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ON THE REPRESENTATION OF THE $^*g$–ME–VECTOR IN $^*g$–MEX_n

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Abstract. An Einstein's connection which takes the form (2.23) is called a $^*g$-ME-connection and the corresponding vector is called a $^*g$-ME-vector. The $^*g$-ME-manifold is a generalized n-dimensional Riemannian manifold X_n on which the differential geometric structure is imposed by the unified field tensor $^*g^{\lambda\nu}$, satisfying certain conditions, through the $^*g$-ME-connection and we denote it by $^*g$-MEX_n. The purpose of this paper is to derive a general representation and a special representation of the $^*g$-ME-vector in $^*g$-MEX_n.

1. Introduction

Einstein [6] proposed a new unified field theory that would include both gravitation and electromagnetism. It may be characterized as a set of geometrical postulates for the space time X_4. However, the geometrical consequences of these postulates are not developed very far by Einstein. Characterizing Einstein's unified field theory as a set of geometrical postulates in X_4, Hlavatý [7] gave its mathematical foundation for the first time. Since then the geometrical consequence of these postulates have been developed very far by numbers of mathematicians and theoretical physicists.

Generalizing X_4 to n-dimensional generalized Riemannian manifold X_n, n-dimensional generalization of this theory, so called *Einstein's n-dimensional unified field theory*(denoted by n-g-UFT), has been attempted by Wrede [11] and Mishra [10]. Corresponding to n-g-UFT, Chung [1] introduced a new unified field theory, called *the Einstein's n-dimensional $^*g$-unified field theory*(denoted by n-$^*g$-UFT), which is more useful than n-g-UFT in some physical aspects.

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On the other hand, Yano [12] and Imai [8,12] assigned a semi-symmetric metric connection to an n-dimensional Riemannian manifold and found many results concerning this manifold. Recently, Chung [3] introduced a new concept of n-dimensional SE-manifold, imposing the semi-symmetric condition to X_n and Ko [9] also introduced a new concept of ME-manifold in n-g-UFT, assigning to X_n a ME-connection which is similar to Yano and Imai's semi-symmetric metric connection.

The purpose of the present paper is to study a general representation of the $^*g$-ME-vector which holds for a general n and all possible classes. Furthermore, we introduce a special kind of representation of X_λ which holds for an even n and for the first class.

2. Preliminaries

This section is a brief collection of the basic concepts, notations, and results which are needed in our subsequent considerations in the present paper. The detailed proof are given in Hlavatý [7].

A. Generalized Riemannian manifold

Let X_n be a generalized n-dimensional Riemannian manifold referred to a real coordinate system x^ν, which obeys coordinate transformations $x^\nu \longrightarrow \bar{x}^\nu$ for which

$$Det\left(\frac{\partial \bar{x}}{\partial x}\right) \neq 0. \tag{2.1}$$

The manifold X_n is endowed which a general real non-symmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}, \tag{2.2}$$

where

$$Det(g_{\lambda\mu}) \neq 0, \qquad Det(h_{\lambda\mu}) \neq 0. \tag{2.3}$$

Hence we may define a unique tensor $h^{\lambda\nu}$ by

$$h_{\lambda\mu}h^{\lambda\nu} = \delta^\nu_\mu. \tag{2.4}$$

The tensor $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and/or lowering indices of tensor in X_n in the usual manner.

The manifold X_n is assumed to be connected by a real general real connection $\Gamma^\nu_{\lambda\mu}$ with the following transformation rule :

$$\bar{\Gamma}^\nu_{\lambda\mu} = \frac{\partial \bar{x}^\nu}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial \bar{x}^\lambda}\frac{\partial x^\gamma}{\partial \bar{x}^\mu}\Gamma^\alpha_{\beta\gamma} + \frac{\partial^2 x^\alpha}{\partial \bar{x}^\lambda \partial \bar{x}^\mu}\right) \tag{2.5}$$

B. n-dimensional ${}^*g$-unified field theory

Hlavatý characterized Einstein's 4-dimensional unified field theory(4-g-UFT) as a set of geometrical postulates in a space-time X_4 for the first time and gave its mathematical foundation. Generalizing this theory, we may consider Einstein's n-dimensional unified field theory. Similarly, our n-dimensional ${}^*g$-unified field field theory(n-${}^*g$-UFT), initiated by Chung [1] and originally suggested by Hlavatý[7], is based on the following three principles.

