.kr