Principle A. The algebraic structure in n-${}^*g$-UFT is imposed on X_n by the basic real tensor ${}^*g^{\lambda\nu}$ defined by

$$g_{\lambda\mu}{}^*g^{\lambda\nu} = g_{\mu\lambda}{}^*g^{\nu\lambda} = \delta^\nu_\mu. \tag{2.6}$$

It may be decomposed into its symmetric part ${}^*h^{\lambda\nu}$ and skew-symmetric part ${}^*k^{\lambda\nu}$:

$$ {}^*g^{\lambda\nu} = {}^*h^{\lambda\nu} + {}^*k^{\lambda\nu}. \tag{2.7}$$

Since $Det({}^*h^{\lambda\nu}) \neq 0$, we may define a unique tensor ${}^*h_{\lambda\mu}$ by

$$ {}^*h_{\lambda\mu}{}^*h^{\lambda\nu} = \delta^\nu_\mu. \tag{2.8}$$

In n-${}^*g$-UFT, *we use both ${}^*h^{\lambda\nu}$ and ${}^*h_{\lambda\mu}$ as a tensors for raising and/or lowering indices of all tensor defined in X_n in the usual manner.*

Principle B. The differential geometric structure is imposed on X_n by the tensor ${}^*g^{\lambda\nu}$ by means of the connection $\Gamma^\nu_{\lambda\mu}$ defined by a system of Einstein's equations

$$D_\omega{}^*g^{\lambda\mu} = -2S_{\omega\alpha}{}^{\mu}{}^*g^{\lambda\alpha}, \tag{2.9}$$

where D_ω denotes the symbol of the covariant derivative with respect to $\Gamma^\nu_{\lambda\mu}$, and $S_{\lambda\mu}{}^\nu$ is *the torsion tensor* of $\Gamma^\nu_{\lambda\mu}$. The connection $\Gamma^\nu_{\lambda\mu}$ satisfying (2.9) is called *an Einstein's connection.* In virtue of (2.6), the system (2.9) is equivalent to the system of the original Einstein's equations

$$D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^\alpha g_{\lambda\alpha}. \tag{2.10}$$

Principle C. In order to obtain ${}^*g^{\lambda\nu}$ involved in the solution for $\Gamma^\nu_{\lambda\mu}$, certain conditions are imposed, which may be condensed to
(2.11a)
$$S_\lambda = S_{\lambda\alpha}{}^\alpha = 0, \qquad R_{[\mu\lambda]} = \partial_{[\mu}X_{\lambda]}, \qquad R_{(\mu\lambda)} = \frac{1}{2}\left(R_{\mu\lambda} + R_{\lambda\mu}\right) = 0,$$

where X_λ is an arbitrary vector, S_λ is *the torsion vector*, and

$$R_{\omega\mu\lambda}{}^\nu = 2\left(\partial_{[\mu}\Gamma^\nu_{|\lambda|\omega]} + \Gamma^\nu_{\alpha[\mu}\Gamma^\alpha_{|\lambda|\omega]}\right), \tag{2.11b}$$

$$R_{\mu\lambda} = R_{\alpha\mu\lambda}{}^\alpha, \qquad V_{\omega\mu} = R_{\omega\mu\alpha}{}^\alpha \tag{2.11c}$$

are curvature tensors of X_n.

The following quantities will be frequently used in our subsequent considerations:

$$ {}^*\mathfrak{g} = Det({}^*g_{\lambda\mu}) \neq 0, \qquad {}^*\mathfrak{h} = Det({}^*h_{\lambda\mu}) \neq 0, \qquad {}^*\mathfrak{k} = Det({}^*k_{\lambda\mu}). \tag{2.12a}$$

$$ {}^*g = \frac{{}^*\mathfrak{g}}{{}^*\mathfrak{h}} \qquad {}^*k = \frac{{}^*\mathfrak{k}}{{}^*\mathfrak{h}}, \tag{2.12b}$$

$$\sigma = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd,} \end{cases} \tag{2.13}$$

$$ {}^{(0)*}k_{\lambda^\nu} = \delta^\nu_\lambda, \quad {}^{(p)*}k_\lambda{}^\nu = {}^{(p-1)*}k_\lambda{}^{\alpha*}k_\alpha{}^\nu, \tag{2.14a}$$

$$K_0 = 1, \quad K_p = {}^*k_{[\alpha_1}{}^{\alpha_1*}k_{\alpha_2}{}^{\alpha_2}\cdots{}^*k_{\alpha_p]}{}^{\alpha_p}, \qquad (p = 1, 2, 3\cdots) \tag{2.14b}$$

$$K_{\omega\mu\nu} = \nabla_\omega {}^*k_{\nu\mu} + \nabla_\mu {}^*k_{\omega\nu} + \nabla_\nu {}^*k_{\omega\mu}, \tag{2.15}$$

where ∇_ω is the symbolic vector of the covariant derivative with respect to the Christoffel symbol ${}^*\left\{ {\nu \atop \lambda\mu} \right\}$ defined by ${}^*h_{\lambda\mu}$.

It has been shown that the following relations hold in X_n ([1],[2],[5]):

$$K_p = \begin{cases} 0 & \text{if } p \text{ is odd,} \\ {}^*k & \text{if } p \text{ is even,} \end{cases} \tag{2.16a}$$

$$Det(M^*h_{\lambda\mu} + {}^*k_{\lambda\mu}) = {}^*\mathfrak{h} \sum_{s=0}^{n-\sigma} K_s M^{n-s}, \qquad (M\, :\, \text{a real number}), \tag{2.16b}$$

$$\sum_{s=0}^{n-\sigma} K_s {}^{(n-s)*}k_\lambda{}^\nu = 0. \tag{2.17}$$

Here and in what follows, the index s is assumed to take the value $0, 2, 4, 6 \cdots$ in the specified range.

It has also been shown that if the equations (2.9) admits a solution $\Gamma^\nu_{\lambda\mu}$, the symmetric part of (2.9) implies that it must be of the form

$$\Gamma^\nu_{\lambda\mu} = {}^*\left\{ {\nu \atop \lambda\mu} \right\} + S_{\lambda\mu}{}^\nu + {}^*U^\nu{}_{\lambda\mu}, \tag{2.18}$$

where

$${}^*U^\nu{}_{\lambda\mu} = S_{\beta(\lambda}{}^{\nu *}k_{\mu)}{}^\beta + S^\nu{}_{\beta(\lambda}{}^*k_{\mu)}{}^\beta - S^\beta{}_{(\lambda\mu)}{}^*k_\beta{}^\nu. \tag{2.19}$$

The skew-symmetric part of (2.9) gives the following relations satisfied by the torsion tensor $S_{\omega\mu\nu}$:

$$B_{\omega\mu\nu} = S_{\omega\mu\nu} + \overset{101}{S}{}_{\omega\mu\nu} + \overset{011}{S}{}_{\omega\mu\nu} + \overset{110}{S}{}_{\omega\mu\nu}, \tag{2.20}$$

where

$$B_{\omega\mu\nu} = \frac{1}{2}\left(K_{\omega\mu\nu} + 3K_{[\alpha\beta\gamma]}{}^*k_\omega{}^{\alpha *}k_\mu{}^{\beta *}k_\nu{}^\gamma\right), \tag{2.21}$$

$$\overset{pqr}{S}{}_{\omega\mu\nu} = S_{\alpha\beta\gamma}{}^{(p)*}k_\omega{}^{\alpha (q)*}k_\mu{}^{\beta (r)*}k_\nu{}^\gamma, \quad (p, q, r = 1, 2, 3 \cdots). \tag{2.22}$$

C. The manifold $^*g$-MEX_n in n-$^*g$-UFT

All results and symbols in this subsection are based on [4].

DEFINITION 2.1 The Einstein's connection $\Gamma^\nu_{\lambda\mu}$ which take the form

$$\Gamma^\nu_{\lambda\mu} = {}^*\left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + 2\delta^\nu_\lambda X_\mu - 2^* g_{\lambda\mu} X^\nu \tag{2.23}$$

for a non-null vector X_λ is called a $^*g$-ME-connection in n-$^*g$-UFT, and X_λ is the corresponding $^*g$-ME-vector.

If X_n admits a $^*g$-ME-connection $\Gamma^\nu_{\lambda\mu}$, it must be of the form (2.18). Hence, comparing (2.18) and (2.23) we have the following relations :

$$S_{\lambda\mu}{}^\nu = 2\delta^\nu_{[\lambda} X_{\mu]} - 2^* k_{\lambda\mu} X^\nu, \tag{2.24}$$

$$^*U^\nu{}_{\lambda\mu} = 2\delta^\nu_{(\lambda} X_{\mu)} - 2^* h_{\lambda\mu} X^\nu. \tag{2.25}$$

THEOREM 2.2. *A necessary and sufficient condition for the system (2.9) to admit a $^*g$-ME-connection $\Gamma^\nu_{\lambda\mu}$ of the form (2.23) is that the tensor field $^*g_{\lambda\mu}$ satisfies the relation*

$$\nabla_\omega {}^* k_{\lambda\mu} = 2\left({}^*h_{\omega[\lambda} {}^* g_{\mu]\beta} - {}^*h_{\omega\beta} {}^* k_{\lambda\mu} \right) C_\alpha B^{\alpha\beta}. \tag{2.26}$$

If this condition is satisfied, then

$$X^\nu = C_\alpha B^{\alpha\nu}, \tag{2.27}$$

where

$$C_\lambda = \nabla_\alpha {}^* k_\lambda{}^\alpha. \tag{2.28}$$

Hence, if the system (2.27) is satisfied, we note that there always exists a unique $^*g$-ME-connection $\Gamma^\nu_{\lambda\mu}$ in our n-$^*g$-UFT. In virtue of (2.23) and (2.27), this connection may be written as

$$\Gamma^\nu_{\lambda\mu} = {}^*\left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + 2\left(\delta^{\nu *}_\lambda h_{\mu\beta} - {}^* g_{\lambda\mu} \delta^\nu_\beta \right) C_\alpha A^{\alpha\beta}. \tag{2.29}$$

In our further considerations in this paper, we use the word "present condition" to describe the situtations that the condition (2.12a) and (2.26) are imposed on the unified field tensor $^*g^{\lambda\nu}$.

DEFINITION 2.3 An n-dimensional generalized Riemannian manifold X_n, on which the differential geometric structure is imposed by the tensor $^*g^{\lambda\nu}$ under the present condition by means of the $^*g$-ME-connection given by (2.29), is called an n-dimensional $^*g$-ME-manifold and denoted by $^*g$-MEX_n.

3. A general representation of the $^*g$-ME-vector in $^*g$-MEX_n

This section is concerned mainly with a general representation of the $^*g$-ME-vector which holds for a general n and all possible classes.

In our further considerations, we use the following abbreviation for an arbitrary real vector A_λ :

$$ {}^{(p)}A_\lambda = {}^{(p)*}k_\lambda{}^\alpha A_\alpha, \tag{3.1a} $$

$$ {}^{(p)}A^\nu = (-1)^{p}\,{}^{(p)*}k_\alpha{}^\nu A^\alpha, \qquad (p = 0, 1, 2, \cdots). \tag{3.1b} $$

We need a symmetric tensor :

$$ P_{\lambda\mu} = {}^{(2)*}k_{\lambda\mu} - {}^*h_{\lambda\mu}, \tag{3.2a} $$

and its unique inverse tensor $Q^{\lambda\nu}$ defined by

$$ P_{\lambda\mu}Q^{\lambda\nu} = \delta^\nu_\mu. \tag{3.2b} $$

We use the following quantities :

$$ N = \frac{1-n}{2}, \tag{3.3a} $$

$$ \widehat{K}_s = \sum_{t=0}^{s} K_t N^{s-t}, \tag{3.3b} $$

$$ Y_\omega = \frac{1}{2} Q^{\nu\mu} B_{\omega\mu\nu}. \tag{3.3c} $$

In virtue of (3.3a) and (3.3b), direct calculations show that

$$ \widehat{K}_s = K_s + \widehat{K}_{s-2}N^2. \tag{3.4} $$

By multiplying A_ν to both sides of (2.17) and using (3.1b), every vector A_ω satisfies the following recurrence relation :

$$ \sum_{s=0}^{n-\sigma} K_s{}^{(n-s)}A_\omega = 0, \tag{3.5a} $$

or equivalently

$$ {}^{(n)}A_\omega + K_2{}^{(n-2)}A_\omega + \cdots + K_{n-\sigma-2}{}^{(\sigma+2)}A_\omega + K_{n-\sigma}{}^{(\sigma)}A_\omega = 0. \tag{3.5b} $$

THEOREM 3.1. *Under the present condition, the following relation holds in $^*g$-MEX_n :*

$$B_{\omega\mu\nu} = -2P_{\nu[\omega}X_{\mu]} + 2^*k_\omega{}^\alpha P_{\alpha\mu}X_\nu. \tag{3.6}$$

Proof. Employing the abbreviations introduced in (2.22) and making use of (2.24) and (3.1a), it following that

$$\begin{aligned} \overset{pqr}{S}{}_{\omega\mu\nu} &= ({}^*h_{\gamma\alpha}X_\beta - {}^*h_{\gamma\beta}X_\alpha - 2^*k_{\alpha\beta}X_\gamma)\,{}^{(p)*}k_\omega{}^{\alpha\,(q)*}k_\mu{}^{\beta\,(r)*}k_\nu{}^\gamma \\ &= (-1)^{r(p+r)*}k_{\omega\nu}{}^{(q)}X_\alpha - (-1)^{q(q+r)*}k_{\nu\mu}{}^{(q)}X_\omega \\ &\qquad\qquad - 2(-1)^{q(p+q+r)*}k_{\omega\mu}{}^{(r)}X_\nu. \end{aligned} \tag{3.7}$$

Consequently, using (3.7) the relation (2.20) is reduced to (3.6) as in the following way :

$$\begin{aligned} B_{\omega\mu\nu} &= S_{\omega\mu\nu} + \overset{101}{S}{}_{\omega\mu\nu} + \overset{011}{S}{}_{\omega\mu\nu} + \overset{110}{S}{}_{\omega\mu\nu} \\ &= 2\left({}^*h_{\nu[\omega} - {}^{(2)*}k_{\nu[\omega}\right)X_{\mu]} + 2\left({}^{(3)*}k_{\omega\mu} - {}^*k_{\omega\mu}\right)X_\nu \\ &= -2P_{\nu[\omega}X_{\mu]} + 2^*k_\omega{}^\alpha P_{\alpha\mu}X_\nu. \end{aligned}$$

□

THEOREM 3.2. *Under the present condition, the following relation holds in $*g$-MEX_n :*

$$^{(p)}X_\omega = {}^{(p-1)}Y_\omega + N^{(p-2)}Y_\omega + N^{2(p-2)}X_\omega, \quad (p = 1, 2, 3, \cdots). \tag{3.8}$$

Proof. Multiplying $Q^{\nu\mu}$ to both sides of (3.6) and making use of (3.2a), we have

$$Q^{\nu\mu}B_{\omega\mu\nu} = (n-1)X_\omega + 2^*k_\omega{}^\alpha X_\alpha = (n-1)X_\omega + 2^{(1)}X_\omega. \tag{3.9a}$$

Comparing (3.3c) and (3.9a) we have the following condition

$$^{(1)}X_\omega = Y_\omega + NX_\omega. \tag{3.9b}$$

Our assertion (3.8) immediately follows from (3.1a) and (3.9). □

Now, we are ready to prove a general representation of a $^*g$-ME-vector in the following theorem.

THEOREM 3.3. *Under the present condition, the ${}^*g$-ME-vector X_ω in ${}^*g$-MEX_n may be given by*

$$(\sigma - 1 - \sigma N)\widehat{K}_{n-\sigma} X_\omega = \sum_{s=0}^{n-\sigma-2} \widehat{K}_s \left({}^{(n-s-1)}Y_\omega + N^{(n-s-2)}Y_\omega \right) + \sigma \widehat{K}_{n-\sigma} Y_\omega. \tag{3.10}$$

Proof. Substituting (3.8) into (3.5b) with A_ω replaced by X_ω and using (3.3b) and (3.4), we have

$$\begin{aligned} &\widehat{K}_0 \left({}^{(n-1)}Y_\omega + N^{(n-2)}Y_\omega \right) + (K_2 + N^2)^{(n-2)}X_\omega + K_4{}^{(n-4)}X_\omega \\ &+ \cdots + K_{(n-\sigma-2)}{}^{(\sigma+2)}X_\omega + K_{n-\sigma}{}^{(\sigma)}X_\omega = 0. \end{aligned} \tag{3.11a}$$

Substituting ${}^{(n-2)}X_\omega$ again from (3.8) into (3.11a), we have

$$\begin{aligned} &\widehat{K}_0 \left({}^{(n-1)}Y_\omega + N^{(n-2)}Y_\omega \right) + \widehat{K}_2 \left({}^{(n-3)}Y_\omega + N^{(n-4)}Y_\omega \right) \\ &+ (K_4 + N^2)^{(n-4)}X_\omega + K_6{}^{(n-6)}X_\omega + \cdots + \widehat{K}_{(n-\sigma-2)}{}^{(\sigma+2)}X_\omega \\ &+ \widehat{K}_{n-\sigma}{}^{(\sigma)}X_\omega = 0. \end{aligned} \tag{3.11b}$$

After $\frac{n-\sigma}{2}$ steps of successive repeated substitutions for ${}^{(p)}X_\omega$, we have

$$\begin{aligned} &\widehat{K}_0 \left({}^{(n-1)}Y_\omega + N^{(n-2)}Y_\omega \right) + \widehat{K}_2 \left({}^{(n-3)}Y_\omega + N^{(n-4)}Y_\omega \right) \\ &+ \widehat{K}_4 \left({}^{(n-5)}Y_\omega + N^{(n-6)}Y_\omega \right) + \cdots \\ &+ \widehat{K}_{(n-\sigma-2)} \left({}^{(\sigma+1)}Y_\omega + N^{(\sigma)}Y_\omega \right) \\ &+ \widehat{K}_{n-\sigma}{}^{(\sigma)}X_\omega = 0. \end{aligned} \tag{3.11c}$$

On the other hand, it follows from (3.1a) and (3.9b) that

$${}^{(\sigma)}X_\omega = \sigma Y_\omega + (\sigma N - \sigma + 1)X_\omega. \tag{3.12}$$

Substituting (3.12) into (3.11c), we finally have the representation (3.10). □

THEOREM 3.4. *there exists a unique $^*g$-ME-vector in $^*g$-MEX_n if and only if the following condition holds for $^*g_{\lambda\mu}$:*

$$\widehat{K}_{n-\sigma} \neq 0. \tag{3.13}$$

Proof. In virtue of (3.10), there exists a unique X_ω if $(\sigma-1-\sigma N)\widehat{K}_{n-\sigma} \neq 0$. Hence the condition (3.13) immediately follows since $(\sigma-1-\sigma N) \neq 0$. □

4. A special representation of the $^*g$-ME-vector in $^*g$-MEX_n

In this section we present a quite different type of a representation of a $^*g$-ME-vector from the general one found in the previous section, which holds in an even-dimensional $^*g$-ME-manifold with a certain special condition imposed on $^*g_{\lambda\mu}$.

In this section we need a tensor $F_{\lambda\mu}$ defined by

$$F_{\lambda\mu} = {}^*k_{\lambda\mu} - 2^{(2)*}k_{\lambda\mu}. \tag{4.1}$$

LEMMA 4.1. *The tensor $F_{\lambda\mu}$ is of rank n if and only if the tensor field $^*g_{\lambda\mu}$ satisfied the following condition:*

$$^*\mathfrak{k}\sum_{s=0}^{n-\sigma} 2^s K_s \neq 0. \tag{4.2}$$

Proof. In virtue of (4.1), we have

$$F_{\lambda\mu} = 2^*k_{\lambda\alpha}\left(\frac{1}{2}{}^*h_{\mu\beta} + {}^*k_{\mu\beta}\right){}^*h^{\alpha\beta}. \tag{4.3}$$

Our assertion follows from the following relation which may be obtained from (4.3) and (2.16b) :

$$Det(F_{\lambda\mu}) = 2^{n*}\mathfrak{k}\left({}^*\mathfrak{h}\sum_{s=0}^{n-\sigma} K_s(\frac{1}{2})^{n-s}\right)\frac{1}{^*\mathfrak{h}} = {}^*\mathfrak{k}\sum_{s=0}^{n-\sigma} 2^s K_s.$$

□

In our further considerations in this section, we restrict ourselves to an even-dimensional $^*g$-ME-manifold and use the word "special condition" to describe the situations that the tensor field $^*g_{\lambda\mu}$ satisfies the condition

$$\sum_{s=0}^{n-\sigma} 2^s K_s \neq 0. \tag{4.4}$$

Therefore, under the special condition the tensor $F_{\lambda\mu}$ is of rank n, so that there exists a unique inverse tensor $G^{\lambda\nu}$ defined by

$$G^{\lambda\nu} F_{\lambda\mu} = G^{\nu\lambda} F_{\mu\lambda} = \delta^{\nu}_{\mu}. \tag{4.5}$$

THEOREM 4.2. *Under the special condition in an even-dimensional $^*g$-ME-manifold, $^*g$-ME-vector X_ω may be given by the following relation:*

$$X^{\nu} = -\frac{1}{2} G^{\nu\lambda} \partial_\alpha (log\ ^*g). \tag{4.6}$$

Proof. Multiplying $g_{\lambda\mu}$ to both sides of (2.9), we have

$$\partial_\omega log\ ^*\mathfrak{g} + 2\Gamma^{\alpha}_{\alpha\omega} = -2S_{\omega\alpha}{}^{\alpha}. \tag{4.7}$$

On the other hand, multiply $^*h_{\lambda\mu}$ to both sides of the symmetric part of (2.9) and making use of (2.12), (2.14) and (2.24) to obtain

$$\partial_\omega log\ ^*\mathfrak{h} + 2\Gamma^{\alpha}_{\alpha\omega} = -2S_{\omega\alpha}{}^{\alpha} - 2\left({}^*k_{\omega\alpha} - 2^{(2)*}k_{\omega\alpha}\right) X^{\alpha}. \tag{4.8}$$

Subtraction of (4.8) from (4.7) and using of (2.12b) and (4.1) gives the following relation:

$$\partial_\omega log\ ^*g = 2\left({}^*k_{\omega\alpha} - 2^{(2)*}k_{\omega\alpha}\right) X_{\alpha} = -2F_{\nu\omega} X^{\nu}. \tag{4.9}$$

The representation (4.6) immediately follows by multiplying $G^{\lambda\omega}$ to both sides of (4.9) and making use of (4.5). □

REMARK 4.3 In virtue of Theorem 4.2, our investigation of the $^*g$-ME-vector under the special condition is reduced to the study of the tensor $G^{\lambda\nu}$. In order to know the $^*g$-ME-vector it is necessary and sufficient to know an explicit representation of $G^{\lambda\nu}$ in terms of $^*g_{\lambda\mu}$.

In our further consideration, we need the abbreviation ${}^{(p)}X^{\lambda\nu}$ for an arbitrary tensor $X^{\lambda\nu}$ and notations $\overset{\dagger}{K}_s$ defined by

$$(4.10) \qquad {}^{(0)}X^{\lambda\nu} = X^{\lambda\nu}, \quad {}^{(p)}X^{\lambda\nu} = {}^{(p)*}k^{\lambda}{}_{\alpha}X^{\alpha\nu} \quad (p = 1, 2, 3, \cdots),$$

$$(4.11) \qquad \overset{\dagger}{K}_s = \frac{1}{4}\sum_{t=0}^{s} \frac{1}{2^t} K_{s-t}.$$

The following relations are immediately consequence of (4.10) and (4.11)

$$(4.12) \qquad {}^{(p)*}k^{\lambda}{}_{\mu}{}^{(q)}X^{\mu\nu} = {}^{(p+q)}X^{\lambda\nu}, \quad (q = 1, 2, 3, \cdots),$$

$$(4.13) \qquad {}^{(p)*}k_{\lambda}{}^{\omega\,(q)}X_{\omega}{}^{\nu} = {}^{(p+q)}X_{\lambda}{}^{\nu},$$

$$(4.14a) \qquad \overset{\dagger}{K}_0 = \frac{1}{4}, \ \ \overset{\dagger}{K}_2 = \frac{1}{4}(K_2 + \frac{1}{4}), \ \ \overset{\dagger}{K}_4 = \frac{1}{4}(K_4 + \frac{1}{4}K_2 + \frac{1}{16}), \cdots,$$

$$(4.14b) \qquad \overset{\dagger}{K}_s = \frac{1}{4}\left(K_s + \overset{\dagger}{K}_{s-2}\right).$$

THEOREM 4.4. *In an even-dimensional* ${}^*g$-MEX_n, *the tensor* ${}^{(p)}G^{\lambda\nu}$ *satisfies the following recurrence relation :*

$$(4.15a) \qquad \sum_{s=0}^{n} K_s{}^{(n-s)}G^{\lambda\nu} = 0,$$

or equivalently

$$(4.15b) \qquad {}^{(n)}G^{\lambda\nu} + K_2{}^{(n-2)}G^{\lambda\nu} + \cdots + K_{n-2}{}^{(2)}G^{\lambda\nu} + K_n G^{\lambda\nu} = 0.$$

Proof. The relations (4.15a) and (4.15b) follow by multiplying $G^{\lambda\mu}$ to both sides of (2.17) and using (4.10). Note that $n - s$ is even, so that ${}^{(n-s)*}k_{\lambda\mu}$ is symmetric. $\square$

THEOREM 4.5. *Under the special condition, the following relations hold in ${}^*g$-MEX_n :*

$$(4.16a)\qquad {}^{(p+2)}G^{\lambda\nu}+\frac{1}{2}{}^{(p+1)}G^{\lambda\nu}+\frac{1}{2}{}^{(p)*}k^{\lambda\nu}=0,\quad (p=0,1,2,\cdots),$$

$$(4.16b)\qquad {}^{(q)}G^{\lambda\nu}=\frac{1}{4}{}^{(q-2)}G^{\lambda\nu}-\frac{1}{2}{}^{(q-2)*}k^{\lambda\nu}+\frac{1}{4}{}^{(q-3)*}k^{\lambda\nu},\quad (q=3,4,5,\cdots).$$

Proof. Substituting of (4.1) into (4.5) and making use of (2.8) gives

$$(4.17)\qquad 2^{(2)}G^{\lambda\mu}+{}^{(1)}G^{\lambda\mu}+{}^*h^{\lambda\mu}=0.$$

The relation (4.16a) may be obtained by multiplying $\frac{1}{2}{}^{(p)*}k^{\nu}{}_{\lambda}$ to both sides of (4.17). Using (4.16a) twice, we have the relation (4.16b). □

LEMMA 4.6. *If the tensor field $G^{\lambda\nu}$ satisfies the following equation under the special condition in ${}^*g$-MEX_n,*

$$(4.18a)\qquad A^{(2)}G^{\lambda\nu}+BG^{\lambda\nu}+\Lambda^{\lambda\nu}=0,$$

then the tensor $G^{\lambda\nu}$ must be of the form

$$(4.18b)\qquad B(A+4B)G^{\lambda\nu}=2AB^*h^{\lambda\nu}+A^{2*}k^{\lambda\nu}-(A+4B)\Lambda^{\lambda\nu}-2A^{(1)}\Lambda^{\lambda\nu},$$

where A,B and $\Lambda^{\lambda\nu}$ are functions of ${}^*g_{\lambda\mu}$.

Proof. Substitution (4.17) into (4.18a) for ${}^{(2)}G^{\lambda\nu}$ gives

$$(4.19a)\qquad A^{(1)}G^{\lambda\nu}=2BG^{\lambda\nu}-A^*h^{\lambda\nu}+2\Lambda^{\lambda\nu}.$$

Multiplying ${}^*k^{\mu}{}_{\lambda}$ to both sides of (4.19a), we have

$$(4.19b)\qquad A^{(2)}G^{\lambda\nu}=2B^{(1)}G^{\lambda\nu}-A^*k^{\lambda\nu}+2\Lambda^{\lambda\nu}.$$

Substitution of (4.17) into (4.19b) for ${}^{(2)}G^{\lambda\nu}$ again gives

$$(4.19c)\qquad (\frac{A}{2}+2B)^{(1)}G^{\lambda\nu}=-\frac{A}{2}{}^*h^{\lambda\nu}+A^*k^{\lambda\nu}-2^{(1)}\Lambda^{\lambda\nu}.$$

Consequently, our assertion (4.18b) follows by elliminating the tensor ${}^{(1)}G^{\lambda\nu}$ from (4.19a) and (4.19c). □

Now, we are ready to prove the following main theorem in this section, which present a representation of the tensor $G^{\lambda\nu}$ under the special condition.

Theorem 4.7. *Under the special condition in an even-dimensional* $^*g$*-*MEX_n*, the tensor* $G^{\lambda\nu}$ *may be given by*

$$2^*k\overset{\dagger}{K}_n G^{\lambda\nu} = \overset{\dagger}{K}_{n-2}\left({}^*k^*h^{\lambda\nu} + 2\overset{\dagger}{K}_{n-2}{}^*k^{\lambda\nu}\right) - 2\overset{\dagger}{K}_n\Lambda^{\lambda\nu} - \overset{\dagger}{K}_{n-2}{}^{(1)}\Lambda^{\lambda\nu}, \tag{4.20}$$

where

$$\Lambda^{\lambda\nu} = \sum_{s=0}^{n-4}\overset{\dagger}{K}_s\left(-2^{(n-2-s)*}k^{\lambda\nu} + {}^{(n-3-s)*}k^{\lambda\nu}\right). \tag{4.21}$$

Proof. Substituting (4.16b) into (4.15b) for ${}^{(n)}G^{\lambda\nu}$ and making use of (4.14), we have

$$\overset{\dagger}{K}_0\left(-2^{(n-2)*}k^{\lambda\nu} + {}^{(n-3)*}k^{\lambda\nu}\right) + 4\overset{\dagger}{K}_2{}^{(n-2)}G^{\lambda\nu} + \cdots + \overset{\dagger}{K}_{n-2}{}^{(2)}G^{\lambda\nu} + \overset{\dagger}{K}_n G^{\lambda\nu} = 0. \tag{4.22a}$$

Substituting again for ${}^{(n-2)}G^{\lambda\nu}$ into (4.22a) from (4.16b) gives

$$\overset{\dagger}{K}_0\left(-2^{(n-2)*}k^{\lambda\nu} + {}^{(n-3)*}k^{\lambda\nu}\right) + \overset{\dagger}{K}_2\left(-2^{(n-4)*}k^{\lambda\nu} + {}^{(n-5)*}k^{\lambda\nu}\right) + 4\overset{\dagger}{K}_4{}^{(n-4)}G^{\lambda\nu} + \cdots + K_{n-2}{}^{(2)}G^{\lambda\nu} + K_n G^{\lambda\nu} = 0. \tag{4.22b}$$

After $\frac{n-2}{2}$ steps of successive repeated substitution for ${}^{(q)}G^{\lambda\nu}$, we have in virtue of (4.21)

$$\Lambda^{\lambda\nu} + 4\overset{\dagger}{K}_{n-2}{}^{(2)}G^{\lambda\nu} + K_n G^{\lambda\nu} = 0. \tag{4.22c}$$

Comparison of (4.22c) with (4.19b) gives

$$A = 4\overset{\dagger}{K}_{n-2}, \qquad B = K_n = {}^*k. \tag{4.23}$$

Consequently, the relation (4.20) follows by substituting (4.23) into (4.18b) and making use of (4.14b). □

Now that we have obtained a representation of $G^{\lambda\nu}$ in Theorem 4.7, under the special condition it is possible for us to represent the $^*g$-ME-vector X^ν in terms of $^*g^{\lambda\nu}$ by only substituting (4.20) into (4.6).

THEOREM 4.8. *Under the special condition in an even-dimensional $^*g$-MEX_n, the $^*g$-ME-vector X^ν may be given by*

$$
\begin{aligned}
4^*k\overset{\dagger}{K}_n X^\nu = &-(\overset{\dagger}{K}_{n-2}(^*k^*h^{\nu\alpha} + 2\overset{\dagger}{K}_{n-2}{}^*k^{\nu\alpha}) \\
&- 2\overset{\dagger}{K}_n\Lambda^{\nu\alpha} + \overset{\dagger}{K}_{n-2}{}^{(1)}\Lambda^{\nu\alpha})\partial_\alpha(log\ ^*g).
\end{aligned}
\tag{4.24}
$$

REMARK 4.9 In virtue of (2.14a), (4.10), (4.14b) and (4.21), we may represent the last two terms on the right-hand side of (4.24) as follows :

(4.25)

$$
\begin{aligned}
&- 2\overset{\dagger}{K}_n\Lambda^{\nu\alpha} + \overset{\dagger}{K}_{n-2}{}^{(1)}\Lambda^{\nu\alpha} \\
&= \sum_{s=0}^{n-4}\overset{\dagger}{K}_s\left(2\overset{\dagger}{K}_{n-2}{}^{(n-1-s)*}k^{\nu\alpha} + {}^*k^{(n-2-s)*}k^{\lambda\nu} - 2\overset{\dagger}{K}_n{}^{(n-3-s)*}k^{\nu\alpha}\right).
\end{aligned}
$$

Therefore, we know that the $^*g$-ME-vector X^ν representation in terms of $^*g_{\lambda\mu}$.

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